A Non-commutative Extension of MELL

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Abstract *We extend multiplicative exponential linear logic* (MELL) *by a non-commutative, self-dual logical operator. The extended system, called* NEL*, is defined in the formalism of the calculus of structures, which is a generalisation of the sequent calculus and provides a more refined analysis of proofs. We should then be able to extend the range of applications of* MELL*, by modelling a broad notion of sequentiality and providing new properties of proofs. We show some proof theoretical results: decomposition and cut elimination. The new operator represents a significant challenge: to get our results we use here for the first time some novel techniques, which constitute a uniform and modular approach to cut elimination, contrary to what is possible in the sequent calculus.*

1 Introduction

Non-commutative logical operators have a long tradition [12, 22, 2, 13, 16, 3], and their proof theoretical properties have been studied in the sequent calculus [7] and in proof nets [8]. Recent research has shown that the sequent calculus is not adequate to deal with very simple forms of non-commutativity [9, 10, 21]. On the other hand, proof nets are not ideal for dealing with exponentials and additives, which are desirable for getting good computational power.

In this paper we show a logical system that joins a simple form of non-commutativity with commutative multiplicatives and exponentials. This is done in the formalism of the calculus of structures [9, 10], which overcomes the difficulties encountered in the sequent calculus and in proof nets. Structures are expressions intermediate between formulae and sequents, and in fact they unify those two latter entities into a single one, thereby allowing more control over mutual dependencies of logical relations.

We perform a proof theoretical analysis for cut elimination, with new tools, and we explore some further important properties, which are not available in more traditional settings and which we can collectively regard as 'modularity'. Despite the complexities of the proof theoretical investigation, the system obtained is very simple. This paper contributes the following new results:

1 We define a propositional logical system, called NEL (Non-commutative exponential linear logic), which extends MELL (multiplicative exponential linear logic [8]) by a non-commutative, self-dual logical operator called *seq*. This system, which was first imagined in [10], is conservative over MELL augmented by the mix and nullary mix rules [1, 6]. System NEL can be immediately understood by anybody acquainted with the sequent calculus, and is aimed at the same range of applications as MELL. In nearly all computer science languages, sequential composition plays a fundamental role,

and it is therefore important to address it in a direct way, in logical representations of those languages. Perhaps surprisingly, parallel composition has been much easier to deal with, due to its commutative nature, which is more similar to the typical nature of traditional logics. The addition of seq opens new syntactic possibilities, for example in dealing with process algebras. It has been used already, in a purely multiplicative setting, to model CCS's prefixing [5]. Furthermore, we show a class of equivalent extensions of NEL, which all enjoy the subformula property. This, together with the finer detail in derivations achieved by the calculus of structures, provides much greater flexibility, as witnessed by the proof theoretical properties mentioned below.

2 We prove for NEL a property called *decomposition* (first pioneered in [10, 19]): we can transform every derivation into an equivalent one, composed of seven derivations carried into seven *disjoint* subsystems of NEL. We can study small subsystems of NEL in isolation and then compose them together with considerable more freedom than in the sequent calculus, where, for example, contraction can not be isolated in a derivation. Decomposition is made available in the calculus of structures by exploiting a new top-down symmetry of derivations. Since it is a basic compositional result, we expect applications to be very broad in range; we are especially excited about the possibilities in the semantics of derivations.

3 We prove cut elimination for NEL by use of decomposition and a new technique that we call *splitting*. In the calculus of structures the traditional methods for proving cut elimination fail, due to the more general applicability of inference rules. The deep reason for this is in how the calculus deals with associativity. Splitting theorems are a uniform means of recovering control over the way logical operators associate; they allow us to manage the complex inductions required. The cut elimination argument becomes modular, because we can reduce the cut rule to several more primitive inference rules, each of which is separately shown admissible by way of splitting. Only one of these rules (an atomic form of cut) is infinitary, all the others enjoy the subformula property and can be used to extend the system without affecting provability. This result should be handy for software analysis and verification.

The points above correspond, respectively, to Sections 2, 3 and 4. Readers who are not interested in the proof theory of system NEL can just read Section 2.

Other systems extending linear logic with non-commutative operators are studied in [3, 18]. These are more traditional systems in the sequent calculus, for which a more limited proof theory can be developed. The calculus of structures allows us to design a much simpler logic, as witnessed by the fact that we have just one self-dual non-commutative operator instead of two dual ones.

It is worth noting that every system that can be expressed in the one-sided sequent calculus can be trivially expressed in the calculus of structures, but the vice versa is not true. The results in this paper help us to establish the calculus of structures as a natural choice for logical systems aimed at computer science. We showed in [10] that the sequent calculus suffers from excessive restrictions, which are not apparent in the traditional systems of classical and intuitionistic logics, but which start to appear in linear logic and are more and more evident when issues such as non-commutativity, locality of inference rules, and various forms of modularity are taken into account. The calculus of structures was in fact conceived, in [9], as a way to overcome the limitations of the sequent calculus in dealing with non-commutativity. Our calculus has later been used successfully in [19] for defining pure MELL and showing decomposition and cut elimination for it. In [4] a completely local definition of classical logic is shown: in that system, not only the cut rule, but also contraction is atomic.

The calculus of structures essentially introduces two new ideas: 1) it makes derivations top-down symmetric and 2) it allows inference rules to be applied anywhere deep inside structures. We are showing, in this and other papers, that it is possible to produce a rich proof theory in our calculus. This formalism is less dependent than the sequent calculus or natural deduction on the original idiosyncrasies of classical (and intuitionistic) logic, and it is actually designed with notions of locality, atomicity and modularity in mind. For these reasons we promote the calculus of structures as a worthy tool for syntactic investigations related to computer science languages.

In the following sections some proof theory is developed for system NEL. We stress the fact that the methods used are general. As stated above, many techniques in this paper are new, but we tested them privately on the systems that have already been studied, namely BV, MELL and classical logic, and in some systems that we are currently investigating, like full linear logic, also in its entirely atomic presentation [20].

The results in this paper are shown in detail in [11].

2 The System

We call *calculus* a formalism, like natural deduction or the sequent calculus, for specifying logical systems. We say (*formal*) *system* to indicate a collection of inference rules in a given calculus.

A system in our calculus requires a language of *structures*, which are intermediate expressions between formulae and sequents. We now define the language for system NEL and its variants. Intuitively, $[S_1, \dots, S_h]$ corresponds to a sequent $\vdash S_1, \dots, S_h$ in linear logic, whose formulae are essentially connected by pars, subject to commutativity (and associativity). The structure $(S_1, \dots, S_h)$ corresponds to the associative and commutative times connection of $S_1, \dots, S_h$. The structure $\langle S_1; \dots; S_h \rangle$ is associative and *non-commutative*: this corresponds to the new logical operator, called *seq*, that we add to those of MELL.

For reasons explained in [9, 10], dealing with seq involves adding the rules mix and its nullary version mix0 (see [1, 6, 14]):

$$\mathsf{mix}\,\frac{\vdash \Phi \quad \vdash \Psi}{\vdash \Phi, \Psi} \qquad \text{and} \qquad \mathsf{mix0}\,\frac{}{\vdash} \quad .$$

This has the effect of collapsing the multiplicative units 1 and $\bot$: we will only have one unit $\circ$ common to par, times and seq. Please notice that mix and mix0 are not an artefact of the calculus of structures. But, as shown by Retoré in [14], they are required when using a self-dual non-commutative connective.

2.1 Definition There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by a, b, ... *Structures* are denoted by S, P, Q, R, T, U, V and X. The structures of the *language* NEL are generated by

$$S ::= a \mid \circ \mid [\underbrace{S, \dots, S}_{>0}] \mid (\underbrace{S, \dots, S}_{>0}) \mid \langle \underbrace{S; \dots; S}_{>0} \rangle \mid ?S \mid !S \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not an atom; $[S_1, \dots, S_h]$ is a *par structure*, $(S_1, \dots, S_h)$ is a *times structure*, $\langle S_1; \dots; S_h \rangle$ is a *seq structure*, $?S$ is a *why-not structure* and $!S$ is an *of-course structure*; $\bar{S}$ is the *negation* of the structure S. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R, T]$ stands for $S\{[R, T]\}$.

Structures come with equational theories establishing some basic, decidable algebraic laws by which structures are indistinguishable. These are analogous to the laws of associativity, commutativity, idempotency, and so on, usually imposed on sequents. The difference is that we merge the notions of formula and sequent, and we extend the equations to formulae. We will show these equational laws together with the inference rules.

2.2 Definition An (*inference*) *rule* is any scheme $\rho \dfrac{T}{R}$, where ρ is the *name* of the rule, T is its *premise* and R is its *conclusion*; R or T, but not both, may be missing. Rule names are denoted by ρ. A (*formal*) *system*, denoted by $\mathscr{S}$, is a set of rules. A *derivation* in a system $\mathscr{S}$ is a finite chain of instances of rules of $\mathscr{S}$, and is denoted by Δ; a derivation can consist of just one structure. The topmost structure in a derivation is called its *premise*; the bottommost structure is called *conclusion*. A derivation Δ whose premise is T, conclusion is R, and whose rules are in $\mathscr{S}$ is denoted by $\begin{array}{c} T \\ \Delta \| \mathscr{S} \\ R \end{array}$.

The typical inference rules are of the kind $\rho \dfrac{S\{T\}}{S\{R\}}$. This rule scheme ρ specifies that if a structure matches R, in a context $S\{\ \}$, it can be rewritten as specified by T, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). A rule corresponds to implementing in the formal system *any axiom* $T \Rightarrow R$, where $\Rightarrow$ stands for the implication we model in the system, in our case linear implication. The case where the context is empty corresponds to the sequent calculus. For example, the linear logic sequent calculus rule $\otimes \dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi}$ could be simulated easily in the calculus of structures by the rule $\otimes' \dfrac{(\Gamma, [A, \Phi], [B, \Psi])}{(\Gamma, [(A, B), \Phi, \Psi])}$, where Φ and Ψ stand for multisets of formulae or their corresponding par structures. The structure Γ stands for the times structure of the other hypotheses in the derivation tree. More precisely, any sequent calculus derivation

$$\begin{array}{c} \vdash \Gamma_1 \quad \cdots \quad \vdash \Gamma_{i-1} \quad \otimes\dfrac{\vdash A, \Phi \quad \vdash B, \Psi}{\vdash A \otimes B, \Phi, \Psi} \quad \vdash \Gamma_{i+1} \quad \cdots \quad \vdash \Gamma_h \\ \Delta \\ \vdash \Sigma \end{array}$$

containing the $\otimes$ rule can by simulated by

$$\begin{array}{c} \otimes' \dfrac{(\Gamma_1, \dots, \Gamma_{i-1}, [A, \Phi], [B, \Psi], \Gamma_{i+1}, \dots, \Gamma_h)}{(\Gamma_1, \dots, \Gamma_{i-1}, [(A, B), \Phi, \Psi], \Gamma_{i+1}, \dots, \Gamma_h)} \\ \Delta \Big\| \\ \Sigma \end{array}$$

in the calculus of structures, where Γ_j, A, B, Φ, Ψ, Δ and Σ are obtained from their counterparts in the sequent calculus by the obvious translation. This means that by this method every system in the one-sided sequent calculus can be ported trivially to the calculus of structures.

Of course, in the calculus of structures, rules could be used as axioms of a generic Hilbert system, where there is no special, structural relation between T and R: then all the good proof theoretical properties of sequent systems would be lost. We will be careful to design rules in a way that is conservative enough to allow us to prove cut elimination, and such that they possess the subformula property.

In our systems, rules come in pairs, $\rho{\downarrow}\,\dfrac{S\{T\}}{S\{R\}}$ (down version) and $\rho{\uparrow}\,\dfrac{S\{\bar{R}\}}{S\{\bar{T}\}}$ (up version). Sometimes rules are self-dual, i.e., the up and down version are identical, in which case we omit the arrows. This duality derives from the duality between $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$. We will be able to get rid of the up rules without affecting provability—after all, $T \Rightarrow R$ and $\bar{R} \Rightarrow \bar{T}$ are equivalent statements in many logics. Remarkably, the cut rule reduces into several up rules, and this makes for a modular decomposition of the cut elimination argument because we can eliminate up rules one independently from the other.

Let us now define system NEL by starting from a top-down symmetric variation, that we call SNEL. It is made by two sub-systems that we will call conventionally *interaction* and *structure*. The interaction fragment deals with negation, i.e., duality. It corresponds to identity and cut in the sequent calculus. In our calculus these rules become mutually top-down symmetric and both can be reduced to their atomic counterparts.

The structure fragment corresponds to logical and structural rules in the sequent calculus; it defines the logical operators. Differently from the sequent calculus, the operators need not be defined in isolation, rather complex contexts can be taken into consideration. In the following system we consider *pairs* of logical relations, one inside the other.

Associativity

$$[\vec{R},[\vec{T}]] = [\vec{R},\vec{T}]$$
$$(\vec{R},(\vec{T})) = (\vec{R},\vec{T})$$
$$\langle \vec{R};\langle \vec{T}\rangle;\vec{U}\rangle = \langle \vec{R};\vec{T};\vec{U}\rangle$$

Singleton

$$[R] = (R) = \langle R\rangle = R$$

Units

$$[\circ,\vec{R}] = [\vec{R}]$$
$$(\circ,\vec{R}) = (\vec{R})$$
$$\langle \circ;\vec{R}\rangle = \langle \vec{R}\rangle$$
$$\langle \vec{R};\circ\rangle = \langle \vec{R}\rangle$$

Exponentials

$$?\circ = !\circ = \circ$$
$$??R = ?R$$
$$!!R = !R$$

Commutativity

$$[\vec{R},\vec{T}] = [\vec{T},\vec{R}]$$
$$(\vec{R},\vec{T}) = (\vec{T},\vec{R})$$

Negation

$$\bar{\circ} = \circ$$
$$\overline{[R,T]} = (\bar{R},\bar{T})$$
$$\overline{(R,T)} = [\bar{R},\bar{T}]$$
$$\overline{\langle R;T\rangle} = \langle \bar{R};\bar{T}\rangle$$
$$\overline{?R} = !\bar{R}$$
$$\overline{!R} = ?\bar{R}$$
$$\bar{\bar{R}} = R$$

Contextual Closure

if $R = T$
then $S\{R\} = S\{T\}$

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a,\bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a,\bar{a})}{S\{\circ\}}$$

Interaction

Structure

$$\mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R;U\rangle,\langle T;V\rangle)}{S\langle (R,T);(U,V)\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R,T]\}}{S[!R,?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R,!T)}{S\{?(R,T)\}}$$

core

non-core

$$\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{\circ\}}$$

$$\mathsf{b}\!\downarrow \frac{S[?R,R]}{S\{?R\}} \qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R,R)}$$

Fig. 1 **Left:** *Syntactic equivalence* $=$ **Right:** *System* SNEL

2.3 Definition The structures of the language NEL are equivalent modulo the relation $=$, defined at the left of Fig. 1. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (sequences may contain ',' or ';' separators as appropriate in the context). At the right of the figure, *system* SNEL is shown (*symmetric non-commutative exponential linear logic*). The rules $\mathsf{ai}\!\downarrow$, $\mathsf{ai}\!\uparrow$, s, $\mathsf{q}\!\downarrow$, $\mathsf{q}\!\uparrow$, $\mathsf{p}\!\downarrow$, $\mathsf{p}\!\uparrow$, $\mathsf{w}\!\downarrow$, $\mathsf{w}\!\uparrow$, $\mathsf{b}\!\downarrow$ and $\mathsf{b}\!\uparrow$, are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *weakening*, *coweakening*, *absorption* and *coabsorption*. The *down fragment* of SNEL is $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{p}\!\downarrow, \mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow\}$, the *up fragment* is $\{\mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{w}\!\uparrow, \mathsf{b}\!\uparrow\}$.

There is a straightforward two-way correspondence between structures not involving seq and formulae of MELL: for example $![(?a,b),\bar{c},!\bar{d}]$ corresponds to $!((?a \otimes b) \mathbin{⅋} c^{\perp} \mathbin{⅋} !d^{\perp})$, and vice versa. Units are mapped into $\circ$, since $1 \equiv \perp$, when mix and mix0 are added to MELL. System SNEL is just the merging of systems SBV and SELS shown in [9, 10, 19]; there one can find details on the correspondence between our systems and linear logic. The reader can check that the equations in Fig. 1 correspond to logical equivalences in MELL, disregarding seq. In particular, $!A \multimap !!A$ and $!!A \multimap !A$ for every MELL formula A, and dually for $?$. The rules s, $\mathsf{q}\!\downarrow$ and $\mathsf{q}\!\uparrow$ are the same as in pomset logic viewed as a calculus of cographs [17].

All equations are typical of a sequent calculus presentation, save those for units, exponentials and contextual closure. Contextual closure just corresponds to equivalence being a congruence, it is a necessary ingredient of the calculus of structures. All other equations can be removed and replaced by rules, as in the sequent calculus. This might prove necessary for certain applications. For our purposes, this setting makes for a much more compact presentation, at a more effective abstraction level.

Negation is involutive and can be pushed to atoms; it is convenient always to imagine it directly over atoms. Please note that negation does not swap arguments of seq, as happens in the systems of Lambek and Abrusci-Ruet. The unit $\circ$ is self-dual and common to par, times and seq. One may think of it as a convenient way of expressing the empty sequence. Rules become very flexible in the presence of the unit. For example, the following notable derivation is valid:

$$\mathsf{q}{\downarrow}\frac{\mathsf{q}{\uparrow}\dfrac{(a,b)}{\langle a;b\rangle}}{[a,b]} \quad = \quad \mathsf{q}{\downarrow}\frac{\mathsf{q}{\uparrow}\dfrac{(\langle a;\circ\rangle,\langle\circ;b\rangle)}{\langle[a,\circ];[\circ,b]\rangle=\langle(a,\circ);(\circ,b)\rangle}}{[\langle a;\circ\rangle,\langle\circ;b\rangle]} \quad .$$

Each inference rule in Fig. 1 corresponds to a linear implication that is sound in MELL plus mix and mix0. For example, promotion corresponds to the implication $!(R \parr T) \multimap (!R \parr ?T)$. Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

2.4 Definition The following rules are called *interaction* and *cut*:

$$\mathsf{i}{\downarrow}\frac{S\{\circ\}}{S[R,\bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow}\frac{S(R,\bar{R})}{S\{\circ\}} \quad ,$$

where R and $\bar{R}$ are called *principal structures.*

The sequent calculus rule $\mathsf{cut}\dfrac{\vdash A,\Phi \quad \vdash A^{\perp},\Psi}{\vdash \Phi,\Psi}$ is realised as

$$\mathsf{i}{\uparrow}\frac{\mathsf{s}\dfrac{\mathsf{s}\dfrac{([A,\Phi],[\bar{A},\Psi])}{[([A,\Phi],\bar{A}),\Psi]}}{[(A,\bar{A}),\Phi,\Psi]}}{[\Phi,\Psi]} \quad ,$$

where Φ and Ψ stand for multisets of formulae or their corresponding par structures. Notice how the tree shape of derivations in the sequent calculus is realised by making use of times structures: in the derivation above, the premise corresponds to the two branches of the cut rule. For this reason, in the calculus of structures rules are allowed to access structures deeply nested into contexts.

The cut rule in the calculus of structures can mimic the classical cut rule in the sequent calculus in its realisation of transitivity, but it is much more general. We believe a good way of understanding it is in thinking of the rule as being about lemmas *in context.* The sequent calculus cut rule generates a lemma valid in the most general context; the new cut rule does the same, but the lemma only affects the limited portion of structure that can interact with it.

We easily get the next two propositions, which say: 1) The interaction and cut rules can be reduced into their atomic forms—note that in the sequent calculus it is possible to reduce interaction to atomic form, but not cut. 2) The cut rule is as powerful as the whole up fragment of the system, and vice versa.

$$\circ\!\downarrow\,\frac{}{\circ}\qquad \mathsf{ai}\!\downarrow\frac{S\{\circ\}}{S[a,\bar a]}\qquad \mathsf{s}\,\frac{S([R,U],T)}{S[(R,T),U]}\qquad \mathsf{q}\!\downarrow\frac{S\langle[R,U];[T,V]\rangle}{S[\langle R;T\rangle,\langle U;V\rangle]}$$

$$\mathsf{p}\!\downarrow\frac{S\{![R,T]\}}{S[!R,?T]}\qquad \mathsf{w}\!\downarrow\frac{S\{\circ\}}{S\{?R\}}\qquad \mathsf{b}\!\downarrow\frac{S[?R,R]}{S\{?R\}}$$

Fig. 2 *System* NEL

2.5 Definition A rule ρ is *derivable* in the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every instance $\rho\,\frac{T}{R}$ there exists a derivation $\begin{array}{c}T\\ \|\mathscr{S}\\ R\end{array}$. The systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation $\begin{array}{c}T\\ \|\mathscr{S}\\ R\end{array}$ there exists a derivation $\begin{array}{c}T\\ \|\mathscr{S}'\\ R\end{array}$, and vice versa.

2.6 Proposition *The rule* $\mathsf{i}\!\downarrow$ *is derivable in* $\{\mathsf{ai}\!\downarrow,\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{p}\!\downarrow\}$, *and, dually, the rule* $\mathsf{i}\!\uparrow$ *is derivable in the system* $\{\mathsf{ai}\!\uparrow,\mathsf{s},\mathsf{q}\!\uparrow,\mathsf{p}\!\uparrow\}$.

Proof Induction on principal structures. We show the inductive cases for $\mathsf{i}\!\uparrow$:

$$\mathsf{i}\!\uparrow\frac{\mathsf{i}\!\uparrow\frac{\mathsf{s}\,\frac{\mathsf{s}\,\frac{S(P,Q,[\bar P,\bar Q])}{S(Q,[(P,\bar P),\bar Q])}}{S[(P,\bar P),(Q,\bar Q)]}}{S(Q,\bar Q)}}{S\{\circ\}}\quad,\qquad \mathsf{i}\!\uparrow\frac{\mathsf{i}\!\uparrow\frac{\mathsf{q}\!\uparrow\frac{S(\langle P;Q\rangle,\langle\bar P;\bar Q\rangle)}{S\langle(P,\bar P);(Q,\bar Q)\rangle}}{S(Q,\bar Q)}}{S\{\circ\}}\qquad\text{and}\qquad \mathsf{i}\!\uparrow\frac{\mathsf{p}\!\uparrow\frac{S(?P,!\bar P)}{S\{?(P,\bar P)\}}}{S\{\circ\}}\quad.$$

The cases for $\mathsf{i}\!\downarrow$ are dual. □

2.7 Proposition *Each rule* $\rho\!\uparrow$ *in* SNEL *is derivable in* $\{\mathsf{i}\!\downarrow,\mathsf{i}\!\uparrow,\mathsf{s},\rho\!\downarrow\}$, *and, dually, each rule* $\rho\!\downarrow$ *in* SNEL *is derivable in the system* $\{\mathsf{i}\!\downarrow,\mathsf{i}\!\uparrow,\mathsf{s},\rho\!\uparrow\}$.

Proof Each instance $\rho\!\uparrow\frac{S\{T\}}{S\{R\}}$ can be replaced by $\mathsf{i}\!\uparrow\frac{\rho\!\downarrow\frac{\mathsf{s}\,\frac{\mathsf{i}\!\downarrow\frac{S\{T\}}{S(T,[R,\bar R])}}{S[R,(T,\bar R)]}}{S[R,(T,\bar T)]}}{S\{R\}}$. □

In the calculus of structures, we call *core* the set of rules, other than atomic interaction and cut, used to reduce interaction and cut to atomic form. Rules, other than interaction and cut, that are not in the core are called *non-core*.

2.8 Definition The *core* of SNEL is $\{\mathsf{s},\mathsf{q}\!\downarrow,\mathsf{q}\!\uparrow,\mathsf{p}\!\downarrow,\mathsf{p}\!\uparrow\}$, denoted by SNELc.

System SNEL is top-down symmetric, and the properties we saw are also symmetric. Provability is an asymmetric notion: we want to observe the possible conclusions that we can obtain from a unit premise. We now break the top-down symmetry by adding an inference rule with no premise, and we join this logical axiom to the down fragment of SNEL.

2.9 Definition The following rule is called *unit*: $\circ\!\downarrow\,\frac{}{\circ}$. *System* NEL is shown in Fig. 2.

As an immediate consequence of Propositions 2.6 and 2.7 we get:

2.10 Theorem *The systems* $\mathsf{NEL}\cup\{\mathsf{i}\!\uparrow\}$ *and* $\mathsf{SNEL}\cup\{\circ\!\downarrow\}$ *are strongly equivalent.*

2.11 Definition A derivation with no premise is called a *proof*, denoted by Π. A system $\mathscr{S}$ *proves* R if there is in the system $\mathscr{S}$ a proof Π whose conclusion is R, written $\begin{array}{c}\Pi \| \mathscr{S} \\ R\end{array}$. We say that a rule ρ is *admissible* for the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof $\begin{array}{c}\| \mathscr{S}\cup\{\rho\} \\ R\end{array}$ there exists a proof $\begin{array}{c}\| \mathscr{S} \\ R\end{array}$. Two systems are *equivalent* if they prove the same structures.

Except for cut and coweakening, systems SNEL and NEL enjoy a subformula property (which we treat as an asymmetric property, by going from conclusion to premise): premises are made of substructures of the conclusions.

To get cut elimination, so as to have a system whose rules all enjoy the subformula property, we could just get rid of ai↑ and w↑, by proving their admissibility for the other rules. But we can do more than that: the whole up fragment of SNEL, except for s (which also belongs to the down fragment), is admissible. This entails a *modular* scheme for proving cut elimination. In Sections 3 and 4 we will sketch a proof of the cut elimination theorem:

2.12 Theorem *System* NEL *is equivalent to every subsystem of* SNEL $\cup$ {∘↓} *which contains* NEL*.*

2.13 Corollary *The rule* i↑ *is admissible for system* NEL*.*

Proof Immediate from Theorems 2.10 and 2.12. □

Any implication $T \multimap R$, i.e. $[\bar{T}, R]$, is connected to derivability by:

2.14 Corollary *For any two structures* T *and* R*, there is a proof* $\begin{array}{c}\| \mathsf{NEL} \\ [\bar{T}, R]\end{array}$ *iff there is a derivation* $\begin{array}{c}T \\ \| \mathsf{SNEL} \\ R\end{array}$*.*

Consistency follows as usual and can be proved by way of the same technique used in [10]. It is also easy to prove that system NEL is a conservative extension of MELL plus mix and mix0 (see [9]). The locality properties shown in [10, 19] still hold in this system, of course. In particular, the promotion rule is local, as opposed to the same rule in the sequent calculus.

3 Decomposition

The new top-down symmetry of derivations in the calculus of structures allows to study properties that are not observable in the sequent calculus. The most remarkable results so far are decomposition theorems. In general, a decomposition theorem says that a given system $\mathscr{S}$ can be divided into n pairwise disjoint subsystems $\mathscr{S}_1, \dots, \mathscr{S}_n$ such that every derivation Δ in system $\mathscr{S}$ can be rearranged as composition of n derivations $\Delta_1, \dots, \Delta_n$, where Δ_i uses only rules of $\mathscr{S}_i$, for every $1 \leqslant i \leqslant n$.

For system SNEL, we have two such results, which both state a decomposition of every derivation into seven subsystems. They can be stated together as follows:

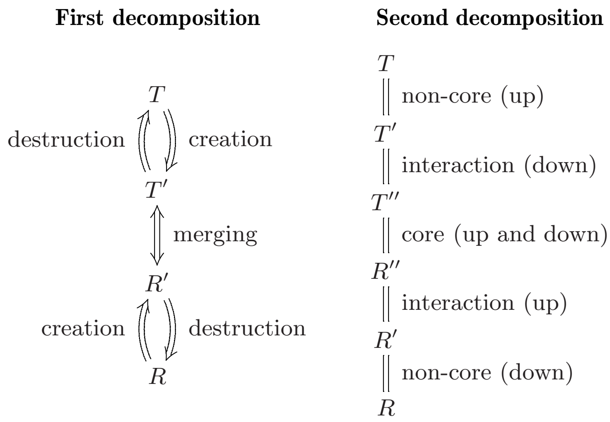


Fig. 3 *Readings of the decomposition theorem*

3.1 Theorem *For every derivation* $\begin{array}{c} T \\ \Vert \mathsf{SNEL} \\ R \end{array}$ *there exist derivations*

$$\begin{array}{c} T \\ \Vert \{\mathsf{b}\uparrow\} \\ T_1 \\ \Vert \{\mathsf{w}\downarrow\} \\ T_2 \\ \Vert \{\mathsf{ai}\downarrow\} \\ T_3 \\ \Vert \mathsf{SNELc} \\ R_3 \\ \Vert \{\mathsf{ai}\uparrow\} \\ R_2 \\ \Vert \{\mathsf{w}\uparrow\} \\ R_1 \\ \Vert \{\mathsf{b}\downarrow\} \\ R \end{array} \quad \textit{and} \quad \begin{array}{c} T \\ \Vert \{\mathsf{b}\uparrow\} \\ T'_1 \\ \Vert \{\mathsf{w}\uparrow\} \\ T'_2 \\ \Vert \{\mathsf{ai}\downarrow\} \\ T'_3 \\ \Vert \mathsf{SNELc} \\ R'_3 \\ \Vert \{\mathsf{ai}\uparrow\} \\ R'_2 \\ \Vert \{\mathsf{w}\downarrow\} \\ R'_1 \\ \Vert \{\mathsf{b}\downarrow\} \\ R \end{array} ,$$

for some structures T_1, T_2, T_3, R_1, R_2, R_3 *and* T'_1, T'_2, T'_3, R'_1, R'_2, R'_3.

The first decomposition can also be read as a decomposition of a derivation into three parts, which can be called *creation*, where the size of the structure is increased, *merging*, where the size of the structure does not change, and *destruction*, where the size of the structure is decreased. The merging part is in the middle of the derivation and (depending on your preferred reading of a derivation) creation and destruction are at the top and at the bottom, as depicted at the left in Fig. 3. In system SNEL the merging part contains the rules s, q↓, q↑, p↓ and p↑, which coincides with the core. In the top-down reading of a derivation, the creation part contains the rules b↑, w↓ and ai↓, and the destruction part consists of b↓, w↑ and ai↑. In the bottom-up reading, creation and destruction are exchanged.

Such a decomposition is not restricted to system SNEL. It also holds for other systems in the calculus of structures, including systems SBV and SELS [10],

$$
\begin{array}{c} T \\ \Big\Vert\, \mathsf{SNEL} \\ R \end{array}
\rightarrow
\begin{array}{c} T \\ \Big\Vert\, \{\mathsf{b}\uparrow\} \\ T_1 \\ \Big\Vert\, \mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ R \end{array}
\rightarrow
\begin{array}{c} T \\ \Big\Vert\, \{\mathsf{b}\uparrow\} \\ T_1 \\ \Big\Vert\, \mathsf{SNEL}\setminus\{\mathsf{b}\downarrow\} \\ R_1 \\ \Big\Vert\, \{\mathsf{b}\downarrow\} \\ R \end{array}
\rightarrow
\begin{array}{c} T \\ \Big\Vert\, \{\mathsf{b}\uparrow\} \\ T_2 \\ \Big\Vert\, \mathsf{SNEL}\setminus\{\mathsf{b}\uparrow\} \\ R_1 \\ \Big\Vert\, \{\mathsf{b}\downarrow\} \\ R \end{array}
\rightarrow \cdots \rightarrow
\begin{array}{c} T \\ \Big\Vert\, \{\mathsf{b}\uparrow\} \\ T_k \\ \Big\Vert\, \mathsf{SNEL}\setminus\{\mathsf{b}\downarrow,\mathsf{b}\uparrow\} \\ R_h \\ \Big\Vert\, \{\mathsf{b}\downarrow\} \\ R \end{array}
$$

Fig. 4 *Permuting* $\mathsf{b}\uparrow$ *up and* $\mathsf{b}\downarrow$ *down*

classical logic [4] and full propositional linear logic.

The second decomposition in Theorem 3.1 states that in any derivation we can separate five homogeneous subsystems, as shown at the right of Fig. 3. In particular, we can separate the non-core part of the system from the core.

We prove the two decomposition statements similarly. The first step is the separation of absorption. For this, the instances of $\mathsf{b}\uparrow$ are permuted over all the other rules. The only problematic case is when

$$
\mathsf{b}\uparrow \dfrac{\mathsf{p}\downarrow \dfrac{S\{![R,T]\}}{S[!R,?T]}}{S[(!R,R),?T]}
\qquad \text{is replaced by} \qquad
\mathsf{b}\downarrow \dfrac{\mathsf{s} \dfrac{\mathsf{s} \dfrac{\mathsf{p}\downarrow \dfrac{\mathsf{b}\uparrow \dfrac{S\{![R,T]\}}{S(![R,T],[R,T])}}{S([!R,?T],[R,T])}}{S[([!R,?T],R),T]}}{S[(!R,R),?T,T]}}{S[(!R,R),?T]} \quad .
$$

Here, a new instance of $\mathsf{b}\downarrow$ is introduced. After all $\mathsf{b}\uparrow$ have reached the top of the derivation, the instances of $\mathsf{b}\downarrow$ are permuted down by the dual procedure, where new instances of $\mathsf{b}\uparrow$ might be introduced; and so on.

The problem is to show that this process, shown in Fig. 4, does terminate eventually, which is done in two steps. First, the assumption of a non-termination is reduced to the existence of a derivation

$$
\begin{array}{c}
([!R_1,?T_1],[!R_2,?T_2],\dots,[!R_n,?T_n]) \\
\Delta \Big\Vert\, \{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\} \\
[(!R_2,?T_1),(!R_3,?T_2),\dots,(!R_1,?T_n)]
\end{array} \quad ,
$$

for some $n \geqslant 1$ and structures $R_1, \dots, R_n, T_1, \dots, T_n$. Second, we show that such a derivation cannot exist. For the first step we mainly rely on the methods used in [19] for a case without seq. We need only a little more effort to deal with the unit. However, the non-existence of the derivation Δ is more difficult to prove for the system $\{\mathsf{s},\mathsf{q}\downarrow,\mathsf{q}\uparrow\}$ than in the case where the rules $\mathsf{q}\downarrow$ and $\mathsf{q}\uparrow$ are not present.

After separating absorption, we have to separate weakening. For the first decomposition, we use the same method used in [19], with the only difference that there are more cases to consider because the units $\mathsf{1}$ and $\bot$ are collapsed. In the second decomposition, weakening is separated in the same way as absorption, but termination is much easier to show.

In the end, the interaction rules are separated from the core, which is similar to the separation of weakening in the first decomposition result.

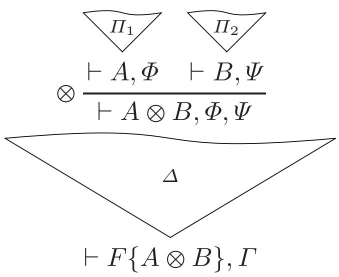
4 Cut Elimination

The classical arguments for proving cut elimination in the sequent calculus rely on the following property: when the principal formulae in a cut are active in both branches, they determine which rules are applied immediately above the cut. This is a consequence of the fact that formulae have a root connective, and logical rules only hinge on that, and nowhere else in the formula.

This property does not hold in the calculus of structures. Further, since rules can be applied anywhere deep inside structures, everything can happen above a cut. This complicates considerably the task of proving cut elimination. On the other hand, a great simplification is made possible in the calculus of structures by the reduction of cut to its atomic form. The remaining difficulty is actually understanding what happens, while going up in a proof, *around* the atoms produced by an atomic cut. The two atoms of an atomic cut can be produced inside any structure, and they do not belong to distinct branches, as in the sequent calculus: complex interactions with their context are possible. As a consequence, our techniques are largely different than the traditional ones.

Two approaches to cut elimination in the calculus of structures have been explored in previous papers: in [10] we relied on permutations of rules, in [4] the authors relied on semantics. In this paper we use a third technique, called *splitting*, which has the advantage of being more uniform than the one based on permutations and which yields a much simpler case analysis. It also establishes a deep connection to the sequent calculus, at least for the fragments of systems that allow for a sequent calculus presentation (in this case, the commutative fragment). Since many systems are expressed in the sequent calculus, our method appears to be entirely general; still it is independent of the sequent calculus and of a complete semantics.

Splitting can be best understood by considering a sequent system with no weakening and absorption (or contraction). Consider for example multiplicative linear logic: If we have a proof of the sequent $\{\vdash F\{A \otimes B\}, \Gamma\}$, where $F\{A \otimes B\}$ is a formula that contains the subformula $A \otimes B$, we know for sure that somewhere in the proof there is one and only one instance of the $\otimes$ rule, which splits A and B along with their context. We are in the following situation:



corresponds to

$$\begin{array}{c} \Pi_2 \| \\ [B,\Psi] \\ \Pi_1 \| \\ \mathsf{s}\,\dfrac{([A,\Phi],[B,\Psi])}{\mathsf{s}\,\dfrac{[([A,\Phi],B),\Psi]}{[(A,B),\Phi,\Psi]}} \\ \Delta \| \\ [F(A,B),\Gamma] \end{array} \quad .$$

We can consider, as shown at the left, the proof for the given sequent as composed of three pieces, Δ, Π_1 and Π_2. In the calculus of structures, many different proofs correspond to the sequent calculus one: they differ for the different possible sequencing of rules, and because rules in the calculus of structures have smaller granularity and larger applicability. But, among all these proofs, there must also be one that fits the scheme at the right of the figure above. This precisely illustrates the idea behind the splitting technique.

The derivation Δ above implements a *context reduction* and a proper splitting. We can state, in general, these principles as follows:

1 Context reduction: If $S\{R\}$ is provable, then $S\{\ \}$ can be reduced to the structure $[\{\ \}, U]$, such that $[R, U]$ is provable. In the example above, $[F\{\ \}, \Gamma]$ is reduced to $[\{\ \}, \Gamma']$, for some Γ'. (Technically, without loss of generality, we have to prefix the whole proof by a $!$.)

2 Splitting: If $[(R, T), P]$ is provable, then P can be reduced to $[P_1, P_2]$, such that $[R, P_1]$ and $[T, P_2]$ are provable. In the example above Γ' is reduced to $[\Phi, \Psi]$. Splitting holds for all logical operators.

Context reduction is in turn proved by splitting, which is then at the core of the matter. The biggest difficulty resides in proving splitting, and this mainly requires finding the right induction measure. The splitting theorems follow, all gathered together in Theorem 4.2.

4.1 Definition We call NELm the system $\mathsf{NEL} \setminus \{\mathsf{w}{\downarrow}, \mathsf{b}{\downarrow}\} = \{\circ{\downarrow}, \mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}\}$.

4.2 Theorem *For all structures R, T and P:*

1 *if $[\langle R; T\rangle, P]$ is provable in* NELm *then there exist P_1, P_2 and* $\begin{array}{c}\langle P_1; P_2\rangle\\ \Vert\,\mathsf{NELm}\\ P\end{array}$ *such that $[R, P_1]$ and $[T, P_2]$ are provable in* NELm;

2 *if $[(R, T), P]$ is provable in* NELm *then there exist P_1, P_2 and* $\begin{array}{c}[P_1, P_2]\\ \Vert\,\mathsf{NELm}\\ P\end{array}$ *such that $[R, P_1]$ and $[T, P_2]$ are provable in* NELm;

3 *if $[!R, P]$ is provable in* NELm *then there exist $P_1, \dots, P_h$, for $h > 0$, and* $\begin{array}{c}[?P_1, \dots, ?P_h]\\ \Vert\,\mathsf{NELm}\\ P\end{array}$ *such that $[R, P_1, \dots, P_h]$ is provable in* NELm;

4 *if $[?R, P]$ is provable in* NELm *then there exist P_1 and* $\begin{array}{c}!P_1\\ \Vert\,\mathsf{NELm}\\ P\end{array}$ *such that $[R, P_1]$ is provable in* NELm.

As a notable consequence of splitting we have:

4.3 Theorem *If $[a, P]$ is provable in* NELm *then there exists* $\begin{array}{c}\bar{a}\\ \Vert\,\mathsf{NELm}\\ P\end{array}$.

Further, we can easily prove that:

4.4 Theorem *A structure R is provable in* NELm *iff $!R$ is provable in* NELm.

The context reduction theorem follows. The previous theorem insures its general applicability, despite the $!$ prefixing:

4.5 Theorem *For all R and $S\{\ \}$ such that $S\{R\}$ is provable in* NELm, *there exists a structure U such that for all X there exist* $\begin{array}{c}![X, U]\\ \Vert\,\mathsf{NELm}\\ !S\{X\}\end{array}$ *and* $\begin{array}{c}\prod\Vert\,\mathsf{NELm}\\ [R, U]\end{array}$.

And now we can sketch the cut elimination argument: Given a proof, the second decomposition theorem moves absorption and weakening down; coabsorption and coweakening go to the top. But coabsorptions and coweakenings at the top of a proof can only be trivial, and disappear. We perform cut elimination, on a proof thus rearranged, by going down the proof and eliminating all instances of ai↑, q↑ and p↑, as they are encountered. This can be verified rather easily by the reader, through a combined use of context reduction and splitting. For example, by using the theorems above one can remove a topmost instance of ai↑ in a proof by the following transformation:

$$
\begin{array}{c}
\Vert \mathsf{NELm} \\
\mathsf{ai}\!\uparrow \dfrac{S(a,\bar{a})}{S\{\circ\}}
\end{array}
\quad\rightarrow\quad
\begin{array}{c}
\mathsf{ai}\!\downarrow \dfrac{\circ\!\downarrow \dfrac{}{\circ}}{![\bar{a},a]} \\
\Vert \mathsf{NELm} \\
![P_1,P_2] \\
\Vert \mathsf{NELm} \\
![\circ,U] \\
\Vert \mathsf{NELm} \\
!S\{\circ\}
\end{array}
\quad\rightarrow\quad
\begin{array}{c}
\Vert \mathsf{NELm} \\
S\{\circ\}
\end{array}
\quad .
$$

We apply context reduction to $S(a,\bar{a})$, which gives us U such that $[(a,\bar{a}),U]$ is provable in NELm. By splitting, U is then further reduced into $[P_1,P_2]$ such that $[a,P_1]$ and $[\bar{a},P_2]$ are provable in NELm. By applying Theorem 4.3 twice, we obtain a proof of $!S\{\circ\}$. By Theorem 4.4, there is then a proof of $S\{\circ\}$ in NELm, which can be plugged into the original proof, so that the topmost ai↑ has disappeared.

The rules q↑ and p↑ can be shown admissible the same way: we reduce the context of a rule instance as dictated by the shape of the structures involved in the rewriting. The pieces so produced are then rearranged—this is conceptually similar to what happens in the sequent calculus.

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules than cut. So, our technique is much more general than cut elimination in the sequent calculus, for two reasons:

1 it applies to operators that admit no sequent calculus definition, as seq;

2 it can be used to show admissibility of non-infinitary rules that involve no negation, like q↑ and p↑.

5 Conclusions and Future Work

We have shown a class of logical systems, built around system NEL, that integrate multiplicative commutativity and non-commutativity, together with exponentials. This has been done in the formalism of the calculus of structures, which allows us to obtain very simple systems. In addition, we get properties of locality, atomicity and modularity that do not hold in other known calculi.

System NEL was originally inspired by Retoré's pomset logic [16]. There is research in progress to show that the multiplicative fragments of his logic and ours coincide. In this case, our system and the work [21] would explain why sequentialising pomset logic has been so hard and unfruitful. It should be possible to extend our system NEL to other logical operators, perhaps to full linear logic, and also to the self-dual modality associated to Retoré's non-commutative operator [15]. In this paper we limited ourselves to the bare necessary to include MELL.

In a forthcoming paper we will show that NEL is Turing-complete. This result establishes an interesting boundary to MELL, whose decidability is still an open problem. If it turns out, as many believe, that MELL is decidable, then the boundary with undecidability is crossed by our simple extension to seq. This would give a precise technical content to the perceived difficulty of getting Turing-equivalence for MELL, namely the trouble in realising the tape of a Turing machine. In this sense, our sequentiality would be even more strongly motivated by a basic computational mechanism.

One of the biggest open problems we have is understanding when and why decomposition theorems work. They seem to have a strong relation to the notion of core system, but we fail to understand the deep reasons for this. For the time being we observe that decomposition theorems hold for all logics we studied so far (classical, linear and several commutative/non-commutative systems).

The calculus of structures generalises the sequent calculus for one-sided sequent systems, which correspond to logics with involutive negation. Preliminary work shows that it is also possible to design intuitionistic systems in the calculus of structures, by way of polarities. This promises to be an active area of research.

Proving cut elimination is more difficult than in the sequent calculus. On the other hand, the methods we used are more general than the traditional ones, and, we believe, unveil some fundamental properties of logical systems that were previously hidden. We make an essential use of a top-down symmetric notion of derivation, which leads to a reduction of the cut rule into constituents which are dual to the common logical rules.

We did not attempt to base our calculus on philosophical grounds. We believe that this can only happen after several systems are thoroughly studied and discussed. For the time being we are still collecting empirical evidence.

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