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A System of Interaction and Structure V: The Exponentials and Splitting

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System NEL is the mixed commutative/non-commutative linear logic BV augmented with linear logic's exponentials, or, equivalently, it is MELL augmented with the non-commutative self-dual connective seq. NEL is presented in deep inference, because no Gentzen formalism can express it in such a way that the cut rule is admissible. Other, recent works, show that system NEL is Turing-complete, it is able to directly express process algebra sequential composition and it faithfully models causal quantum evolution. In this paper, we show cut elimination for NEL, based on a technique that we call *splitting*. The splitting theorem shows how and to what extent we can recover a sequent-like structure in NEL proofs. Together with a 'decomposition' theorem, proved in the previous paper of this series, splitting yields a cut-elimination procedure for NEL.

1. Introduction

This is the fifth in a series of papers dedicated to the proof theory of a self-dual, non-commutative, linear connective, called *seq*, in the context of linear logic. The addition of seq to multiplicative linear logic yields a logic that we call BV. This logic is the main subject of study of this series of papers. BV is conjectured to be the same as pomset logic, studied by Retoré in (Ret97) and other papers.

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BV has been defined in (Gug07) (the first paper of this series), where a sound, complete and cut-free system for BV is given, together with a cut-elimination procedure. The proof system of BV radically departs from the traditional sequent calculus methodology, and adopts instead *deep inference* as the design principle. In short, this means that proofs can be freely composed by the same connectives used for formulae, or, equivalently, inference rules can be applied arbitrarily deep inside formulae. The cited paper provides an introduction to deep inference, which, as a matter of fact, has been originally conceived precisely for capturing BV.

The formalism adopted in the papers of the series so far, including this one, is called the *calculus of structures*, or *CoS*. This is, conceptually, the simplest formalism in deep inference, being just a special form of term rewriting. More sophisticated formalisms in deep inference are emerging, in particular *open deduction* (GGP10). This formalism improves on CoS because it has an increased algebraic flavour and allows for a speed-up in the size of proofs. However, these differences do not affect cut-elimination procedures, in the sense that from a cut-elimination procedure in CoS, one can trivially obtain a cut-elimination procedure in open deduction. For this reason, the results in this paper are broadly valid inside the deep-inference paradigm.

The use of deep inference is necessary: Alwen Tiu showed in (Tiu06) (the second paper of this series) that it is impossible for the sequent calculus to provide a sound, complete and cut-free proof system for BV. This is proved by exhibiting an infinite set of BV tautologies with a cleverly designed structure, such that any bounded-depth, cut-free inference system (in particular, any sequent calculus system) is either unsound or incomplete on the set of tautologies. The design of these tautologies exploits the ability of the seq connective, together with the usual 'par' disjunction of linear logic, to bury at arbitrary depths inside formulae certain key structures that have to be 'unlocked', by inference, before any other parts of the formula are touched. We stress the fact that this behaviour is independent of the logical formalism employed to describe it. So, deep inference appears to be the most natural choice of proof-system design methodology, because of its ability to apply inference at arbitrary depths inside formulae.

BV might be considered exotic as a logic, but it has a very natural algebraic nature, that already found applications in diverse fields. We mention here: 1) its ability to capture very precisely the sequential connective of Milner's CCS (and so of other process algebras) (Bru02); 2) a better axiomatisation of causal quantum computation than linear logic provides (BPS08; BGI$^+$10); and 3) a new class of categorical models (BPS09). We know that the proof system BV is NP-complete (Kah07), and its feasibility for proof search has been studied (Kah04).

This fifth paper, together with the fourth paper (SG09) in the series, is devoted to the proof theory of system BV when it is enriched with linear logic's exponentials. We call NEL (non-commutative exponential linear logic) the resulting system. We can also consider NEL as MELL (multiplicative exponential linear logic (Gir87)) augmented with seq. NEL, which was first presented in (GS02), is conservative over both BV and over MELL augmented by the mix and nullary mix rules (FR94; Ret93; AJ94). Like BV, NEL cannot be expressed with a cut-free proof system outside of deep inference, because of

Tiu's counterexample in (Tiu06) (mentioned above), and because NEL is conservative over BV.

An important feature of NEL is that it is Turing-complete, as shown in (Str03c). This makes for an interesting comparison with MELL, whose complexity is currently unknown. MELL is expected to be decidable, but several years of research have not settled the question. If MELL turns out to be decidable, then seq would be the decisive factor that allows us to cross the border between decidability and Turing-completeness. This would have an intuitive explanation in the fact that seq can be employed to simulate the structure of a Turing machine tape, which is precisely what MELL, which is fully commutative, apparently cannot do.

Further interest in NEL comes from the possibility of enhancing, in a natural way, the range of applications of BV. Similarly to what happens when exponentials are added to multiplicative linear logic, the augmented expressivity of NEL over BV can be employed to better capture process algebras, for example, and this is indeed an active area of research. We might equally expect that NEL will further improve our ability to describe quantum evolution phenomena, and we expect to find enriched categorical models over those for BV.

Each of the two NEL papers in the series is devoted to a theorem: *splitting* in this paper and *decomposition* in the previous paper (SG09). Together, the two theorems immediately yield a cut-elimination procedure, and the cut-elimination result is claimed in this paper.

Splitting (first pioneered in (Gug07)) is, in a sense, a way of rebuilding into deep inference the structure of Gentzen sequent-calculus proofs, to the extent possible in the presence of par and seq. The technique consists in blocking the access of inference rules to a part of the formula to be proved, however deep; then, we remove from the context of this blocked part as much 'logical material' as possible. In other words, we prove as much as we can, of a given formula, in the presence of a part that has been blocked. The splitting theorem states properties of what is left of the context of the blocked part, in relation to the shape of the blocked part. It turns out that the splitting property is nothing else than a generalization of the shape of Gentzen calculi rules. It precisely coincides with them when we stipulate that the blocked part of a formula is at the shallowest possible level.

Splitting is, *a priori*, a hard theorem to prove, but, thanks to the decomposition theorem proved in (SG09), we only need to prove it for a fragment of NEL, and this is what we do in this paper. Once splitting is available, cut elimination follows immediately.

The main results of this paper have already been presented, without proof, in (GS02) (for several years, the proofs of the statements have been available in a manuscript on the web).

2. The System

Let us briefly recall system NEL as defined in (SG09); the reader can find in that paper more details and introductory comments.

Associativity

$$\begin{array}{rcl}[\vec{R} \parr [\vec{T}] \parr \vec{U}] &=& [\vec{R} \parr \vec{T} \parr \vec{U}]\\ (\vec{R} \otimes (\vec{T}) \otimes \vec{U}) &=& (\vec{R} \otimes \vec{T} \otimes \vec{U})\\ \langle \vec{R} \lhd \langle \vec{T} \rangle \lhd \vec{U} \rangle &=& \langle \vec{R} \lhd \vec{T} \lhd \vec{U} \rangle\end{array}$$

Commutativity

$$\begin{array}{rcl}[\vec{R} \parr \vec{T}] &=& [\vec{T} \parr \vec{R}]\\ (\vec{R} \otimes \vec{T}) &=& (\vec{T} \otimes \vec{R})\end{array}$$

Unit

$$\begin{array}{rcl}[\circ \parr \vec{R}] &=& [\vec{R}]\\ (\circ \otimes \vec{R}) &=& (\vec{R})\\ \langle \circ \lhd \vec{R} \rangle &=& \langle \vec{R} \rangle\\ \langle \vec{R} \lhd \circ \rangle &=& \langle \vec{R} \rangle\end{array}$$

Singleton

$$[R] = (R) = \langle R \rangle = R$$

Negation

$$\begin{array}{rcl}\bar{\circ} &=& \circ\\ \overline{[R_1 \parr \cdots \parr R_h]} &=& (\bar{R}_1 \otimes \cdots \otimes \bar{R}_h)\\ \overline{(R_1 \otimes \cdots \otimes R_h)} &=& [\bar{R}_1 \parr \cdots \parr \bar{R}_h]\\ \overline{\langle R_1 \lhd \cdots \lhd R_h \rangle} &=& \langle \bar{R}_1 \lhd \cdots \lhd \bar{R}_h \rangle\\ \overline{?R} &=& !\bar{R}\\ \overline{!R} &=& ?\bar{R}\\ \bar{\bar{R}} &=& R\end{array}$$

Contextual Closure

$$\text{if } R = T \text{ then } S\{R\} = S\{T\}$$

Fig. 1. Basic equations for the syntactic equivalence =

Definition 2.1. There are countably many *positive* and *negative atoms*. They, positive or negative, are denoted by a, b, ... *Structures* are denoted by S, P, Q, R, T, U, V, W, X and Z. The structures of the *language* NEL are generated by

$$S ::= a \mid \circ \mid [\underbrace{S \parr \cdots \parr S}_{>0}] \mid (\underbrace{S \otimes \cdots \otimes S}_{>0}) \mid \langle \underbrace{S \lhd \cdots \lhd S}_{>0} \rangle \mid ?S \mid !S \mid \bar{S} \quad ,$$

where $\circ$, the *unit*, is not an atom and $\bar{S}$ is the *negation* of the structure S. Structures with a hole that does not appear in the scope of a negation are denoted by $S\{\ \}$. The structure R is a *substructure* of $S\{R\}$, and $S\{\ \}$ is its *context*. We simplify the indication of context in cases where structural parentheses fill the hole exactly: for example, $S[R \parr T]$ stands for $S\{[R \parr T]\}$. The structures of the language NEL are equivalent modulo the relation =, defined in Figure 1. There, $\vec{R}$, $\vec{T}$ and $\vec{U}$ stand for finite, non-empty sequences of structures (elements of the sequences are separated by $\parr$, $\lhd$, or $\otimes$, as appropriate in the context).

Definition 2.2. In Figure 2, *system* SNEL (*symmetric non-commutative exponential linear logic*) is shown. The rules ai↓, ai↑, s, q↓, q↑, p↓, p↑, e↓, e↓, w↓, w↑, b↓, b↑, g↓, and g↑ are called respectively *atomic interaction*, *atomic cut*, *switch*, *seq*, *coseq*, *promotion*, *copromotion*, *empty*, *coempty*, *weakening*, *coweakening*, *absorption*, *coabsorption*, *digging*, and *codigging*. The *down fragment* of SNEL is {ai↓, s, q↓, p↓, e↓, w↓, b↓, g↓}, the *up fragment* is {ai↑, s, q↑, p↑, e↑, w↑, b↑, g↑}. Then, *system* NEL is shown in Figure 3, where the rule ∘↓ is called *unit*.

All inference rules in SNEL are of the kind

$$\rho \frac{S\{T\}}{S\{R\}} \quad ,$$

$$\mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad \mathsf{ai}\!\uparrow \frac{S(a \otimes \bar{a})}{S\{\circ\}}$$

$$\mathsf{s}\,\frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]}$$

$$\mathsf{q}\!\downarrow \frac{S\langle [R \parr U] \lhd [T \parr V]\rangle}{S[\langle R \lhd T\rangle \parr \langle U \lhd V\rangle]} \qquad \mathsf{q}\!\uparrow \frac{S(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle)}{S\langle (R \otimes T) \lhd (U \otimes V)\rangle}$$

$$\mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]} \qquad \mathsf{p}\!\uparrow \frac{S(?R \otimes !T)}{S\{?(R \otimes T)\}}$$

$$\mathsf{e}\!\downarrow \frac{S\{\circ\}}{S\{!\circ\}} \qquad \mathsf{e}\!\uparrow \frac{S\{?\circ\}}{S\{\circ\}}$$

$$\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{w}\!\uparrow \frac{S\{!R\}}{S\{\circ\}}$$

$$\mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}} \qquad \mathsf{b}\!\uparrow \frac{S\{!R\}}{S(!R \otimes R)}$$

$$\mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}} \qquad \mathsf{g}\!\uparrow \frac{S\{!R\}}{S\{!!R\}}$$

Fig. 2. System SNEL

$$\circ\!\downarrow \frac{}{\circ} \qquad \mathsf{ai}\!\downarrow \frac{S\{\circ\}}{S[a \parr \bar{a}]} \qquad \mathsf{e}\!\downarrow \frac{S\{\circ\}}{S\{!\circ\}}$$

$$\mathsf{s}\,\frac{S([R \parr U] \otimes T)}{S[(R \otimes T) \parr U]} \qquad \mathsf{q}\!\downarrow \frac{S\langle [R \parr U] \lhd [T \parr V]\rangle}{S[\langle R \lhd T\rangle \parr \langle U \lhd V\rangle]} \qquad \mathsf{p}\!\downarrow \frac{S\{![R \parr T]\}}{S[!R \parr ?T]}$$

$$\mathsf{w}\!\downarrow \frac{S\{\circ\}}{S\{?R\}} \qquad \mathsf{b}\!\downarrow \frac{S[?R \parr R]}{S\{?R\}} \qquad \mathsf{g}\!\downarrow \frac{S\{??R\}}{S\{?R\}}$$

Fig. 3. System NEL

saying that if a structure matches R, in a context $S\{\ \}$, it can be rewritten as specified by T, in the same context $S\{\ \}$ (or vice versa if one reasons top-down). The unit rule of NEL is special in this respect, as it has no context; however, we only use it for convenience, and we could easily do without by slightly adapting the notion of proof.

A derivation Δ is a chain of consecutive applications of instances of inference rules. A derivation with no premise is called a *proof*, denoted by Π. A system $\mathscr{S}$ *proves* R if there is in the system $\mathscr{S}$ a proof Π whose conclusion is R. We use the notations

$$\begin{array}{c} T \\ \mathscr{S} \Big\| \Delta \\ R \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathscr{S} \Big\| \Pi \\ R \end{array} \quad .$$

to denote a derivation Δ with premise T and conclusion R, whose rules are in $\mathscr{S}$, and a proof Π in $\mathscr{S}$ whose conclusion is R, respectively.

Definition 2.3. A rule ρ is *derivable* in the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every instance $\rho \dfrac{T}{R}$ there exists a derivation from T to R in $\mathscr{S}$. We say that a rule ρ is *admissible* for the system $\mathscr{S}$ if $\rho \notin \mathscr{S}$ and for every proof in $\mathscr{S} \cup \{\rho\}$ there is a proof in $\mathscr{S}$ with the same conclusion. Two systems are *equivalent* if they prove the same structures. Two systems $\mathscr{S}$ and $\mathscr{S}'$ are *strongly equivalent* if for every derivation from T to R in $\mathscr{S}$ there is a derivation from T to R in $\mathscr{S}'$, and vice versa.

Notice that interaction and cut are atomic in SNEL; we can define their general versions as follows.

Definition 2.4. The following rules are called *interaction* and *cut*:

$$\mathsf{i}{\downarrow} \frac{S\{\circ\}}{S[R \parr \bar{R}]} \qquad \text{and} \qquad \mathsf{i}{\uparrow} \frac{S(R \otimes \bar{R})}{S\{\circ\}} \quad ,$$

where R and $\bar{R}$ are called *principal structures.*

The following two propositions have been shown in (SG09).

Proposition 2.5. The rule $\mathsf{i}{\downarrow}$ is derivable in $\{\mathsf{ai}{\downarrow}, \mathsf{s}, \mathsf{q}{\downarrow}, \mathsf{p}{\downarrow}, \mathsf{e}{\downarrow}\}$, and, dually, the rule $\mathsf{i}{\uparrow}$ is derivable in the system $\{\mathsf{ai}{\uparrow}, \mathsf{s}, \mathsf{q}{\uparrow}, \mathsf{p}{\uparrow}, \mathsf{e}{\uparrow}\}$.

Proposition 2.6. Each rule $\rho{\uparrow}$ in SNEL is derivable in $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\downarrow}\}$, and, dually, each rule $\rho{\downarrow}$ in SNEL is derivable in the system $\{\mathsf{i}{\downarrow}, \mathsf{i}{\uparrow}, \mathsf{s}, \rho{\uparrow}\}$.

As an immediate consequence of Propositions 2.5 and 2.6 we get:

Proposition 2.7. The systems $\mathsf{NEL} \cup \{\mathsf{i}{\uparrow}\}$ and $\mathsf{SNEL} \cup \{\circ{\downarrow}\}$ are strongly equivalent.

In the remainder of this paper we will give the proof of the cut elimination theorem, which, as usual in deep inference, means proving that the up-fragment is admissible:

Theorem 2.8 (Cut Admissibility). System NEL is equivalent to $\mathsf{SNEL} \cup \{\circ{\downarrow}\}$.

The following corollaries are immediate consequences of cut admissibility.

Corollary 2.9. The rule $\mathsf{i}{\uparrow}$ is admissible for system NEL.

$$
\begin{array}{c} T \\ \mathsf{SNEL} \Big\| \Delta \\ R \end{array}
\quad \leadsto \quad
\begin{array}{r} T \\ \{\mathsf{g}\uparrow\} \Big\| \\ T_1 \\ \{\mathsf{b}\uparrow\} \Big\| \\ T_2 \\ \{\mathsf{w}\uparrow\} \Big\| \\ T_3 \\ \{\mathsf{e}\downarrow\} \Big\| \\ T_4 \\ \{\mathsf{ai}\downarrow\} \Big\| \\ T_5 \\ \{\mathsf{s}, \mathsf{q}\downarrow, \mathsf{q}\uparrow, \mathsf{p}\downarrow, \mathsf{p}\uparrow\} \Big\| \\ R_5 \\ \{\mathsf{ai}\uparrow\} \Big\| \\ R_4 \\ \{\mathsf{e}\uparrow\} \Big\| \\ R_3 \\ \{\mathsf{w}\downarrow\} \Big\| \\ R_2 \\ \{\mathsf{b}\downarrow\} \Big\| \\ R_1 \\ \{\mathsf{g}\downarrow\} \Big\| \\ R \end{array}
$$

Fig. 4. Decomposition of derivations (Theorem 2.11)

Corollary 2.10. For any two structures T and R, we have

$$
\begin{array}{c} T \\ \mathsf{SNEL} \Big\| \\ R \end{array}
\quad \text{if and only if} \quad
\begin{array}{c} \mathsf{NEL} \Big\|\!\overline{} \\ [\bar{T} \parr R] \end{array} \quad .
$$

Our cut elimination proof relies on the following theorem, which is a special case of a more general one, and whose proof can be found in (SG09).

Theorem 2.11 (Decomposition). Every derivation Δ in SNEL can be rewritten as shown in Figure 4.

3. Splitting

Three approaches to cut elimination in deep inference have been explored in previous papers, but none of them can be applied in our case.

First, we could hope to rely on semantics, as Brünnler and Tiu did in (BT01) for classical logic. However, so far, there is no provability semantics for NEL (in the sense, for example, of phase spaces (Oka99) or other model theoretic semantics) that we could use for a completeness argument.

Second, we could hope to do as in (Brü03), where Brünnler presents a simple syntactic method that employs the atomicity of cut together with certain proof theoretical properties of classical logic. More recent papers that refine his method by using *atomic flows* are (GG08; GGS10). However, this approach cannot be used for NEL, because contraction cannot be applied to arbitrary formulas as in classical logic.

Third, we could hope to do as in (GS01; Str03b), where we relied on permutations of rules. However, traditional techniques based on simple notions of rule-instance permutation cannot work. To use permutation arguments for NEL requires a more general notion of permutation than the usual one. The following remark shows an example that illustrates the point and gives some hints to anybody who would like to proceed in that direction.

Remark 3.1. Let P and Q be arbitrary provable structures with $P \neq \circ \neq Q$. Consider then the following proof, where two proofs of P and Q are stacked one on top of the other and then we interweave two structures over the atoms a, b and c and their duals, and where the bottom rule instance is a $\mathsf{q}\!\uparrow$:

$$
\begin{array}{rc}
\mathsf{NEL} & \Vert \\
 & P \\
\mathsf{NEL} & \Vert \\
 & (P \otimes Q) \\
2\cdot\mathsf{ai}\!\downarrow & \overline{} \\
 & (P \otimes \langle [b \parr \bar{b}] \lhd [c \parr \bar{c}] \rangle \otimes Q) \\
\mathsf{q}\!\downarrow & \overline{} \\
 & (P \otimes [\langle b \lhd c \rangle \parr \langle \bar{b} \lhd \bar{c} \rangle] \otimes Q) \\
\mathsf{s} & \overline{} \\
 & (P \otimes [\langle b \lhd c \rangle \parr (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q)]) \\
\mathsf{ai}\!\downarrow & \overline{} \\
 & (P \otimes \langle [a \parr \bar{a}] \lhd [\langle b \lhd c \rangle \parr (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q)] \rangle) \\
\mathsf{q}\!\downarrow & \overline{} \\
 & (P \otimes [\langle a \lhd b \lhd c \rangle \parr \langle \bar{a} \lhd (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q) \rangle]) \\
\mathsf{s} & \overline{} \\
 & [(P \otimes \langle a \lhd b \lhd c \rangle) \parr \langle \bar{a} \lhd (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q) \rangle] \\
\mathsf{q}\!\uparrow & \overline{} \\
 & [\langle (P \otimes \langle a \lhd b \rangle) \lhd c \rangle \parr \langle \bar{a} \lhd (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q) \rangle]
\end{array}
\qquad (1)
$$

We can further suppose that the atoms a, b, c and their duals do not appear in P and Q. If so, any permutation of the bottom $\mathsf{q}\!\uparrow$ instance over the rule instances immediately above it leads to an unprovable structure, because some two dual atoms $a/\bar{a}$ or $b/\bar{b}$ or $c/\bar{c}$ would become connected by a connective different from a $\parr$. In order for a proof to exist, any rule instance changing the mutual logical relations of the $a/\bar{a}$, $b/\bar{b}$ and $c/\bar{c}$ atoms in the conclusion must be an instance of $\mathsf{q}\!\uparrow$. As an example of the contrary, consider

$$
\mathsf{q}\!\downarrow \frac{\langle [(P \otimes \langle a \lhd b \rangle) \parr \langle \bar{a} \lhd (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q) \rangle] \lhd c \rangle}{[\langle (P \otimes \langle a \lhd b \rangle) \lhd c \rangle \parr \langle \bar{a} \lhd (\langle \bar{b} \lhd \bar{c} \rangle \otimes Q) \rangle]}
\qquad (2)
$$

where c and $\bar{c}$ are not able to be joined in any $\mathsf{ai}\downarrow$ going upwards. This represents a difficulty for any argument based on permuting up the $\mathsf{q}\uparrow$ rule.

However, we can observe that the entire proofs of P and Q could be permuted below the $\mathsf{q}\uparrow$ instance, step by step. The two structures P and Q act as sort of "locks" for the substructure made of a, b and c atoms (and their duals). Once the locks are open, by independently proving them and so reducing them to the unit, the permuting up of the $\mathsf{q}\uparrow$ instance can take place.

So, an approach to proof normalisation based on permutations is not to be ruled out completely, but appears to be complicated and is not pursued in this paper.

By using the idea in this example, it is not difficult to construct another example (different from the one used in (Tiu06)) that shows the necessity of deep inference for designing cut-free systems for logics incorporating a self-dual non-commutative connective, like NEL.

Luckily, for proving cut admissibility for NEL we can use the technique that has been called *splitting* in (Gug07). As the cited paper explains, this technique establishes a clear connection with the sequent calculus, at least for the fragments of proof systems that allow for a sequent calculus presentation (in our case, the commutative fragment).

We refer the reader to (Gug07) for an intuitive explanation of splitting. We only repeat here that the technique relies on two separate phases:

1 Context reduction: If a structure $S\{R\}$ is provable, then $S\{\ \}$ can be reduced (by performing inference steps going upwards in the derivation) to the structure $[\{\ \} \mathbin{⅋} U]$, for some U, such that $[R \mathbin{⅋} U]$ is provable (the hole can be filled by any structure that does not play an active part in inference steps).

2 Splitting: If $[(R \otimes T) \mathbin{⅋} P]$ is provable, then P can be reduced to $[P_1 \mathbin{⅋} P_2]$, such that $[R \mathbin{⅋} P_1]$ and $[T \mathbin{⅋} P_2]$ are provable.

Context reduction is in turn proved by splitting, which is then at the core of the matter. In this section we will state and prove splitting proper, and in the next one we tackle context reduction.

For notational convenience, we define *system* NELc to be the system obtained from NEL by removing the rules for weakening, absorption, and digging (called *non-core* in (SG09):

$$\mathsf{NELc} = \mathsf{NEL} \setminus \{\mathsf{w}\downarrow, \mathsf{b}\downarrow, \mathsf{g}\downarrow\} = \{\circ\downarrow, \mathsf{ai}\downarrow, \mathsf{s}, \mathsf{q}\downarrow, \mathsf{p}\downarrow, \mathsf{e}\downarrow\} \quad .$$

The following lemma has been called 'shallow splitting' in (Gug07). The proof is very similar, and we do not give it in full here. But note that we organize the case analysis here in a different way than in (Gug07), which allows us to reuse it for the Lemmas 3.3 and 3.4 below.

Lemma 3.2 (Splitting). Let R, T, P be any NEL structures.

(i) If $[(R \otimes T) \mathbin{⅋} P]$ is provable in NELc, then there are structures P_R and P_T, such that

$$\begin{array}{c}[P_R \mathbin{⅋} P_T]\\ \mathsf{NELc}\,\Big\Vert\\ P\end{array} \qquad\text{and}\qquad \begin{array}{c}\overline{}\\ \mathsf{NELc}\,\Big\Vert\\ [R \mathbin{⅋} P_R]\end{array} \qquad\text{and}\qquad \begin{array}{c}\overline{}\\ \mathsf{NELc}\,\Big\Vert\\ [T \mathbin{⅋} P_T]\end{array} \quad .$$

(ii) If $[\langle R \lhd T\rangle \parr P]$ is provable in NELc, then there are structures P_R and P_T, such that

$$\begin{array}{c}\langle P_R \lhd P_T\rangle\\ \mathsf{NELc}\Big\Vert\\ P\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\Big\Vert\\ [R \parr P_R]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\Big\Vert\\ [T \parr P_T]\end{array} .$$

Proof. We prove both statements simultaneously by structural induction on the number of atoms in the conclusion and the length (number of rule instances) of the proof, ordered lexicographically. Without loss of generality, assume $R \neq \circ \neq T$ (otherwise both statements are trivially true).

(i) Consider the bottommost rule instance ρ in the proof of $[(R \otimes T) \parr P]$. We can distinguish between three different kinds of cases:

(a) The first kind appears when the redex of ρ is inside R, T or P. Then we have the following situation:

$$\begin{array}{c}\mathsf{NELc}\Big\Vert\,\Pi\\ \rho\,\dfrac{[(R' \otimes T) \parr P]}{[(R \otimes T) \parr P]}\end{array}$$

where we can apply the induction hypothesis to Π because it is one rule shorter (if $\rho = \mathsf{ai}{\downarrow}$ also the conclusion is smaller). We get

$$\begin{array}{c}[P_{R'} \parr P_T]\\ \mathsf{NELc}\Big\Vert\,\Delta_P\\ P\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\Big\Vert\,\Pi_R\\ [R' \parr P_R]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\Big\Vert\,\Pi_T\\ [T \parr P_T]\end{array}$$

From Π_R, we can get

$$\begin{array}{c}\mathsf{NELc}\Big\Vert\,\Pi'_R\\ \rho\,\dfrac{[R' \parr P_R]}{[R \parr P_R]}\end{array}$$

and we are done. If the redex of ρ is inside T or P, the situation is similar.

(b) In the second kind of case the substructure $(R \otimes T)$ is inside the redex of ρ, but is not modified by ρ. These cases can be compared with the 'commutative cases' in the usual sequent calculus cut elimination argument. We show only one representative example (a complete case analysis can be found in (Gug07) and (Str03a)): Suppose we have

$$\begin{array}{c}\mathsf{NELc}\Big\Vert\,\Pi\\ \mathsf{q}{\downarrow}\,\dfrac{[\langle[(R \otimes T) \parr P_1 \parr P_3] \lhd P_2\rangle \parr P_4]}{[(R \otimes T) \parr \langle P_1 \lhd P_2\rangle \parr P_3 \parr P_4]}\end{array}$$

We can apply the induction hypothesis to Π because it is one rule shorter (the size of the conclusion does not change). This gives us

$$\begin{array}{c}\langle Q_1 \lhd Q_2\rangle\\ \mathsf{NELc}\,\Big\Vert\,\Delta_1\\ P_4\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_1\\ [(R\otimes T)\parr P_1\parr P_3\parr Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_2\\ [P_2\parr Q_2]\end{array}$$

We can apply the induction hypothesis again to Π_1, because now the number of atoms in the conclusion is strictly smaller (because we can assume that the instance of $\mathsf{q}\!\downarrow$ is not trivial). We get

$$\begin{array}{c}[P_R\parr P_T]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1\parr P_3\parr Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_R\\ [R\parr P_R]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_T\\ [T\parr P_T]\end{array}$$

From Δ_1, Δ_2 and Π_2 we can build the following derivation

$$\begin{array}{c}[P_R\parr P_T]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_2\\ [P_1\parr P_3\parr Q_1]\\ \mathsf{NELc}\,\Big\Vert\,\Pi_2\\ \mathsf{q}\!\downarrow\dfrac{[\langle[P_1\parr Q_1]\lhd[P_2\parr Q_2]\rangle\parr P_3]}{[\langle P_1\lhd P_2\rangle\parr P_3\parr\langle Q_1\lhd Q_2\rangle]}\\ \mathsf{NELc}\,\Big\Vert\,\Delta_1\\ [\langle P_1\lhd P_2\rangle\parr P_3\parr P_4]\end{array}$$

and we are done. All other cases in this group are similar.

(c) In the last type of case the substructure $(R\otimes T)$ is destroyed by ρ. These cases can be compared to the 'key cases' in a standard sequent calculus cut elimination argument. We have only one possibility. The most general situation is as follows:

$$\begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi\\ \mathsf{s}\,\dfrac{[([(R_1\otimes T_1)\parr P_1]\otimes R_2\otimes T_2)\parr P_2]}{[(R_1\otimes R_2\otimes T_1\otimes T_2)\parr P_1\parr P_2]}\end{array}$$

where one of R_1 and R_2 might be $\circ$, but not both of them (similarly for T_1 and T_2). As before, we can apply the induction hypothesis to Π and get

$$\begin{array}{c}[Q_1\parr Q_2]\\ \mathsf{NELc}\,\Big\Vert\,\Delta_1\\ P_2\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_1\\ [(R_1\otimes T_1)\parr P_1\parr Q_1]\end{array} \quad\text{and}\quad \begin{array}{c}\mathsf{NELc}\,\Big\Vert\,\Pi_2\\ [(R_2\otimes T_2)\parr Q_2]\end{array}$$

We can apply the induction hypothesis again to Π_1 and Π_2. (Because we assume that the instance of s is not trivial, the conclusions are strictly smaller than the one of the original proof.) We get:

$$\begin{array}{c} [P_{R_1} \parr P_{T_1}] \\ \mathsf{NELc} \Big\Vert \Delta_3 \\ [P_1 \parr Q_1] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \Pi_{R_1} \\ [R_1 \parr P_{R_1}] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \Pi_{T_1} \\ [T_1 \parr P_{T_1}] \end{array}$$

and

$$\begin{array}{c} [P_{R_2} \parr P_{T_2}] \\ \mathsf{NELc} \Big\Vert \Delta_4 \\ Q_2 \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \Pi_{R_2} \\ [R_2 \parr P_{R_2}] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \Pi_{T_2} \\ [T_2 \parr P_{T_2}] \end{array}$$

Now let $P_R = [P_{R_1} \parr P_{R_2}]$ and $P_T = [P_{T_1} \parr P_{T_2}]$. We can build

$$\begin{array}{c} [P_{R_1} \parr P_{R_2} \parr P_{T_1} \parr P_{T_2}] \\ \mathsf{NELc} \Big\Vert \Delta_4 \\ [P_{R_1} \parr P_{T_1} \parr Q_2] \\ \mathsf{NELc} \Big\Vert \Delta_3 \\ [P_1 \parr Q_1 \parr Q_2] \\ \mathsf{NELc} \Big\Vert \Delta_1 \\ [P_1 \parr P_2] \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \Pi_{R_1} \\ [R_1 \parr P_{R_1}] \\ \mathsf{NELc} \Big\Vert \Pi_{R_2} \\ \mathsf{s}\,\dfrac{[(R_1 \otimes [R_2 \parr P_{R_2}]) \parr P_{R_1}]}{[(R_1 \otimes R_2) \parr P_{R_1} \parr P_{R_2}]} \end{array}$$

and a similar proof of $[(T_1 \otimes T_2) \parr P_{T_1} \parr P_{T_2}]$, and we are done.

(ii) The case for $[\langle R \lhd T \rangle \parr P]$ is similar to the one for $[(R \otimes T) \parr P]$, and we leave it to the reader. □

We can now tackle modalities, for which we can exhibit, too, a splitting lemma.

Lemma 3.3 (Splitting for Modalities). Let R and P be any NEL structures.

(i) If $[!R \parr P]$ is provable in NELc, then there are structures $P_1, \dots, P_h$ for some $h \geq 0$, such that

$$\begin{array}{c} [?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc} \Big\Vert \\ P \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \\ [R \parr P_1 \parr \cdots \parr P_h] \end{array} \quad .$$

(ii) If $[?R \parr P]$ is provable in NELc, then there is a structure P_R, such that

$$\begin{array}{c} !P_R \\ \mathsf{NELc} \Big\Vert \\ P \end{array} \quad\text{and}\quad \begin{array}{c} \mathsf{NELc} \overline{\Big\Vert} \\ [R \parr P_R] \end{array} \quad .$$

Proof. The proof is similar to the previous one. We use the same induction measure and the same pattern in the case analysis as before.

(i) We consider again the bottommost rule instance ρ in the proof of $[!R \parr P]$, and we have the same three classes of cases as in the proof of Lemma 3.2.

(a) The redex of ρ is inside R or P. This case is the same as in the proof of Lemma 3.2.

(b) The substructure $!R$ is inside the redex of ρ, but is not changed by ρ. This case is almost literally the same as for Lemma 3.2. We only have to replace $(R \otimes T)$ by $!R$, and

$$\begin{array}{c}[P_R \mathbin{⅋} P_T]\\ \mathsf{NELc}\,\Big\| \,\Delta_2\\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array} \qquad \text{by} \qquad \begin{array}{c}[?P_1 \mathbin{⅋} \cdots \mathbin{⅋} ?P_h]\\ \mathsf{NELc}\,\Big\| \,\Delta_2\\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array}$$

(As for the previous lemma, the full details can be found in (Str03a).)

(c) The substructure $!R$ is destroyed by ρ. There are two possibilities ($\rho = \mathsf{e}\!\downarrow$ and $\rho = \mathsf{p}\!\downarrow$):

$$\begin{array}{c}\mathsf{NELc}\,\Big\|\\ \mathsf{e}\!\downarrow \dfrac{[\circ \mathbin{⅋} P]}{[!\circ \mathbin{⅋} P]}\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\Big\|\,\Pi\\ \mathsf{p}\!\downarrow \dfrac{[![R \mathbin{⅋} P_1] \mathbin{⅋} Q_2]}{[!R \mathbin{⅋} ?P_1 \mathbin{⅋} Q_2]}\end{array}$$

For $\rho = \mathsf{e}\!\downarrow$ we are done immediately by letting $h = 0$. For $\rho = \mathsf{p}\!\downarrow$ we can apply the induction hypothesis to Π and get structures $P_2, \dots, P_h$ such that

$$\begin{array}{c}[?P_2 \mathbin{⅋} \cdots \mathbin{⅋} ?P_h]\\ \mathsf{NELc}\,\Big\|\\ Q_2\end{array} \qquad \text{and} \qquad \begin{array}{c}\mathsf{NELc}\,\Big\|\\ [R \mathbin{⅋} P_1 \mathbin{⅋} P_2 \mathbin{⅋} \cdots \mathbin{⅋} P_h]\end{array}$$

We immediately get

$$\begin{array}{c}[?P_1 \mathbin{⅋} ?P_2 \mathbin{⅋} \cdots \mathbin{⅋} ?P_h]\\ \mathsf{NELc}\,\Big\|\\ [?P_1 \mathbin{⅋} Q_2]\end{array} \quad .$$

(ii) As before, consider the bottommost rule instance ρ in the proof of $[?R \mathbin{⅋} P]$.

(a) The redex of ρ is inside R or P. This case is the same as before.

(b) The substructure $?R$ is inside the redex of ρ, but is not changed by ρ. As before, this case is almost literally the same as in the proof of Lemma 3.2. This time we have to replace $(R \otimes T)$ by $?R$, and

$$\begin{array}{c}[P_R \mathbin{⅋} P_T]\\ \mathsf{NELc}\,\Big\| \,\Delta_2\\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array} \qquad \text{by} \qquad \begin{array}{c}!P_R\\ \mathsf{NELc}\,\Big\| \,\Delta_2\\ [P_1 \mathbin{⅋} P_3 \mathbin{⅋} Q_1]\end{array}$$

(c) The substructure $?R$ is destroyed by ρ. For this case there is only one possibility:

$$\begin{array}{c}\mathsf{NELc}\,\Big\|\,\Pi\\ \mathsf{p}\!\downarrow \dfrac{[![R \mathbin{⅋} P_1] \mathbin{⅋} P_2]}{[?R \mathbin{⅋} !P_1 \mathbin{⅋} P_2]}\end{array}$$

We can apply part (i) of the lemma and get

$$\mathsf{NELc}\,\Big\Vert\,\Delta\ \frac{[?Q_1 \parr \cdots \parr ?Q_h]}{P_2} \qquad\text{and}\qquad \mathsf{NELc}\,\Big\Vert\,\Pi_R\ \frac{}{[R \parr P_1 \parr Q_1 \parr \cdots \parr Q_h]}$$

Now let $P_R = [P_1 \parr Q_1 \parr \ldots \parr Q_h]$. We can build

$$\begin{array}{c} ![P_1 \parr Q_1 \parr \cdots \parr Q_h] \\ \{\mathsf{p}\!\downarrow\}\ \Vert \\ [!P_1 \parr ?Q_1 \parr \cdots \parr ?Q_h] \\ \mathsf{NELc}\ \Vert\ \Delta \\ [!P_1 \parr P_2] \end{array}$$

as desired. □

Lemma 3.4 (Splitting for Atoms). Let a be any atom and P be any NEL structure.

If there is a proof $\mathsf{NELc}\,\Vert\ [a \parr P]$ then there is a derivation $\begin{array}{c}\bar{a}\\ \mathsf{NELc}\ \Vert \\ P\end{array}$.

Proof. After the previous two proofs this is an almost trivial exercise: The case (a) is as before, and for (b), we have to replace $(R \otimes T)$ by a, and

$$\begin{array}{c} [P_R \parr P_T] \\ \mathsf{NELc}\ \Vert\ \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} \qquad\text{by}\qquad \begin{array}{c} \bar{a} \\ \mathsf{NELc}\ \Vert\ \Delta_2 \\ [P_1 \parr P_3 \parr Q_1] \end{array} .$$

For case (c), the only possibility is

$$\begin{array}{c} \mathsf{NELc}\ \Vert\ \Pi' \\ \mathsf{ai}\!\downarrow \dfrac{P_1}{[a, \bar{a}, P_1]} \end{array}$$

from which we immediately get

$$\begin{array}{c} \bar{a} \\ \mathsf{NELc}\ \Vert \\ [\bar{a}, P_1] \end{array} .$$

as desired. □

4. Context Reduction

The idea of context reduction is to reduce a problem that concerns an arbitrary (deep) context $S\{\ \}$ to a problem that concerns only a shallow context $[\{\ \} \parr P]$. In the case of cut elimination, for example, we will then be able to apply splitting.

Before giving the statement, we need to define the *modality depth* of a context $S\{\ \}$ to be the number of ! and ? in whose scope the $\{\ \}$ occurs. In the following lemma, the $\{\ \}$ is treated as ordinary atom.

Lemma 4.1 (Context Reduction). Let R be a NEL structure and $S\{\ \}$ be a context. If $S\{R\}$ is provable in NELc, then there is a structure P_R, such that

$$\begin{array}{c} !\cdots![\{\ \} \parr P_R] \\ \mathsf{NELc} \Big\| \Delta \\ S\{\ \} \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc} \Big\| \Pi \\ [R \parr P_R] \end{array}$$

where the number of ! in front of $[\{\ \} \parr P_R]$ is the modality depth of $S\{\ \}$.

Proof. We proceed by structural induction on the context $S\{\ \}$. The base case when $S\{\ \} = \{\ \}$ is trivial. Now we can distinguish four cases

(a) $S\{\ \} = [(S'\{\ \} \otimes T) \parr P]$ where,without loss of generality, $T \neq \circ$. Note that we do allow $P = \circ$. We can apply splitting (Lemma 3.2) to the proof of $[(S'\{R\} \otimes T) \parr P]$ and get:

$$\begin{array}{c} [P_S \parr P_T] \\ \mathsf{NELc} \Big\| \Delta_P \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc} \Big\| \Pi_S \\ [S'\{R\} \parr P_S] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc} \Big\| \Pi_T \\ [T \parr P_T] \end{array}$$

Because $T \neq \circ$ we can now apply the induction hypothesis to Π_S and get:

$$\begin{array}{c} !\cdots![\{\ \} \parr P_R] \\ \mathsf{NELc} \Big\| \Delta' \\ [S'\{\ \} \parr P_S] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc} \Big\| \Pi \\ [R \parr P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !\cdots![\{\ \} \parr P_R] \\ \mathsf{NELc} \Big\| \Delta' \\ [S'\{\ \} \parr P_S] \\ \mathsf{NELc} \Big\| \Pi_T \\ \mathsf{s}\,\dfrac{[(S'\{\ \} \otimes [T \parr P_T]) \parr P_S]}{[(S'\{\ \} \otimes T) \parr P_S \parr P_T]} \\ \mathsf{NELc} \Big\| \Delta_P \\ [(S'\{\ \} \otimes T) \parr P] \end{array}$$

as desired.

(b) The cases $S\{\ \} = [\langle S'\{\ \} \triangleleft T\rangle \parr P]$ and $S\{\ \} = [\langle T \triangleleft S'\{\ \}\rangle \parr P]$ are handled similarly to (a).

(c) If $S\{\ \} = [!S'\{\ \} \parr P]$, then we can apply splitting (Lemma 3.3) to the proof of $[!S'\{R\} \parr P]$ and get:

$$\begin{array}{c} [?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi_S \\ [S'\{R\} \parr P_1 \parr \cdots \parr P_h] \end{array} .$$

By applying the induction hypothesis to Π_S we get P_R such that

$$\begin{array}{c} ! \cdots ![\{\ \} \parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ [S'\{\ \} \parr P_1 \parr \cdots \parr P_h] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert}\,\Pi \\ [R \parr P_R] \end{array}$$

From this we can build

$$\begin{array}{c} !! \cdots ![\{\ \} \parr P_R] \\ \mathsf{NELc}\,\Big\Vert\,\Delta' \\ ![S'\{\ \} \parr P_1 \parr \cdots \parr P_h] \\ \{\mathsf{p}\!\downarrow\}\,\Big\Vert \\ [!S'\{\ \} \parr ?P_1 \parr \cdots \parr ?P_h] \\ \mathsf{NELc}\,\Big\Vert\,\Delta_P \\ [!S'\{\ \} \parr P] \end{array}$$

Note that in this case the number of ! in front of $[\{\ \} \parr P_R]$ increases.

(d) The case where $S\{\ \} = [?S'\{\ \} \parr P]$ is similar to (c). □

5. Elimination of the Up Fragment

In this section, we will first show four lemmas, which are all easy consequences of splitting and which say that the core up rules of system SNEL are admissible if they are applied in a shallow context $[\{\ \} \parr P]$. Then we will show how context reduction is used to extend these lemmas to any context. As a result, we get a proof of cut elimination that can be considered modular, in the sense that the four core up rules $\mathsf{ai}\!\uparrow$, $\mathsf{q}\!\uparrow$, $\mathsf{p}\!\uparrow$, and $\mathsf{e}\!\uparrow$ are shown to be admissible, one independently from the other.

Lemma 5.1. Let P be a structure and let a be an atom. If $[(a \otimes \bar{a}) \parr P]$ is provable in NELc, then P is also provable in NELc.

Proof. Apply splitting to the proof of $[(a \otimes \bar{a}) \parr P]$. This yields:

$$\begin{array}{c} [P_a \parr P_{\bar{a}}] \\ \mathsf{NELc}\,\Big\Vert \\ P \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [a \parr P_a] \end{array} \qquad \text{and} \qquad \begin{array}{c} \mathsf{NELc}\,\overline{\Big\Vert} \\ [\bar{a} \parr P_{\bar{a}}] \end{array} .$$

By applying Lemma 3.4, we get a derivation from $\bar{a}$ to P_a and one from a to $P_{\bar{a}}$. From these we can build our proof

$$
\begin{array}{c}
\circ\!\downarrow \dfrac{}{\circ} \\
\mathsf{ai}\!\downarrow \dfrac{}{[\bar{a} \parr a]} \\
\mathsf{NELc} \Big\| \\
[P_a \parr P_{\bar{a}}] \\
\mathsf{NELc} \Big\| \\
P
\end{array}
$$

as desired. □

Lemma 5.2. Let R, T, U, V and P be any NEL structures. If $[(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle) \parr P]$ is provable in NELc, then $[\langle (R \otimes T) \lhd (U \otimes V)\rangle \parr P]$ is also provable in NELc.

Proof. By applying splitting several times to the proof of $[(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle) \parr P]$, we get structures P_R, P_T, P_U, and P_V such that

$$
\begin{array}{c}
[\langle P_R \lhd P_U\rangle \parr \langle P_T \lhd P_V\rangle] \\
\mathsf{NELc} \Big\| \\
P
\end{array}
\qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[R \parr P_R]
\end{array}
\qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[U \parr P_U]
\end{array}
\qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[T \parr P_T]
\end{array}
\qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[V \parr P_V]
\end{array}
$$

By putting things together, we can build the proof

$$
\begin{array}{c}
\mathsf{NELc} \Big\| \\
\mathsf{s},\mathsf{s},\mathsf{s},\mathsf{s} \dfrac{\langle ([R \parr P_R] \otimes [T \parr P_T]) \lhd ([U \parr P_U] \otimes [V \parr P_V])\rangle}{\langle [(R \otimes T) \parr P_R \parr P_T] \lhd [(U \otimes V) \parr P_U \parr P_V]\rangle} \\
\mathsf{q}\!\downarrow,\mathsf{q}\!\downarrow \dfrac{}{[\langle (R \otimes T) \lhd (U \otimes V)\rangle \parr \langle P_R \lhd P_U\rangle \parr \langle P_T \lhd P_V\rangle]} \\
\mathsf{NELc} \Big\| \\
[\langle (R \otimes T) \lhd (U \otimes V)\rangle \parr P]
\end{array}
$$

as desired. □

Lemma 5.3. Let R, T and P be any NEL structures. If $[(?R \otimes !T) \parr P]$ is provable in NELc, then $[?(R \otimes T) \parr P]$ is also provable in NELc.

Proof. As above, we apply splitting several times to the proof of $[(?R \otimes !T) \parr P]$ and get structures P_R, $P_1, \ldots, P_h$ such that:

$$
\begin{array}{c}
[!P_R \parr ?P_1 \parr \cdots \parr ?P_h] \\
\mathsf{NELc} \Big\| \\
P
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[R \parr P_R]
\end{array}
\qquad \text{and} \qquad
\begin{array}{c}
\mathsf{NELc} \Big\| \\
[T \parr P_1 \parr \cdots \parr P_h]
\end{array}
$$

By putting things together, we can build the proof

$$\begin{array}{c} \text{NELc} \Big\Vert \\ \mathsf{s},\mathsf{s}\,\dfrac{!([R \parr P_R] \otimes [T \parr P_1 \parr \cdots \parr P_h])}{![(R \otimes T) \parr P_R \parr P_1 \parr \cdots \parr P_h]} \\ \{\mathsf{p}\!\downarrow\} \Big\Vert \\ [?(R \otimes T) \parr !P_R \parr ?P_1 \parr \cdots \parr ?P_h] \\ \text{NELc} \Big\Vert \\ [?(R \otimes T) \parr P] \end{array}$$

as desired. □

Lemma 5.4. Let P be any NEL structure. If $[?\circ \parr P]$ is provable in NELc, then $[\circ \parr P]$ is also provable in NELc.

Proof. This is now a trivial exercise, that we leave to the reader. □

By the use of context reduction (Lemma 4.1), we can extend the statements of Lemmas 5.1–5.4 from shallow contexts $[\{\ \} \parr P]$ to arbitrary contexts $S\{\ \}$, which is done by the following lemma.

Lemma 5.5. Let R, T, U and V be any structures, let a be an atom and let $S\{\ \}$ be any context. Then we have the following

(i) If $S(a \otimes \bar{a})$ is provable in NELc, then so is $S\{\circ\}$.
(ii) If $S(\langle R \lhd U\rangle \otimes \langle T \lhd V\rangle)$ is provable in NELc, then so is $S\langle (R \otimes T) \lhd (U \otimes V)\rangle$.
(iii) If $S(?R \otimes !T)$ is provable in NELc, then so is $S\{?(R \otimes T)\}$.
(iv) If $S\{?\circ\}$ is provable in NELc, then so is $S\{\circ\}$.

Proof. All four statements are proved similarly. We will here show only the third: Let a proof of $S(?R \otimes !T)$ be given and apply context reduction, to get a structure P, such that

$$\begin{array}{c} ! \cdots ![\{\ \} \parr P] \\ \text{NELc} \Big\Vert \Delta \\ S\{\ \} \end{array} \qquad \text{and} \qquad \begin{array}{c} \text{NELc} \Big\Vert \Pi \\ [(?R \otimes !T) \parr P] \end{array}$$

By Lemma 5.3 there is a proof Π' of $[?(R \otimes T) \parr P]$. By plugging $?(R \otimes T)$ into the hole of Δ, we can build

$$\begin{array}{c} \{\circ\!\downarrow, \mathsf{e}\!\downarrow\} \Big\| \\ !\cdots!\circ \\ \mathsf{NELc} \Big\| \Pi' \\ !\cdots![?(R \otimes T) \parr P] \\ \mathsf{NELc} \Big\| \Delta \\ S\{?(R \otimes T)\} \end{array}$$

It is obvious that the other statements are proved in the same way. □

Lemma 5.6. If a structure R is provable in $\mathsf{NELc} \cup \{\mathsf{ai}\!\uparrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{e}\!\uparrow\}$ then it is also provable in NELc.

Proof. The instances of the rules $\mathsf{ai}\!\uparrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{e}\!\uparrow$ are removed one after the other (starting with the topmost one) via Lemma 5.5. □

Now we can very easily give a proof for the cut elimination theorem for the system NEL.

Proof of Theorem 2.8 Cut elimination is obtained in two steps:

$$\begin{array}{c} \circ\!\downarrow \dfrac{}{\circ} \\ \mathsf{SNEL} \Big\| \\ R \end{array} \xrightarrow{1} \begin{array}{c} \mathsf{NELc} \cup \{\mathsf{ai}\!\uparrow, \mathsf{q}\!\uparrow, \mathsf{p}\!\uparrow, \mathsf{e}\!\uparrow\} \Big\| \\ R' \\ \{\mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{g}\!\downarrow\} \Big\| \\ R \end{array} \xrightarrow{2} \begin{array}{c} \mathsf{NELc} \Big\| \\ R' \\ \{\mathsf{w}\!\downarrow, \mathsf{b}\!\downarrow, \mathsf{g}\!\downarrow\} \Big\| \\ R \end{array}$$

Step 1 is an application of the decomposition (Theorem 2.11). The instances of $\mathsf{g}\!\uparrow, \mathsf{b}\!\uparrow, \mathsf{w}\!\uparrow$ disappear because their premise must be the unit $\circ$, which is impossible. Step 2 is just Lemma 5.6. □

This technique shows how admissibility can be proved uniformly, both for cut rules (the atomic ones) and the other up rules, which are actually very different rules from the cut. So, our technique is more general than cut elimination in the sequent calculus, for two reasons:

1 it applies to connectives that admit no sequent calculus definition, as seq;
2 it can be used to show admissibility of non-infinitary rules that involve no negation, like $\mathsf{q}\!\uparrow$ and $\mathsf{p}\!\uparrow$.

6. Perspectives

We think that the techniques developed here for splitting can be exported to the many modal logics already available in deep inference (some of which have no known cut-free presentation in Gentzen formalisms). The reason is that linear logic modalities have a similar behaviour to those of modal logic. This is particularly obvious if we observe that the promotion rule of NEL is the same as the K rule of all modal logics in deep inference (corresponding to the K axiom of basic modal logic). Of course, contraction in linear logic, and in NEL, is restricted, but the splitting theorems, crucially, do not make any use of it.

Apart from its use in getting cut elimination, splitting is a powerful tool for reducing proof search non-determinism in deep inference proof systems. This is explored in the works (Kah06; Kah08).

We are currently investigating, in the context of the INRIA ARC project REDO, the relations between splitting and the focusing technique in linear logic (And92; Mil96), which is at the basis of ludics (Gir01). It appears that focusing can be justified and greatly generalized by splitting in deep inference. It seems like splitting is a way to explore the duality between any subformula and its context, so unveiling a new logical symmetry.

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