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Constructing free Boolean categories

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Abstract

*By Boolean category we mean something which is to a Boolean algebra what a category is to a poset. We propose an axiomatic system for Boolean categories, similar to but differing in several respects from the one given very recently by Führmann and Pym. In particular everything is done from the start in a *-autonomous category and not a linear distributive one, which simplifies issues like the Mix rule. An important axiom, which is introduced later, is a "graphical" condition, which is closely related to denotational semantics and the Geometry of Interaction. Then we show that a previously constructed category of proof nets is the free "graphical" Boolean category in our sense. This validates our categorical axiomatization with respect to a real-life example. Another important aspect of this work is that we do not assume a-priori the existence of units in the *-autonomous categories we use. This has some retroactive interest for the semantics of linear logic, and is motivated by the properties of our example with respect to units.*

1. Introduction

Unlike other mathematicians, proof theorists have access to very few canonical objects. All mathematicians have the integers, the reals, the rationals. Geometers have projective planes and spheres, algebraists have polynomial rings and permutation groups. Indeed, algebraists have access to the *concept* of a group and of a ring, which have been stable for more than a hundred years. In contrast, a proof theorist is always ready to tweak a definition like that of the sequent calculus to suit his needs. We say *the* sequent calculus but there is no such thing.

Logicians have Boolean and Heyting algebras, but they are of limited interest to proof theorists since they identify too many things: In a Boolean or Heyting algebra two formulas, a seemigly complex one and a seemingly trivial one, can turn out to have identical denotations—and things are the same, if not worse, for proofs.

We know that much information about a proof is kept if we replace posets by categories. A celebrated example of this is Freyd's proof [14] that higher order intuitionsistic logic has the existence and disjunction properties (as a constructive logic should) purely by observing the free elementary topos, and using this very property of freeness. The free topos is a canonical object if there ever was one.

The free elementary topos is one of the many, many examples of a "Heyting category", which is to categories what a Heyting algebra is to posets: a bicartesian closed category. Until very recently it was absolutely mysterious how one could define "Boolean categories" in the same manner. For a long time the only known natural definition of a Boolean category collapsed to a poset. This was first corrected by following closely the approach to term systems for classical logic: in order to prevent collapse, introduce asymetries, which is what is done for example in Selinger's control categories [18] (which correspond to the $\lambda\mu-$calculus [17]) or the models of Girard's LC [7] and the closely related work of Streicher and Reus on continuations [20], which introduce restrictions by the means of polarities.

But then there appeared several approaches [6, 5, 13, **?**] to Boolean categories that do keep the symmetry we associate with Booleanness: these categories are all equipped with an contravariant involution, and except for the last example they are *-autonomous categories. The present paper is concerned with one of these, which was given a concrete construction in [13]. It is a remarkably simple object, a candidate for canonicity: a "beefed up" Boolean algebra. It is surprising that it was not discovered before.

In this paper we present a series of axioms for Boolean categories, in order of increasing strength. We then show that this category of $\mathbb{B}$-nets [13] for a set of atomic formulas is the free Boolean category for the strongest combination of axioms, with the atoms as generators. On the way to establishing this result, we will introduce axioms little by little. With hindsight we can say that they fall in three classes:

- the "general" axioms, for which it is fair to say that they should hold in any "really Boolean" model of classical logic. All axioms but three in this paper belong to this class, and they are presented first.

- the "semantical" axiom, which is the property of "graphicality". As the presentation implies this axiom gives the model in [13] its semantical character and relates it to coherences spaces and the Geometry of Interaction. It is a very strong axiom, and we now think it masks some important properties of intrinsic Booleanness.

- two axioms that can be switched off with very interesting results. One we call "loop killing" for a rather geometric reasons. Its status and flavor is intriguing. It can be expressed in an intuitionistic model, where is says that composing (multiplying) the church numerals two and zero gives zero. Thus it this context it has no meaning whatsoever. But in our Boolean world, it seems to have some real power of its own, as we will see; moreover it is now clear that there are "Boolean" categories where it does not hold.

  The other one of these independent-minded axioms is Δ-∇-strength. In more traditional fields of algebra it is automatic: when an object is equipped with both a monoid (algebra) and comonoid (co-algebra) structure, it is always required that an operation in one structure be a morphism in the other structure; this is the definition of a bialgebra. In the model we study this bialgebraic condition holds, but it does not seem to be intrinsic to Booleannes and we now have new semantics where it does not hold.

Our axiomatic approach differs from that of Führmann and Pym [5, 6] in several respects. It is completely 1-categorical and does not use something like an order enrichment. Also, we start with a *-autonomous category and show how to extract (several) weakly distributive categories it contains, while they start with a weakly distributive category and then complete it to a *-automous one by adding structure. The present approach cannot really be compared with [**?**], because in the latter work the operation of curryfying (or transposition) is not intrinsic, but can only be simulated.

A side effect of our work is a novel answer to the problem of defining a *-autonomous category that does not have units, which we need to interpret logics without constants. This retroactively applies to multiplicative [1] and multiplicative-additive [9] proof nets.

2. The axioms

It is very well known how to model a multiple-premiss, single-conclusion linear calculus in a symmetric monoidal category that has the $\multimap$ adjoint operator. It is also well-known how to have multiple premisses, and/or a negation. If we want zero premiss, it is natural to think of the tensor unit as source as representing an empty family of premises: an empty context. But if we have the unit in the category, shouldn't we also have it in the logic? The standard approach to this question is found in [1], where the existence of a unit $\mathbf{I}$ is assumed in the category that is used for the semantics, but its use is very restricted: it can only appear as the source of a semantical map. There is a problem, though: for example, the category of ordinary multiplicative proof nets without units cannot be used to interpret itself as a theory! We propose a solution to this problem: replace the unit with a functor to $\mathbf{Set}$, which would be the covariant functor represented by the unit, if only there was a unit. This seems to be a very trivial change, but it has interesting consequences.

2.1. *-autonomous categories without units

We will define autonomous (SMC) and *-autonomous categories not to have units by default. This spares us from having *-autonomous categories without units with units.

From now on $\mathscr{C}$ denotes a (small) category. We denote the composition of two maps f, g by either gf or $g \circ f$, depending on readability; the order is the standard (functional, as opposed to diagrammatic) order. Given $X \in \mathscr{C}$, we will write either X or 1_X to represent the identity map on it, also depending on readability. We use the standard notation for the covariant representable functor associated with X, i.e., $h^X = \mathrm{Hom}_{\mathscr{C}}(X, -)$, and h_X for the contravariant representable $\mathrm{Hom}_{\mathscr{C}}(-, X)$.

The arguments in the following section need familiarity with Yoneda's Lemma: given a functor $F\colon \mathscr{C} \to \mathbf{Set}$ there is a natural bijective correspondence between $F(X)$ and the set of natural transformations $h^X \to F$.

2.1.1 Definition A category $\mathscr{C}$ *has tensors* if it is equipped with a bifunctor $- \otimes -$ with the usual associativity and symmetry isomorphisms

$$\mathsf{assoc}_{A,B,C}\colon A \otimes (B \otimes C) \to (A \otimes B) \otimes C$$
$$\mathsf{twist}_{A,B}\colon A \otimes B \to B \otimes A$$

that obey the usual "pentagon" and "hexagon" (see [16, p.158,p.180]).

Note that we do not ask for a unit in that definition. Nonetheless the "coherence" theorem for symmetric monoidal categories [15] does also hold in our case, or more precisely everything in it that does not deal with units. In particular, we can simply write $A \otimes B \otimes C \otimes D$ for $((A \otimes B) \otimes C) \otimes D$ or $A \otimes ((B \otimes C) \otimes D)$, or even $(B \otimes D) \otimes (A \otimes C)$, because there is a uniquely defined isomorphism between any two of them.

If it exists, we denote the usual right adjoint to tensoring as $(-) \multimap (-)$ and it defines the usual bivariant bifunctor. We will denote the "internal covariant representable functor" defined by X as $H^X = X \multimap (-) \colon \mathscr{C} \to \mathscr{C}$. The following two natural isomorphisms are trivial but important, and they are natural in both X and Y:

$$H^X H^Y \cong H^{X \otimes Y} \quad \text{and} \quad h^X H^Y \cong h^{X \otimes Y} \;. \tag{1}$$

It is very well known that a functor $\mathscr{C} \to \mathbf{Set}$ can be profitably seen as a "generalized object" of $\mathscr{C}$; we call such a thing a *virtual object* of $\mathscr{C}$ and we emphasize this fact by writing it as $h^{\mathbb{A}}$, which is a functor, and would be the representable functor associated to the object $\mathbb{A}$ if the latter only existed. Given $X \in \mathscr{C}$, maps $\mathbb{A} \to X$ should morally be in bijective correspondence with natural transformations $h^X \to h^{\mathbb{A}}$, and the latter are truly in bijective correspondence with elements of $h^{\mathbb{A}}(X)$ and this allows us to write an $s \in h^{\mathbb{A}}(X)$ as

$$\mathbb{A} \overset{s}{\dashrightarrow} X \quad .$$

In general a dotted arrow will mean that at least one of the source or target is virtual, and it is to be interpreted as a reverse-direction natural transformation between the corresponding functors. For example, given $f \colon X \to Y$ and $t = (h^{\mathbb{A}} f)(s)$, we can write this as a commutative diagram

$$\begin{array}{ccccc} & & \mathbb{A} & & \\ & {\scriptstyle s}\swarrow & & \searrow{\scriptstyle t} & \\ X & & \xrightarrow[f]{} & & Y \end{array} \quad ,$$

which justifies the notation $t = f \circ s$, or simply $t = fs$. But we have to be very careful on how to extend the $\otimes, \multimap$ structure to virtuals. At least one thing works: given a virtual object A and a real one X we can define a virtual object $\mathbb{A} \otimes X$, by composing their "representables" (the reader should check that this makes perfect sense, by plugging an object of $\mathscr{C}$ in the functors): $h^{\mathbb{A} \otimes X} = h^{\mathbb{A}} H^X$.

So we can only *left*- tensor a virtual object, and only to get a virtual one.[1] This construction in natural in both variables: given $s \colon \mathbb{A} \dashrightarrow \mathbb{B}$ and $f \colon X \to Y$ then there is an obvious $s \otimes f \colon \mathbb{A} \otimes X \dashrightarrow \mathbb{B} \otimes Y$. Suppose we have a "virtual left unit" $\mathbb{I}$; if it were real we would have a natural isomorphism $\lambda \colon \mathbb{I} \otimes (-) \cong (-)$; this translates, given a real $f \colon X \to X'$, as a commuting square

$$\begin{array}{ccc} h^{X'} & \xrightarrow{h^f} & h^X \\ {\scriptstyle h^{\lambda_{X'}}}\big\downarrow{\scriptstyle \cong} & & {\scriptstyle \cong}\big\downarrow{\scriptstyle h^{\lambda_X}} \\ h^{\mathbb{I}} H^{X'} & \xrightarrow[h^{\mathbb{I}} H^f]{} & h^{\mathbb{I}} H^X \end{array}$$

Since this is a diagram of functors we can plug any map $Y \to Y'$ in there; it is then easy to see that having a "virtual left unit law" isomorphism is equivalent to having an isomorphism $\mathrm{Hom}_{\mathscr{C}}(X, Y) \cong h^{\mathbb{I}}(X \multimap Y)$, natural in both X and Y. This is the point of the whole exercise: a "proof" of an object X can be seen as an element of $h^{\mathbb{I}}(X)$ because a proof of $X \multimap Y$ will just be a map $X \to Y$.

The unit isomorphism in a monoidal category has to interact well with the associativity iso [16, p.159]; this can be translated here:

$$\begin{array}{ccc} \mathbb{I} \otimes X \otimes Y & \xrightarrow{\cong} & \mathbb{I} \otimes Y \otimes X \\ {\scriptstyle \lambda_{X \otimes Y}}\vdots\downarrow & & \vdots\downarrow{\scriptstyle \lambda_{Y \otimes X}} \\ X \otimes Y & \xrightarrow[\cong]{} & Y \otimes X \end{array}$$

and it is not hard to show, using the fact that we already have $\multimap$, that this holds. But there is a missing condition: the unit laws also have to hold *with the unit itself*: the two ways of going from $\mathbb{I} \otimes \mathbb{I}$ to $\mathbb{I}$ have to coincide. We cannot construct this directly; the equivalent condition for us is to require, given $s \colon \mathbb{I} \dashrightarrow X$ and $t \colon \mathbb{I} \dashrightarrow Y$, that the following diagram of (mostly) virtual maps commutes:

$$\begin{array}{ccccccc} & {\scriptstyle s}\nearrow & X & \overset{\cong}{\dashrightarrow} & \mathbb{I} \otimes X & \overset{t \otimes X}{\dashrightarrow} & Y \otimes X \\ \mathbb{I} & & & & & & \big\downarrow{\scriptstyle \cong} \\ & {\scriptstyle t}\searrow & Y & \underset{\cong}{\dashrightarrow} & \mathbb{I} \otimes Y & \underset{s \otimes Y}{\dashrightarrow} & X \otimes Y \end{array} \tag{2}$$

Thus when this happens we have a uniquely defined $s \otimes t \colon \mathbb{I} \dashrightarrow X \otimes Y$. One can then show that the operation $s, t \mapsto s \otimes t \colon h^{\mathbb{I}}(X) \times h^{\mathbb{I}}(Y) \longrightarrow h^{\mathbb{I}}(X \otimes Y)$ agrees well with associativity and twist; in other words, given X, Y, Z with $s \in h^{\mathbb{I}}(X)$, $t \in h^{\mathbb{I}}(Y)$ and $r \in h^{\mathbb{I}}(Z)$ we always have that $t \otimes s = \mathsf{twist}_{X,Y} \circ (s \otimes t)$ and $(s \otimes t) \otimes r = \mathsf{assoc}_{X,Y,Z} \circ (s \otimes (t \otimes r))$. This allows us to simply write $s \otimes t \otimes r \colon \mathbb{I} \dashrightarrow X \otimes Y \otimes Z$. In technical parlance $h^{\mathbb{I}}$ would be a monoidal functor $(\mathscr{C}, \otimes) \to (\mathbf{Set}, \times)$ if $\mathscr{C}$ had a unit (when $\mathscr{C}$ does have a unit $\mathbf{I}$ then $h^{\mathbf{I}}$ is always monoidal).

Notice that it is perfectly natural to write $s \otimes Y$ or $s \otimes 1_Y$ for the (real) horizontal map $Y \to X \otimes Y$ at the bottom of the diagram. In the same way $X \otimes t$ or $1_X \otimes t$ can stand for the map $X \to X \otimes Y$ which is the top horizontal map followed by the twist.

[1] More precisely: everything is a composition of functors, and there can be as many "internal representables" $\mathscr{C} \to \mathscr{C}$ as we want but exactly one $\mathscr{C} \to \mathbf{Set}$, which has to appear at leftmost end. But since we have a symmetry we can play notational tricks; if the logic were non-commutative, we would have access to two implications, which would allow us to attain similar effects.

2.1.2 Definition A category $\mathscr{C}$ with tensors is an *autonomous category* if is has the structure in the previous paragraphs: the adjoint $\multimap$ and the functor $h^{\mathbb{I}}$ along with the natural iso $h^{\mathbb{I}}(X \multimap Y) \cong \mathrm{Hom}_{\mathscr{C}}(X, Y)$, which obeys Equation (2). The $\mathscr{C}$ is a **-autonomous category* if in addition it has a functor $(-)^{\perp}\colon \mathscr{C}^{op} \to \mathscr{C}$ which is an involution (for simplicity we will later assume that $X^{\perp\perp} = X$, but it could also be a natural isomorphism), and which obeys $X \multimap Y \cong (Y \otimes X^{\perp})^{\perp}$.

2.1.3 Proposition *If $\mathscr{C}$ is an autonomous category in the sense above and such that $h^{\mathbb{I}}$ is representable, then $\mathscr{C}$ is autonomous (SMC) in the usual sense (with the usual units).*

Proof: Let $\mathbf{I}$ be the object that represents $h^{\mathbb{I}}$, i.e., $h^{\mathbf{I}}$ is naturally isomorphic to $h^{\mathbb{I}}$. Then we have, for any X, Y:

$$\begin{array}{rcl} \mathrm{Hom}(X, Y) & \cong & h^{\mathbb{I}}(X \multimap Y) \\ & \cong & h^{\mathbf{I}}(X \multimap Y) \;=\; \mathrm{Hom}(\mathbf{I}, X \multimap Y) \\ & \cong & \mathrm{Hom}(\mathbf{I} \otimes X, Y)\ . \end{array}$$

By Yoneda we get an iso $\mathbf{I} \otimes X \cong X$; it is then easy to check, using Yoneda again and our definition of the natural transformation h^{λ}, that this iso obeys all the requirements for the unit of a monoidal category. □

In a *-autonomous category, we can define another bifunctor $- \parr -$ (called *cotensor* or *par*) to be the de Morgan dual of $- \otimes -$, i.e., $X \parr Y = (Y^{\perp} \otimes X^{\perp})^{\perp}$.[2] Then we have $X \multimap Y \cong X^{\perp} \parr Y$.

If $\mathscr{C}$ is *-autonomous we also have a "virtual bottom", that we write $h_{\bot\!\!\!\bot}$, given by $h_{\bot\!\!\!\bot}(X) = h^{\mathbb{I}}(X^{\perp})$, and as for $h^{\mathbb{I}}$, thinking of it as an object $\bot\!\!\!\bot$ of $\mathscr{C}$ allows us to write

$$X \overset{s}{\dashrightarrow} \bot\!\!\!\bot$$

for an element $s \in h_{\bot\!\!\!\bot}(X)$. As before, we also get $u \parr v \parr w\colon X \parr Y \parr T \dashrightarrow \bot\!\!\!\bot$ for $u \in h_{\bot\!\!\!\bot}(X)$ and $v \in h_{\bot\!\!\!\bot}(Y)$ and $w \in h_{\bot\!\!\!\bot}(Z)$.[3]

Given maps $f\colon A \to B \parr C$ and $g\colon A \otimes B^{\perp} \to C$ where g is the curryfication of f, we say that f and g are *transposes* of each other. More generally, for any objects $A_1, \ldots, A_n$, a map $f\colon A_1^{\perp} \otimes \cdots \otimes A_k^{\perp} \to A_{k+1} \parr \cdots \parr A_n$ uniquely determines a map $g\colon A_{p(1)}^{\perp} \otimes \cdots \otimes A_{p(l)}^{\perp} \to A_{p(l+1)} \parr \cdots \parr A_{p(n)}$, where $1 \le k, l < n$ and $p\colon \{1, \ldots, n\} \to \{1, \ldots, n\}$ is an arbitrary permutation. Obviously f determines in this way a whole family of maps, and we will call such a family an *equivariant family over* $A_1, \ldots, A_n$ [19, 12]. A member of such a family is called a *representative* and it determines the whole family. Given $A_1, \ldots, A_n$ and f as above we write $[\![f]\!]$ to denote the equivariant family determined by f. If we let $l = 0$ in the situation above, we get $\hat{f}\colon \mathbb{I} \dashrightarrow A_1 \parr \cdots \parr A_n$, that we call the *name of the equivariant family*. For $l = n$, we get its *coname* $\check{f}\colon A_1^{\perp} \otimes \cdots \otimes A_n^{\perp} \dashrightarrow \bot\!\!\!\bot$. Important examples are the name and the coname of the identity:

$$\mathbb{I} \overset{\hat{1}_A}{\dashrightarrow} A^{\perp} \parr A \qquad \text{and} \qquad A \otimes A^{\perp} \overset{\check{1}_A}{\dashrightarrow} \bot\!\!\!\bot$$

If we transpose the identity $1_{B \parr C}\colon B \parr C \to B \parr C$, we get the evaluation map $\mathsf{eval}\colon (B \parr C) \otimes C^{\perp} \to B$. Taking the tensor of this with $1_A\colon A \to A$ and transposing back gives us a map $\mathsf{switch}\colon A \otimes (B \parr C) \to (A \otimes B) \parr C$, that is natural in all three arguments, and that we call the *switch map* [8, 2]. For the sake of simplicity (and since we are working in the symmetric world), we will also use switches that are obtained by composing with the twistmap (for $\otimes$ as well as for $\parr$). In a similar way we obtain the maps $\mathsf{tens}\colon (A \parr B) \otimes (C \parr D) \to A \parr (B \otimes C) \parr D$ and $\mathsf{cotens}\colon A \otimes (B \parr C) \otimes D \to (A \otimes B) \parr (C \otimes D)$. Note that they are dual to each other and that they both can be obtained by composing two switches. Switch is self-dual.

Although the units are only "virtual", all the standard properties of *-autonomous categories can be proved in the standard way. For example, we have that the following commutes:

$$\begin{array}{ccc} A \otimes B \otimes (B^{\perp} \parr A^{\perp}) & & \\ \cong \big\downarrow & \overset{\check{1}_{A \otimes B}}{\searrow} & \\ B \otimes (B^{\perp} \parr A^{\perp}) \parr A & & \bot\!\!\!\bot \\ \mathsf{cotens}_{B, B^{\perp}, A^{\perp}, A} \big\downarrow & \overset{\check{1}_B \otimes \check{1}_A}{\nearrow} & \\ (B \otimes B^{\perp}) \parr (A \otimes A^{\perp}) & & \end{array} \qquad (3)$$

and the proof goes the same way as for example in [12].

A functor between autonomous or *-autonomous categories should preserve everything on the nose. But this cannot entirely achieved here because of the $h^{\mathbb{I}}$ functor.

2.1.4 Definition Let $\mathscr{C}$ and $\mathscr{D}$ be autonomous (*-autonomous) categories. we define an *autonomous functor* (**-autonomous functor*) from $\mathscr{C}$ to $\mathscr{D}$ to be a pair (F, α) where $F\colon \mathscr{C} \to \mathscr{D}$ is a functor that preserves $\otimes$ and $\multimap$ (and $(-)^{\perp}$) on the nose, and where α is a natural isomorphism $h^{\mathbb{I}}_{\mathscr{D}} \circ F \to h^{\mathbb{I}}_{\mathscr{C}}$.

So it should seem that the category of autonomous categories is slightly "looser" when real units are replaced by virtual units. But this makes no difference for us, as the free unitless *-autonomous categories we will construct will be free in the usual, strictest possible sense (as well as those with weak units, see just below).

[2] Most of the times we will reverses the order when taking the negation, but not always.

[3] Strictly speaking we should use different arrows shape to denote these virtual maps, because they deal with contravariant functors to $\mathbf{Set}$ and not covariant ones, and the two kinds cannot be mixed at all. But there is no risk of such a thing happening here, given the quite conservative use we make of this notation.

2.2. Weak units

2.2.1 Definition Let $\mathscr{C}$ be autonomous in the sense above. A *weak unit* in $\mathscr{C}$ is a pair $(\mathbf{I}, e)$ where $e \colon \mathbf{I} \to \mathbf{I}$ is an itempotent map such that splitting h^e in $\mathbf{Set}^{\mathscr{C}}$ gives $h^{\mathbb{I}}$:

$$h^{\mathbf{I}} \longrightarrow h^{\mathbb{I}} \longrightarrow h^{\mathbf{I}} \tag{4}$$

It is well-known that composing with an idempotent is a process of normalization. Let X, Y and $s \colon \mathbf{I} \to X \multimap Y$ be given. We can always normalize s by taking se, and we can say that s is in normal form if $s = se$. The definition above says that there is a natural bijective correspondence between the maps $X \to Y$ and the maps $\mathbf{I} \to X \multimap Y$ that are in normal form. For any X we can transform the virtual maps into real ones, in the following way:

$$\begin{array}{ccccc} \mathbf{I} \otimes X & \dashrightarrow & \mathbb{I} \otimes X & \dashrightarrow & \mathbf{I} \otimes X \\ & {\scriptstyle \ell_X} \searrow & \Big\downarrow {\scriptstyle \lambda_X} & \nearrow {\scriptstyle \ell^*_X} & \\ & & X & & \end{array}$$

thus getting two maps ℓ_X, ℓ^*_X with $\ell_X \ell^*_X = 1_X$ and $\ell^*_X \ell_X = e \otimes X$. These are obviously natural in X. The virtual map $\hat{\mathbf{I}} \colon \mathbb{I} \dashrightarrow \mathbf{I}$ induced by (4) is called the *canonical proof of the* $\mathbf{I}$. If $\mathbf{I}$ is a real unit, then the idempotent e is just the identity $1_{\mathbf{I}}$.

2.2.2 Definition An autonomous functor (*-autonomous functor) *preserves the weak unit* $(\mathbf{I}, e)$ if it preserves both the object and the idempotent.

Weak units can be used to give "elementary" axiomatization of the ideas of the previous section; we can even define the concept of a "weakly monoidal category", where the unit isomorphism would be replaced by an embedding-projection pair; it is easy to tweak the standard axioms for that purpose. But they are highly non-canonical: as soon as we have a weak unit we can construct many other weak units from it. Also, having weak units is the same as saying that splitting the idempotents in $\mathscr{C}$ [14] would give us an ordinary symmetrical monoidal closed category. But these are very semantical constructions and we work with syntax: if we discuss autonomous categories without units or with weak units, it is not only because we have constructions that obey these axioms, but in addition *that these constructions do not involve quotienting by equivalence relations* [19].

Notice that an autonomous category can have several weak units as well as a real one at the same time. What matters is which one is denoted by $\mathbf{I}$.

2.3. Going Classical

Let now $\mathscr{C}$ be *-autonomous. We will change the notation, and use $-\wedge-$ for the tensor and $-\vee-$ for the cotensor. The virtual unit and virtual bottom will be denoted by $\mathbb{t}$ and $\mathbb{f}$, called *virtual truth* and *virtual falsehood*, respectively. In case there are actual objects in the category playing the roles of the units (or weak units), they are denoted by $\mathbf{t}$ and $\mathbf{f}$, respectively. Notice that both, $- \wedge -$ and $- \vee -$, come with their own associativity and twist isos (see Definition 2.1.1); but we will in both cases simply write assoc and twist. The dual of an object A will be denoted $\bar{A}$.

Unsurprisingly, $\wedge$-comonoids and $\vee$-monoids are going to be important. But since we do not have real units for $\wedge, \vee$, we need to adapt the standard definitions of (co)monoid. In order to define the counit to a $\wedge$-comonoid X, which should be a map $X \to \mathbf{t}$ we (unsurprisingly) replace it by a natural transformation $\Pi^X \colon h^{\mathbb{t}} \to h^X$, which we call an X*-pre-projection*. Suppose $A \in \mathscr{C}$. We can construct

$$h^A \xrightarrow{\;\cong\;} h^{\mathbb{t}} H^A \xrightarrow{\;\Pi^X H^A\;} h^X H^A \xrightarrow{\;\cong\;} h^{X \wedge A} \quad ,$$

where the first iso comes from Definition 2.1.2 and the second iso is just (1). By Yoneda we get a map $\Pi^X_A \colon X \wedge A \to A$ which is natural in A, i.e., for $f \colon A \to B$, the diagram

$$\begin{array}{ccc} X \wedge A & \xrightarrow{\;\Pi^X_A\;} & A \\ {\scriptstyle X \wedge f} \Big\downarrow & & \Big\downarrow {\scriptstyle f} \\ X \wedge B & \xrightarrow[\;\Pi^X_B\;]{} & B \end{array} \tag{5}$$

commutes, and thus an X-pre-projection can be seen as natural transformation $\Pi^X \colon X \wedge (-) \to (-)$.

2.3.1 Definition A *cocommutative* $\wedge$*-comonoid* in $\mathscr{C}$ is a triple (X, Δ_X, Π^X) such that $\Delta_X \colon X \to X \wedge X$ is coassociative and cocommutative, i.e.,

$$\begin{aligned} (X \wedge \Delta_X) \circ \Delta_X &= \mathsf{assoc}_{X,X,X} \circ (\Delta_X \wedge X) \circ \Delta_X \\ \Delta_X &= \mathsf{twist}_{X,X} \circ \Delta_X \quad , \end{aligned} \tag{6}$$

and such that $\Pi^X \colon h^{\mathbb{t}} \to h^X$ obeys

$$\Pi^X_X \circ \Delta_X \;=\; 1_X \colon X \to X \; . \tag{7}$$

2.3.2 Definition A *pre-*K*-autonomous category* is a *-autonomous category $\mathscr{K}$, in which every object X is equipped with a cocommutative $\wedge$-comonoid structure (X, Δ_X, Π^X) such that for all A, B, X, and Y, we have

$$\begin{array}{ccc} & X \wedge Y & \\ {\scriptstyle \Delta_X \wedge \Delta_Y} \swarrow & & \searrow {\scriptstyle \Delta_{X \wedge Y}} \\ X \wedge X \wedge Y \wedge Y & \xrightarrow[\;X \wedge \mathsf{twist}_{X,Y} \wedge Y\;]{} & X \wedge Y \wedge X \wedge Y \end{array} \tag{8}$$

and

$$\Pi^X_A \wedge 1_B \;=\; \Pi^X_{A \wedge B} \colon X \wedge A \wedge B \to A \wedge B \; . \tag{9}$$

and such that *all isos preserve this* $\wedge$*-comonoid structure.*

We call Δ_X and Π^X the *diagonal* and *projection* on X. By duality we also have maps $\nabla_X : X \vee X \to X$, called *codiagonal*, and a natural transformation $\amalg^X : (-) \to (-) \vee X$, which we call the *coprojection*, and they give an associative, commutative $\vee$-monoid structure on X, in an obvious sense, slightly different from the standard definition, obeying the dual of equations (8) and (9).

A word on notation: we write $\Pi_A^{\Box X}$ for the map $A \wedge X \to A$ obtained by precomposing Π_A^X with the twistmap. In the same line of thought, $\Pi_A^{X\Box}$ is just Π_A^X, and more generally, an expression like $\Pi_{A,B}^{X\Box Y\Box Z}$ is the uniquely defined composite projection $X \wedge A \wedge Y \wedge B \wedge Z \to A \wedge B$. Uniqueness follows from the commutativity of

$$\begin{array}{ccc} A \wedge (X \wedge B) & \xrightarrow{\mathsf{assoc}} & (A \wedge X) \wedge B \\ & {\scriptstyle A \wedge \Pi_B^{X\Box}} \searrow \quad \swarrow {\scriptstyle \Pi_A^{\Box X} \wedge B} & \\ & A \wedge B & \end{array} , \qquad (10)$$

which is an immediate consequence of (9). By duality for every A, X there are $\amalg_A^{\Box X} : A \to A \vee X$ and $\amalg_A^{X\Box} : A \to X \vee A$ which are natural in A. We write $\amalg_A^X$ for $\amalg_A^{\Box X}$.

2.3.3 Definition In a pre-K-autonomous category a map $f : X \to Y$ is a *quasientropy* if it preserves the counits of $\wedge$-comonoids and the unit of $\vee$-monoids, that is:

$$\begin{array}{ccc} X \wedge A & \xrightarrow{f \wedge 1_A} & Y \wedge A \\ & {\scriptstyle \Pi_A^X} \searrow \quad \swarrow {\scriptstyle \Pi_A^Y} & \\ & A & \end{array} \quad \text{and} \quad \begin{array}{ccc} & A & \\ & {\scriptstyle \amalg_A^X} \swarrow \quad \searrow {\scriptstyle \amalg_A^Y} & \\ A \vee X & \xrightarrow[1_A \vee f]{} & A \vee Y \end{array}$$

both commute for every A. The map f is said to be *strong*, if in addition[4] it preserves the codiagonals as well as the diagonals, that is:

$$(f \wedge f) \circ \Delta_X = \Delta_Y \circ f \quad \text{and} \quad f \circ \nabla_X = \nabla_Y \circ (f \vee f) \;.$$

Thus a strong map between two objects is on that preserves the whole of the monoid and comonoid structures.

2.3.4 Definition

- A $\mathsf{K}^\flat$-*autonomous category* is a pre-K-autonomous category in which Δ, Π, and switch are quasientropies (and thus $\nabla, \amalg$ too), and quasientropies are closed under $\wedge$ and $\vee$. A $\mathsf{K}^\flat$-*autonomous functor* is a *-autonomous functor that preserves the obvious monoid and comonoid structures.

[4] There is a use for maps that preserve only the binary operations and not the 0-ary ones, but we will not need them here.

- It is a $\mathsf{K}^\natural$-*autonomous category* if the usual units are present and the comonoid structure on $\mathbf{t}$ is the standard degenerate one, obtained from the coherence isos. A $\mathsf{K}^\natural$-*autonomous functor* is a $\mathsf{K}^\flat$-autonomous functor that preserves the units in the usual sense.

- We speak of a $\mathsf{K}^\sharp$-*autonomous category* if the units are weak; we change the preceding condition with the requirement that $\ell_X = \Pi_X^{\mathbf{t}} : \mathbf{t} \wedge X \longrightarrow X$ and that

$$\begin{array}{ccc} \mathbb{t} & \xrightarrow{\hat{\mathbf{t}}} & \mathbf{t} \\ & {\scriptstyle \hat{\mathbf{t}} \wedge \hat{\mathbf{t}}} \searrow & \downarrow {\scriptstyle \Delta_{\mathbf{t}}} \\ & & \mathbf{t} \wedge \mathbf{t} \end{array} \qquad (11)$$

commutes, where $\hat{\mathbf{t}}$ is the *canonical proof of* $\mathbf{t}$ (see Section 2.2) , and the two conditions say that it is strong and a quasientropy. A $\mathsf{K}^\sharp$-*autonomous functor* is a $\mathsf{K}^\flat$-autonomous functor that preserves the weak units.

We simply say K-autonomous category and K-autonomous functor if the discussion is independent from the units.

In a K-autonomous category $\mathscr{K}$, the subcategory $\mathsf{Q}\mathscr{K}$ of quasientropies (with the same objects) inherits the two monoidal structures, switch, and also the involution. It is not *-autonomous in general, but it is weakly distributive [3].

Given two objects A and X, we define $\Lambda_A^X : A \wedge \bar{A} \to X$ by transposing $\amalg_A^{X\Box} : A \to X \vee A$, and $\mathrm{V}_A^X : X \to \bar{A} \vee A$ by transposing $\Pi_A^{\Box X} : A \wedge X \to A$.

2.3.5 Proposition *The transpose of* $\Pi_A^{\Box X} : A \wedge X \to A$ *is* $\amalg_A^{\bar{X}\Box} : A \to \bar{X} \vee A$.

Proof: The dual of $\Pi_A^{\Box X} : A \wedge X \to A$ is $\amalg_{\bar{A}}^{\bar{X}\Box} : \bar{A} \to \bar{X} \vee \bar{A}$ by definition, and it belongs to the same equivariant family. But we can transpose the latter to get $\mathrm{V}_A^X : X \to \bar{A} \vee A$ and, using symmetry of the latter definition, transpose again to get $\amalg_A^{\bar{X}\Box} : A \to \bar{X} \vee A$ (notice that V_A^X can be given

$$X \xrightarrow{\amalg} \mathbb{t} \xrightarrow{\hat{1}} \bar{A} \vee A$$

a virtual factorization) □

2.3.6 Proposition *Let* X, A, B *and* $f : A \to B$ *be given. Then*

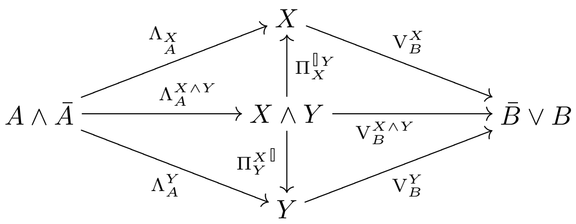
$$\begin{array}{ccc} X & \xrightarrow{\mathrm{V}_B^X} & \bar{B} \vee B \\ {\scriptstyle \mathrm{V}_A^X} \downarrow & & \downarrow {\scriptstyle \bar{f} \vee B} \\ \bar{A} \vee A & \xrightarrow[\bar{A} \vee f]{} & \bar{A} \vee B \end{array} \qquad (12)$$

commutes.

Proof: Transpose (5). □

2.3.7 Proposition *For any A, B, X, the map $\mathrm{V}_B^X \circ \Lambda_A^X : A \wedge \bar{A} \to \bar{B} \vee B$ is independent from X.*

Proof: Look at the following:



Taking their transposes, we see that the left triangles commute because projections are quasientropies, and the right triangles do because projections commute with projections. ◻

By doing a double transposition on $\mathrm{V}_A \circ \Lambda_B : B \wedge \bar{B} \to \bar{A} \wedge A$ we get the *mix map* $\mathsf{mix}_{A,B} : A \wedge B \to A \vee B$.

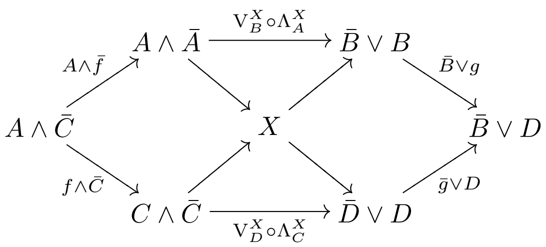
2.3.8 Proposition *The following is equal to* $\mathsf{mix}_{A,B}$

$$A \wedge B \xrightarrow{A \wedge \Pi_B^{X}} A \wedge (X \vee B) \xrightarrow{\mathsf{switch}} (A \wedge X) \vee B \xrightarrow{\Pi_A^{X} \vee B} A \vee B$$

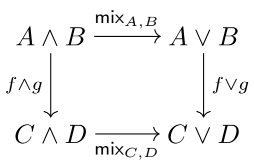
Proof: Transpose $\mathrm{V}_A^X \circ \Lambda_B^X$ twice and use the definition of switch. ◻

2.3.9 Proposition *The map $\mathsf{mix}_{A,B} : A \wedge B \to A \vee B$ is natural in A and B.*

Proof: This follows immediately from Proposition 2.3.8. But alternatively we could proceed as follows: Let $f : A \to C$ and $g : B \to D$, and look at

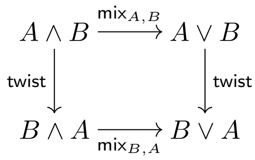


The two triangles commute by definition and the two quadrangles commute because of Proposition 2.3.6. The outer hexagon can be transposed to give



and that completes the proof. ◻

It is also very easy to see that mix agrees with the twistmap, i.e.



(13)

This gives us a unique map $f \between g : A \wedge B \to C \vee D$, which we call the *disjoint sum* of f and g. This operation is obviously stable under transposes:

2.3.10 Proposition *Le $f : A \wedge B \to C$ and $f' : A' \wedge B' \to C'$ be given, and let $g : B \to \bar{A} \vee C$ and $g' : B' \to \bar{A}' \vee C'$ be their transposes, respectively. Then $g \between g' : B \wedge B' \to A \vee C \vee A' \vee C'$ is the transpose of $f \between f' : A \wedge B \wedge A' \wedge B' \to C \vee C'$.*

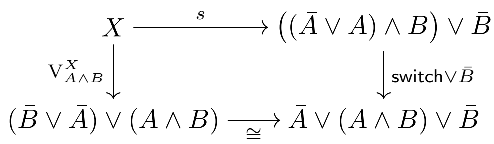
Proof: By uniqueness of transposes. ◻

We also have the following:

2.3.11 Proposition *In a* K*-autonomous category, the map $\mathsf{mix}_{A,B}$ is a quasientropy for every A and B.*

Proof: Use Proposition 2.3.8, together with Definition 2.3.4. ◻

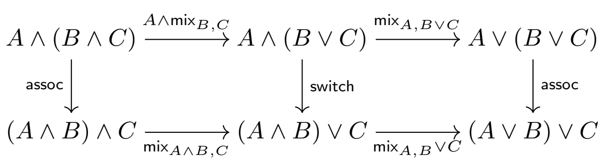
2.3.12 Lemma *Let X, A, B be given, and let $s : X \to \big((\bar{A} \vee A) \wedge B\big) \vee \bar{B}$ be the transpose of $\mathrm{V}_A^X \vee 1_B : X \wedge B \to (\bar{A} \vee A) \wedge B$. Then*



commutes.

Proof: Transpose (12) and use the definition of switch. ◻

2.3.13 Proposition *Given A, B, and C, then the following commutes:*



Proof: Apply Lemma 2.3.12 twice on each square and transpose. ◻

For those into things monoidal, this says that mix would furnish the necessary structure for identity to be a monoidal functor $(\mathscr{K}, \wedge) \to (\mathscr{K}, \vee)$—if we had units, naturally. A consequence of this is that there is a unique way to define a natural n-ary mix map

$$\mathsf{mix}_{A_1,\ldots,A_n} = 1_{A_1} \parr \ldots \parr 1_{A_n} : \quad A_1 \wedge \cdots \wedge A_n \longrightarrow A_1 \vee \cdots \vee A_n \ .$$

Let $f, g \colon A \to B$ be given. We define

$$f + g = \nabla_B \circ (f \parr g) \circ \Delta_A \colon A \to B \quad .$$

It is easy to show, using (co)-associativity and (co)-commutativity of Δ and ∇, along with naturality of mix, that the operation $+$ on maps is associative and commutative.[5] Thus every $\mathrm{Hom}_{\mathscr{C}}(A, B)$ has a commutative semigroup structure. In the view of Proposition 2.3.10 this semigroup structure is also present for $h^{\mathbb{tt}}(X)$. For $h, k \colon \mathbb{tt} \dashrightarrow X$ define $h + k = \nabla_X \circ (h \parr k) \colon \mathbb{tt} \dashrightarrow X$, where $h \parr k = \mathsf{mix}_{X,X} \circ (h \wedge k)$. It immediately follows that $\widehat{f+g} = \hat{f} + \hat{g} \colon \mathbb{tt} \dashrightarrow \bar{A} \vee B$, for every $f, g \colon A \to B$.

2.3.14 Proposition *Let $f, g \colon A \to B$ and $h, k \colon B \to C$. If h is strong, then*

$$h \circ (f + g) = hf + hg \ , \tag{14}$$

and if f is strong then

$$(h + k) \circ f = hf + kf \ . \tag{15}$$

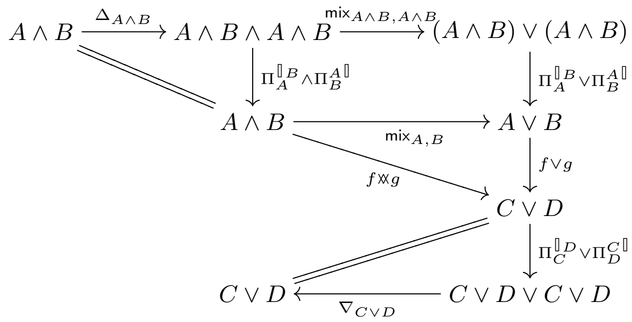
Proof: Immediately from the definitions. ☐

Note that it does *not* follow that (14) and (15) hold in general. In other words we do not have that K-autonomous category $\mathscr{K}$ is enriched over commutative semigroups.

But we can consider the subcategory $\mathsf{S}\mathscr{K}$ of all strong maps. It shares its objects with $\mathscr{K}$ since all isos (and therefore all identities) are strong. Clearly $\mathsf{S}\mathscr{K}$ has the semigroup enrichment.

2.3.15 Proposition *Let $f \colon A \to C$ and $g \colon B \to D$ be given. Then $f \parr g = (\Pi_C^{\square D} \circ f \circ \Pi_A^{\square B}) + (\Pi_D^{C\square} \circ g \circ \Pi_B^{A\square})$.*

Proof: Chase

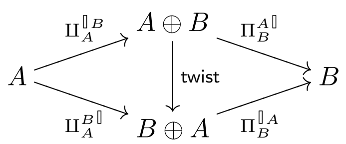


The leftmost path is $f \parr g$ and the rightmost path is $(\Pi_C^{\square D} \circ f \circ \Pi_A^{\square B}) + (\Pi_D^{C\square} \circ g \circ \Pi_B^{A\square})$. The square is naturality of mix and the triangles commute by definition. ☐

2.4. Going graphical

Let $\mathscr{K}$ be a K-autonomous category. We define $\mathscr{K}^{\oplus}$ to be the category obtained from $\mathscr{K}$ by formally inverting the mix maps. In other words, for every pair of objects A, B we add a map $\mathsf{mix}^{-1}_{A,B} \colon A \vee B \to A \wedge B$ such that $\mathsf{mix}_{A,B} \circ \mathsf{mix}^{-1}_{A,B} = 1_{A \vee B}$ and $\mathsf{mix}^{-1}_{A,B} \circ \mathsf{mix}_{A,B} = 1_{A \wedge B}$. Looking at the diagram in Proposition 2.3.13 we get a new diagram whose horizontal arrows now go in the reverse direction. This new diagram also commutes for trivial reason; thus it identifies the two associativities and switch. In the same way, the horizontal arrows in (13) can be inverted. The outcome of this is that not only are the bifunctors $\wedge$ and $\vee$ identified in $\mathscr{K}^{\oplus}$, but that this new bifunctor $\oplus$ inherits *a single* symmetric monoidal structure from its two parents: they are identified too. This gives us the right to write $f \oplus g$ for $f \wedge g$, $f \vee g$, as well as $f \parr g$.

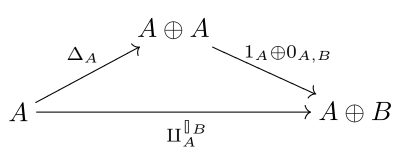
For trivial reasons the following diagram commutes:



. (16)

This uniquely determines a map $0_{A,B} \colon A \to B$, that we call the *zero map*.

2.4.1 Lemma *In $\mathscr{K}^{\oplus}$, the following commutes:*



. (17)

[5] In the theory of bialgebras and Hopf algebras this operations is traditionally called convolution [11]. This name suggests the use of both additive and multiplicative operations, and it may not apply here, since in the present work it is hard to tell what is additive and what is multiplicative (and it is simply a pointwise sum for our proof nets).

Proof: We have:

$$\begin{aligned}(A \oplus 0_{A,B}) \circ \Delta_A &= (A \oplus \Pi_B^{A\Box}) \circ (A \oplus \amalg_A^{\Box B}) \circ \Delta_A \\ &= (\Pi_A^{\Box A} \oplus B) \circ (A \oplus \amalg_A^{\Box B}) \circ \Delta_A \\ &= \amalg_A^{\Box B} \circ \Pi_A^{\Box A} \circ \Delta_A \\ &= \amalg_A^{\Box B} \circ 1_A \\ &= \amalg_A^{\Box B}\end{aligned}$$

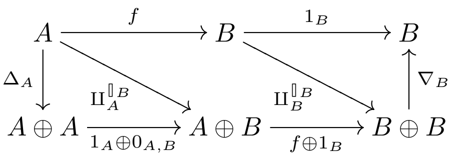
The first equation is the definition of $0_{A,B}$, the second one is (10), the third one is naturality of $\amalg^{\Box B}$, and the fourth one is (7). ☐

2.4.2 Proposition *The map $0_{A,B}$ is a quasientropy.*

Proof: Immediately from the definition of $0_{A,B}$. ☐

2.4.3 Proposition *Let $f\colon A \to B$. Then $f + 0_{A,B} = f$.*

Proof: Look at:

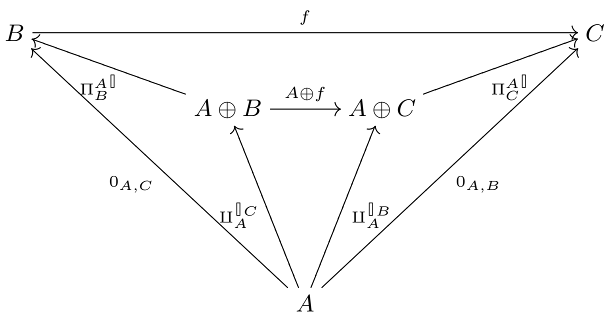


. (18)

The upper path is f and the lower path is $f + 0_{A,B}$. The left triangle is (17), the right triangle is (7), and the middle quadrangle is naturality of $\amalg^{\Box B}$. ☐

2.4.4 Proposition *For every quasientropy $f\colon B \to C$, we have $f \circ 0_{A,B} = 0_{A,C}$ and $0_{C,D} \circ f = 0_{B,D}$.*

Proof: The first equation is shown by:



The middle triangle commutes because f is a quasientropy. The second equation is similar. ☐

2.4.5 Remark The expression "zero map" has two meanings, and they usually coincide, but hot here. A zero map can be one which is a unit for the semigroup structure on hom-sets, as here. It can also mean a (family of) map(s) that "absorbs" every other map with which it is pre- or post-composed. This means, usually, that the category containts an object which is both terminal and initial; if no such object exists, just split the idempotents and one will appear. But here the second definition of zero map holds *only for the subcategory of quasientropies,* and not in general. We would like to emphasize that for us zero maps are just a means to state some important properties, and, unlike [4] we do not attribute them any logical meaning.

2.4.6 Theorem *In $\mathscr{K}^\oplus$ the diagram*

$$A \underset{\Pi_A^{\Box B}}{\overset{\amalg_A^{\Box B}}{\rightleftarrows}} A \oplus B \underset{\Pi_B^{A\Box}}{\overset{\amalg_B^{A\Box}}{\leftrightarrows}} B \qquad (19)$$

obeys the standard biproduct equations, i.e.,

$$\begin{aligned}1_{A\oplus B} &= \amalg_A^{\Box B}\Pi_A^{\Box B} + \amalg_B^{A\Box}\Pi_B^{A\Box} \\ 1_A &= \Pi_A^{\Box B}\,\amalg_A^{\Box B} \\ 1_B &= \Pi_B^{A\Box}\amalg_B^{A\Box}\end{aligned}$$

Proof: The first equation is a direct consequence of Proposition 2.3.15. The second one is shown by:

$$\begin{aligned}1_A &= \Pi_A^{\Box A} \circ \Delta_A \\ &= \Pi_A^{\Box B} \circ (A \oplus 0_{A,B}) \circ \Delta_A \\ &= \Pi_A^{\Box B} \circ \amalg_A^{\Box B}\end{aligned}$$

The first equation is (7), the second one uses that $0_{A,B}$ is a quasientropy, and the third one is Lemma 2.4.1. ☐

Notice that we do not have biproducts in $\mathscr{K}^\oplus$, i.e., the Πs do not form a product diagram and the $\amalg$s do not form a coproduct diagram in general.

2.4.7 Proposition *The transpose of $0_{A\oplus B,C}\colon A \oplus B \to C$ is $0_{A,\bar{B}\oplus C}\colon A \to \bar{B} \oplus C$.*

Proof: Use Proposition 2.3.5 and the definition of the zero map. ☐

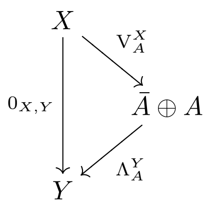
In the view of Proposition 2.4.7, we can make the following construction:

If we transpose $0_{A,B}$ and compose with the projection, we get a (virtual) map

$$\mathbb{tt} \overset{\hat{0}_{A,B}}{\dashrightarrow} \bar{A} \oplus B \xrightarrow{\Pi_B^{\Box \bar{A}}} B \quad , \qquad (20)$$

that we denote by 0_B. Clearly this is independent from A. By duality we get $B \dashrightarrow \mathbb{ff}$, which we also denote by 0_B.

2.4.8 Definition The category $\mathscr{K}^\oplus$ is said to be *pregraphical* if



(21)

commutes for all X, Y, and A.

We immediately have

2.4.9 Proposition *If $\mathscr{K}^{\oplus}$ is pre-graphical, then*

$$\begin{array}{ccc} \mathfrak{tt} & \overset{0_Y}{\dashrightarrow} & Y \\ & {\scriptstyle \hat{\imath}_A} \searrow \quad \nearrow {\scriptstyle \wedge^Y_A} & \\ & \bar{A} \oplus A & \end{array} \quad \textit{and} \quad \begin{array}{ccc} X & \overset{0_X}{\dashrightarrow} & \mathfrak{ff} \\ & {\scriptstyle \vee^X_A} \searrow \quad \nearrow {\scriptstyle \check{\imath}_A} & \\ & \bar{A} \oplus A & \end{array} \tag{22}$$

commute for all X, Y, and A.

Proof: Use (20). □

2.4.10 Definition Let $\mathscr{K}$ be a K-autonomous category. We say that $\mathscr{K}$ is *graphical* if $\mathscr{K}^{\oplus}$ is pre-graphical and the canonical functor $\mathsf{G}_{\mathscr{K}}\colon \mathscr{K} \to \mathscr{K}^{\oplus}$ is faithful. We say that $\mathscr{K}$ is *purely graphical* if in addition $\mathsf{G}_{\mathscr{K}}$ is full.

2.4.11 Definition A K-autonomous category $\mathscr{K}$ is called *loop-killing*, if

$$\begin{array}{ccc} \mathfrak{tt} & \overset{\hat{\imath}_A}{\dashrightarrow} & \bar{A} \vee A \\ & & \downarrow {\scriptstyle \Delta_{\bar{A}\vee A}} \\ {\scriptstyle \hat{\imath}_A} \vdots & & (\bar{A} \vee A) \wedge (\bar{A} \vee A) \\ & & \downarrow {\scriptstyle \mathsf{tens}} \\ \bar{A} \vee A & \underset{\bar{A}\vee\check{\imath}_A\vee A}{\longleftarrow} & \bar{A} \vee (A \wedge \bar{A}) \vee A \end{array} \tag{23}$$

commutes.

2.4.12 Proposition *A graphical* K*-autonomous category $\mathscr{K}$ is loop-killing, if and only if*

$$\begin{array}{ccc} A & \xrightarrow{A\oplus\hat{\imath}_{\bar{A}}} & A \oplus A \oplus \bar{A} \\ & & \downarrow {\scriptstyle \nabla_A \oplus \bar{A}} \\ {\scriptstyle 1_A} \downarrow & & A \oplus \bar{A} \\ & & \downarrow {\scriptstyle \Delta_A \oplus \bar{A}} \\ A & \underset{A\oplus\check{\imath}_A}{\longleftarrow} & A \oplus A \oplus \bar{A} \end{array} \tag{24}$$

commutes in $\mathscr{K}^{\oplus}$.

Proof: This follows immediately from the faithfulness of $\mathsf{G}_{\mathscr{K}}$ and transposing diagram (24) by using (8) together with the fact that Δ and ∇ are dual. □

Graphicality is quite a powerful property. One can easily show that in a graphical K-autonomous category, the maps Π, $\amalg$, and switch are strong, and that the strong maps are closed under $\wedge$ and $\vee$.[6] But note that it does not follow that Δ and ∇ are strong.[7] Full graphicality is an even more powerful property, since it enters the realm of degeneracy: it obviously identifies $\wedge, \vee$ in $\mathscr{K}$. But it is useful technically.

[6] In fact, for showing these facts, a much weaker property than graphicality (the presence of a *medial map* [2]) is sufficient. But since graphicality implies medial and is needed anyway, we do not deal with medial in this paper.

[7] We do not need this fact here and a proof of it would go beyond the scope of this paper.

2.4.13 Definition A K-autonomous category $\mathscr{K}$ is called Δ-∇*-strong* if Δ and ∇ are strong.

2.4.14 Remark In a graphical K-autonomous category which is Δ-∇-strong, the subcategory $\mathsf{S}\mathscr{K}$ of strong maps behaves quite nicely: not only is it weakly distributive, but in addition, since every object is equipped with both a monoid and comonoid structure *which is preserved by every map*, the category has binary products and coproducts, and the semigroup structure on the hom-sets of $\mathsf{S}\mathscr{K}$ is an enrichment in the usual sense. This works in reverse: the properties just stated suffice to show Δ-∇-strength and graphicality.

2.4.15 Theorem *In a graphical* K*-autonomous category which is loop-killing and Δ-∇-strong, we have that $1_A + 1_A = 1_A$.*

Proof: For the sake of simplicity, we show the statement for $\mathscr{K}^{\oplus}$. By graphicality it follows for $\mathscr{K}$.

$$\begin{aligned} 1_A &= (A \oplus \check{\imath}_A) \circ (\Delta_A \oplus \bar{A}) \circ (\nabla_A \oplus \bar{A}) \circ (A \oplus \hat{\imath}_{\bar{A}}) \\ &= (A \oplus \check{\imath}_A) \circ (\nabla_A \oplus \nabla_A \oplus \bar{A}) \\ &\quad \circ (A \oplus \mathsf{twist} \oplus A \oplus \bar{A}) \\ &\quad \circ (\Delta_A \oplus \Delta_A \oplus \bar{A}) \circ (A \oplus \hat{\imath}_{\bar{A}}) \\ &= (A \oplus \check{\imath}_A \oplus \check{\imath}_A) \circ (A \oplus A \oplus \mathsf{twist} \oplus \bar{A}) \\ &\quad \circ (\nabla_A \oplus A \oplus A \oplus \Delta_{\bar{A}}) \\ &\quad \circ (A \oplus \mathsf{twist} \oplus A \oplus \bar{A}) \\ &\quad \circ (\Delta_A \oplus A \oplus A \oplus \nabla_{\bar{A}}) \\ &\quad \circ (A \oplus A \oplus \mathsf{twist} \oplus \bar{A}) \circ (A \oplus \hat{\imath}_{\bar{A}} \oplus \hat{\imath}_{\bar{A}}) \\ &= (A \oplus \check{\imath}_A) \circ (\nabla_A \oplus A \oplus \bar{A}) \\ &\quad \circ (A \oplus \mathsf{twist} \oplus \bar{A}) \\ &\quad \circ (\Delta_A \oplus A \oplus \bar{A}) \circ (A \oplus \hat{\imath}_{\bar{A}}) \\ &= 1_A \circ \nabla_A \circ (1_A \oplus 1_A) \circ \Delta_A \circ 1_A \\ &= 1_A + 1_A \end{aligned}$$

The first equation is just (24). The second one is Δ-∇-strength together with (8). The third equation uses that Δ and ∇ are dual, together with (3) and its dual. The fourth equation uses again (24), and the fifth equation is a twisted form of $1_A = (\check{\imath}_A \vee A) \circ \mathsf{switch} \circ (A \wedge \hat{\imath}_A)$ which holds in every *-autonomous category. The last equation holds by definition. □

Note that this proof does not make any use of the projections nor the notion of quasientropy, i.e., is independent from the treatment of the units.

2.4.16 Corollary *In a graphical* $\mathsf{K}^{\sharp}$*-autonomous category $\mathscr{K}$ which is loop-killing and Δ-∇-strong, we have that $\hat{\mathbf{t}} + \hat{\mathbf{t}} = \hat{\mathbf{t}}$.*

Proof: We have in $\mathscr{K}^{\oplus}$:

$$\begin{aligned}\hat{\mathbf{t}} + \hat{\mathbf{t}} &= \nabla_{\mathbf{t}} \circ (\hat{\mathbf{t}} \oplus \hat{\mathbf{t}}) \\ &= \nabla_{\mathbf{t}} \circ \Delta_{\mathbf{t}} \circ \hat{\mathbf{t}} \\ &= \nabla_{\mathbf{t}} \circ (1_{\mathbf{t}} \oplus 1_{\mathbf{t}}) \circ \Delta_{\mathbf{t}} \circ \hat{\mathbf{t}} \\ &= 1_{\mathbf{t}} \circ \hat{\mathbf{t}} \\ &= \hat{\mathbf{t}}\end{aligned}$$

The first equation is the definition of $+$, the second equation is (11), the third one is trivial, the fourth one is Theorem 2.4.15, and the last one is trivial again. By graphicality, the equation holds in $\mathscr{K}$. □

2.4.17 Definition A K-autonomous category is *idempotent* if $f + f = f$ for every map f.

In such a category every hom-set is equipped with a semilattice structure, given by $f \leq g$ iff $f + g = g$ (see also [5]).

Note that Theorem 2.4.15 does *not* imply that a graphical, loop-killing, and Δ-∇-strong K-autonomous category is idempotent. However by an inductive argument, which is implicitely contained in the construction of the next section, one can show that the *free* graphical, loop-killing, and Δ-∇-strong K-autonomous category is idempotent.

For the time being, we can show for all objects A, B, C, that $\Delta_A + \Delta_A = \Delta_A$ and $\Pi^B_A + \Pi^B_A = \Pi^B_A$, as well as $\mathsf{switch}_{A,B,C} + \mathsf{switch}_{A,B,C} = \mathsf{switch}_{A,B,C}$, as the interested reader might verify.

3. Proof nets

We will recall the notion of proof nets that has been introduced in [13]. We consider only the case of $\mathbb{B}$-nets.

3.1. Cut-free prenets

For a given set $\mathscr{A} = \{a, b, c, \ldots\}$ of *propositional variables*, the set of $\mathsf{K}^{\sharp}$*-formulas* over $\mathscr{A}$ is generated from the set $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{t}, \mathbf{f}\}$ via the binary connectives $\wedge$ (*conjunction*) and $\vee$ (*disjunction*). Here $\bar{\mathscr{A}} = \{\bar{a}, \bar{b}, \bar{c}, \ldots\}$ is the set of *negated propositional variables*, and $\mathbf{t}$ and $\mathbf{f}$ are the *constants* representing "true" and "false", respectively. The elements of the set $\mathscr{A} \cup \bar{\mathscr{A}} \cup \{\mathbf{t}, \mathbf{f}\}$ are called *atoms*. The formulas in which the constants do not appear are called $\mathsf{K}^{\flat}$*-formulas*. A finite list of formulas $\Gamma = A_1, A_2, \ldots, A_n$ is called a *sequent*. We will consider formulas as binary trees (and sequents as forests), whose leaves are decorated by atoms, and whose inner nodes are decorated by the connectives. Given a formula A or a sequent Γ, we write $\mathscr{L}(A)$ or $\mathscr{L}(\Gamma)$, respectively, to denote its set of leaves. For simplicity, we will suppose, that this is actually the set $\{1, \ldots, n\}$ if there are n leaves. We can accomplish this by agreeing that for example if $\mathscr{L}(A) = \{1, \ldots, n\}$ and $\mathscr{L}(B) = \{1, \ldots, m\}$, then $\mathscr{L}(A \wedge B) = \{1, \ldots, n+m\}$ with $\mathscr{L}(A)$ and $\mathscr{L}(B)$ embedded as complementary subsets $\{1, \ldots, n\}$ and $\{n+1, \ldots, n+m\}$. We will write a_u to say that the leaf u is decorated by the atom a. If no ambiguity is possible, we will omit the index or the decoration, i.e., just write a or u for a_u.

We define the negation $\bar{A}$ of a formula A as follows:

$$\begin{array}{lll} \bar{\bar{a}} = a & \bar{\mathbf{t}} = \mathbf{f} & \overline{(A \wedge B)} = \bar{B} \vee \bar{A} \\ \bar{a} = \bar{a} & \bar{\mathbf{f}} = \mathbf{t} & \overline{(A \vee B)} = \bar{B} \wedge \bar{A} \end{array} \tag{25}$$

Here a ranges over the set $\mathscr{A}$, and there is a slight abuse of notation. However, from now on we will use a to denote an arbitrary atom (including constants), and $\bar{a}$ to denote its negation according to (25). Note that (25) implies $\bar{\bar{A}} = A$ for all A.

3.1.1 Definition A *linking* for a sequent Γ is an undirected graph P whose set of vertices is $\mathscr{L}(\Gamma)$ and whose set of edges obeys the following condition: whenever there is an edge between two leaves $u, v \in \mathscr{L}(\Gamma)$, denoted as $u \frown v$, then one of the following two cases holds:

- either, u is decorated by an atom a and v by its dual $\bar{a}$,
- or, $u = v$ and it is decorated by $\mathbf{t}$.

A *prenet*[8] consists of a sequent Γ and a linking P for it. It will be denoted by $P \triangleright \Gamma$ or

$$\begin{array}{c} P \\ \triangledown \\ \Gamma \end{array} .$$

Since no ambiguity is possible, we will identify a linking with its set of edges. Here is an example:

$$\begin{array}{c} \{ \mathbf{t}_1 \frown \mathbf{t}_1 , a_4 \frown \bar{a}_5 , \mathbf{t}_7 \frown \mathbf{t}_7 , \mathbf{t}_7 \frown \mathbf{f}_8 \} \\ \triangledown \\ \mathbf{t}_1 \vee (a_2 \wedge \mathbf{t}_3), (a_4 \vee (\bar{a}_5 \vee \mathbf{f}_6)) \wedge (\mathbf{t}_7 \vee \mathbf{f}_8) \end{array} \tag{26}$$

Recall that the indices simply represent the leaves (e.g., $\mathbf{t}_1$ stands for the leaf 1 which is decorated by $\mathbf{t}$). We can also think of the indices as distinguishing different atom occurrences, i.e., a_2 and a_4 are not different atoms but different occurrences of the same atom a. Here is another example:

$$\begin{array}{c} \{ \bar{b}_1 \frown b_5 , \bar{b}_1 \frown b_8 , \bar{b}_4 \frown b_5 , \bar{b}_4 \frown b_8 , a_2 \frown \bar{a}_3 , a_6 \frown \bar{a}_7 \} \\ \triangledown \\ \bar{b}_1 \wedge a_2, \bar{a}_3 \wedge \bar{b}_4, b_5 \wedge a_6, \bar{a}_7 \wedge b_8 \end{array} \tag{27}$$

Figure 1 shows several graphical presentations of (27): The left-most diagram shows it in the proof net tradition as as it

[8] What we call *prenet* is sometimes also called a *proof structure.*

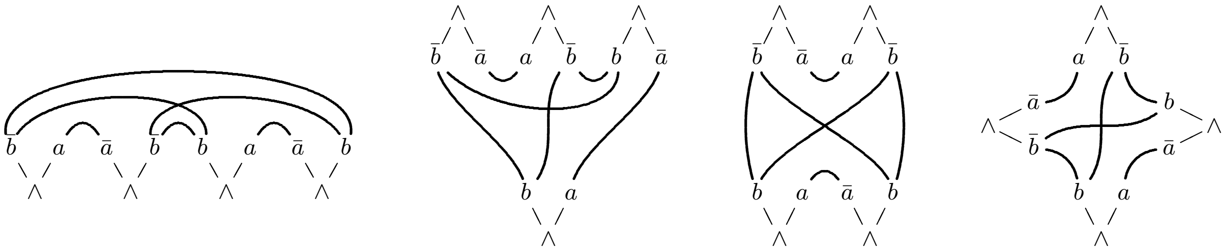


Figure 1. Several ways of drawing the same prenet

has been done in [13]. The two middle ones are "two-sided" versions. But we think that for the intuition it is better to think of it as someting like in the right-most diagram, where the formulas live in a circle around the linking graph.

On the set of prenets we define the following two operations: Let $P \rhd \Gamma$ and $Q \rhd \Gamma$ and $R \rhd \Theta$ be given. Then $(P + Q) \rhd \Gamma$ is obtained by taking the union of the two graphs P and Q (the set of vertices does not change), and $(P \oplus R) \rhd \Gamma, \Theta$ is obtained by taking the disjoint union of the two graphs (i.e., they are simply put next to each other).

Let $P \rhd \Gamma$ be a prenet and $L \subseteq \mathscr{L}(\Gamma)$ an arbitrary subset of leaves. Then $P|_L$ denotes the subgraph of P induced by L. We also have a subforest $\Gamma' = \Gamma|_L$ of Γ, whose set of leaves is precisely L and such that an inner node s of Γ is in $\Gamma|_L$ if *one or two* of its children is in $\Gamma|_L$. We will say that $P|_L \rhd \Gamma'$ is a *sub-prenet* of $P \rhd \Gamma$. Since this sub-prenet is entirely determined by Γ', we can also write it as $f|_{\Gamma'} \rhd \Gamma'$ without mentioning L any further.

3.2. Cuts and cut elimination

A *cut* is a formula of the shape $A \between \bar{A}$, where $\between$ is called the cut connective. It is allowed only at the place of the root of a formula tree. A *prenet with cuts* is a prenet $P \rhd \Gamma$, where Γ may contain cuts. On these, the cut reduction relation $\to$ is defined by

$$\begin{array}{rcl} P \rhd (A \wedge B) \between (\bar{B} \vee \bar{A}), \Gamma & \to & P \rhd A \between \bar{A}, B \between \bar{B}, \Gamma \\ P \rhd a_u \between \bar{a}_v, \Gamma & \to & (P|_\Gamma + Q) \rhd \Gamma \end{array}$$

where

$$\begin{array}{rcl} Q & = & \{\, i \frown j \mid i, j \in \mathscr{L}(\Gamma) \text{ and } i \frown u\,,\, v \frown j \in P \} \cup \\ & & \{\, i \frown i \mid i \in \mathscr{L}(\Gamma) \text{ and } i \frown u\,,\, v \frown v \in P \} \cup \\ & & \{\, j \frown j \mid j \in \mathscr{L}(\Gamma) \text{ and } u \frown u\,,\, v \frown j \in P \} \end{array}$$

3.2.1 Theorem *The cut reduction relation on prenets is confluent and terminating.*

Proof: See [13]. □

3.3. Prenet categories

An important consequence of Theorem 3.2.1 is that we can construct a category of prenets: The objects are the formulas and the arrows are the two-conclusion prenets. More precisely, any prenet $P \rhd \bar{A}, B$ is an arrow from A to B. The composition of two arrows $P \rhd \bar{A}, B$ and $Q \rhd \bar{B}, C$ is defined by eliminating the cut from $P \oplus Q \rhd \bar{A}, B \between \bar{B}, C$.

Furthermore, for each formula A we can define the *identity net* $I_A \rhd \bar{A}, A$ inductively as follows: For every atom a we have $\{\, \bar{a} \frown a \,\} \rhd \bar{a}, a$, and for compound formulas we have: $I_{A \wedge B} \rhd \bar{B} \vee \bar{A}, A \wedge B$ and $I_{A \vee B} \rhd \bar{B} \wedge \bar{A}, A \vee B$, where $I_{A \wedge B} = I_{A \vee B} = I_A \oplus I_B$. It is easy to see that the identity prenet behaves indeed as identity with respect to the composition defined in the previous section. This is enough to give us a category of prenets. We denote this category by $\mathbf{Pre}^\flat(\mathscr{A})$, respectively $\mathbf{Pre}^\sharp(\mathscr{A})$, if the objects are the $\mathsf{K}^\flat$-, respectively the $\mathsf{K}^\sharp$-formulas, generated from $\mathscr{A}$. $\mathbf{Pre}^\flat(\mathscr{A})$ is a full subcategory of $\mathbf{Pre}^\sharp(\mathscr{A})$.

3.3.1 Proposition *For every $\mathscr{A}$, the category $\mathbf{Pre}^\flat(\mathscr{A})$ is a $\mathsf{K}^\flat$-autonomous category, and $\mathbf{Pre}^\sharp(\mathscr{A})$ is a $\mathsf{K}^\sharp$-autonomous category.*

Proof: The maps assoc and twist are given by the obvious prenets. The functor $h^\natural$ is given by letting $h^\natural(A)$ be the set of all prenets $P \rhd A$; notice that this always contains the empty linking. The duality functor is defined on the objects as in (25), and on arrows by assigning to $P \rhd \bar{A}, B$ the net $P \rhd B, \bar{A}$. We obviously have the natural bijections $\mathrm{Hom}(A \wedge B, C) \cong \mathrm{Hom}(B, \bar{A} \vee C)$ and $h^\natural(\bar{A} \vee B) \cong \mathrm{Hom}(A, B)$, given by $P \rhd \bar{B} \vee \bar{A}, C \mapsto P \rhd \bar{B}, \bar{A} \vee C$ and $P \rhd \bar{A} \vee B \mapsto P \rhd \bar{A}, B$. For every formula A we have the diagonal net $\Delta_A \rhd \bar{A}, A \wedge A$, where $\Delta_a = \{\, \bar{a}_1 \frown a_2\,,\, \bar{a}_1 \frown a_3 \,\}$ and $\Delta_{A \wedge B} = \Delta_{A \vee B} = \Delta_A \oplus \Delta_B$. For every formula A (resp. sequent Γ) we can define the zero-linking which is just the graph without any edges. We will denote that with 0_A (resp. 0_Γ). For obtaining the projection net $\Pi^X_A \rhd \bar{X} \vee \bar{A}, A$, let $\Pi^X_A = 0_{\bar{X}} \oplus I_A$. It is easy to see that all the necessary properties and equations hold.

In the case of $\mathbf{Pre}^{\sharp}(\mathscr{A})$ (i.e, when $\mathbf{t},\mathbf{f}$ are present), let e be the linking $\{\,\mathbf{t}\frown\mathbf{t}\,\} \rhd \mathbf{f},\mathbf{t}$. It is easy to see that cutting e with itself yields e again; moreover, given any $P \rhd A$ then cutting e with $P \rhd \mathbf{f}, A$ will give us $P \rhd \mathbf{f}, A$ again; thus for any formula A the set $h^{\mathbb{t}}(A)$ is in bijective correspondence with the set of maps $s\colon \mathbf{t} \to A$ such that $se = s$. □

3.3.2 Proposition $\mathbf{Pre}^{\flat}(\mathscr{A})$ *and* $\mathbf{Pre}^{\sharp}(\mathscr{A})$ *are purely graphical, loop-killing, and* Δ-∇*-strong.*

Proof: In both categories $\wedge$ and $\vee$ are isomorphic. Note that in particular the prenet $0_{\bar{A},B} \rhd \bar{A}, B$ plays the role of the zero-map $0_{A,B}\colon A \to B$ in our categories.

Δ-∇-strength and the equations (21) and (23) can be shown by performing cut elimination. □

3.4. Prenets and equivariant families

Purely graphical K-autonomous categories are pretty absurd creatures, since they implement the same structure twice under the different names of $\wedge$ and $\vee$. But they are useful for us, because we are working in a situation where all the structure is preserved on the nose. If we took the viewpoint that two equivalent categories are "the same" category, then we could replace the categories $\mathbf{Pre}^{\flat}(\mathscr{A})$ and $\mathbf{Pre}^{\sharp}(\mathscr{A})$ by much simpler beasts, where objects would be sets M decorated with atoms, and a map $M \to N$ would be a binary relation on the disjoint sum $M + N$, and composition would be defined with our version of the Execution Formula [13, Section 5].

Let $\mathscr{K}$ be a purely graphical, loop-killing, and Δ-∇-strong K-autonomous category, and $G^{\circ}\colon \mathscr{A} \to \mathrm{Obj}(\mathscr{K})$ a map that chooses an object $a^{\bullet}$ of $\mathscr{K}$ for every atom $a \in \mathscr{A}$. It is obvious how to extend this map to every formula of the logic, since we want things to be preserved on the nose. We can now give a construction that assigns to every prenet $P \rhd \Gamma$ with $\Gamma = A_1, \ldots, A_n$ an equivariant family over $A_1, \ldots, A_n$ in $\mathscr{K}$, and this in a unique way. We will start with the cut-free case and then extend the construction to the prenets with cuts.

3.4.1 Definition Let $P \rhd \Gamma$ be given and let $u \in \mathscr{L}(\Gamma)$ and $\mathscr{S}(u) = \{v \in \mathscr{L}(\Gamma) \mid u \frown v \in P\}$. We call u *celibate* if $|\mathscr{S}(u)| = 0$, we say u is *monogamous* if $|\mathscr{S}(u)| = 1$, and *polygamous* if $|\mathscr{S}(u)| \geq 2$. The *size* of $P \rhd \Gamma$ is the sum of

- the number of $\wedge$-nodes and $\vee$-nodes in Γ,
- the number of polygamous and celibate leaves in Γ,
- the number of edges in P.

Note that the monogamous leaves are not counted.

3.4.2 Notation The associativity of Δ allows us to omit the parentheses in $X \to X \wedge X \wedge X$. More generally, for every $k > 0$ we can unambiguously define the map

$$\Delta^k_X\colon X \longrightarrow X^{\wedge k}$$

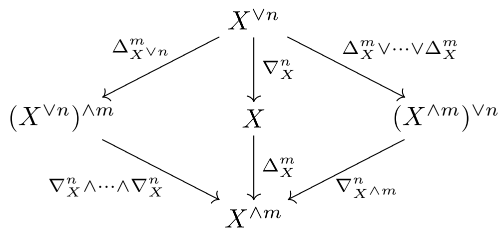
where $X^{\wedge k}$ is an abbreviation for

$$\underbrace{X \wedge X \wedge \cdots \wedge X}_{k} \quad ,$$

by $k-1$ applications of Δ_X. Obviously $\Delta^2_X = \Delta_X$ and $\Delta^1_X = 1_X$. Dually we define

$$X^{\vee k} = \underbrace{X \vee X \vee \cdots \vee X}_{k}$$

and get $\nabla^k_X\colon X^{\vee k} \longrightarrow X$. In a Δ-∇-strong K-autonomous category, we have that the following commutes



(28)

This can be shown by induction on $n + m$ and using Δ-∇-strength.[9]

3.4.3 Equivariant family construction (cut-free case) The unique family $[\![g]\!]$ that we are going to construct will be denoted by $[\![P \rhd \Gamma]\!]^{\bullet}$. We proceed by induction on the size of $P \rhd \Gamma$. We have the following cases:

0. If there are no edges in P, then $[\![P \rhd \Gamma]\!]^{\bullet}$ is a family all whose members are zero maps (see Proposition 2.4.7). It is easy to see that this is the only possible choice since the prenet without edges in P are the zero maps in our prenet category, and zero maps are uniquely defined by the categorical structure (compare with the proofs of Propositions 3.3.1 and 3.3.2).

1. If $P \rhd \Gamma$ is $\{\,\bar{a}\frown a\,\} \rhd a, \bar{a}$ for some atom a, then $[\![P \rhd \Gamma]\!]^{\bullet}$ is determined by the identity on a. This is obviously uniquely defined.

2. If it is $\{\,\mathbf{t}\frown\mathbf{t}\,\} \rhd \mathbf{t}$ (thus $\mathscr{K}$ is $\mathsf{K}^{\sharp}$-autonomous and there is $e\colon \mathbf{t} \to \mathbf{t}$ in $\mathscr{K}$ satisfying the requirement of 2.2.1), then purely equational considerations force $\{\,\mathbf{t}\frown\mathbf{t}\,\} = \hat{\mathbf{t}} \in h^{\mathbb{t}}(\mathbf{t})$ in the category of prenets to be mapped to $\hat{\mathbf{t}} \in h^{\mathbb{t}}(\mathbf{t})$ in $\mathscr{K}$ (see also Section 2.2 and Definition 2.3.4).

[9] Note that the outer quadrangle in (28) also commutes in a graphical K-autonomous category. But this is not the case for the inner two quadrangles, for which Δ-∇-strength is crucial.

3. If one of the root nodes in the prenet is a $\vee$, say $A_1 = B \vee C$, then $P \rhd B, C, A_2, \dots, A_n$ is of smaller size than π. By induction hypothesis we immediately have the representative $\bar{A}_2^\bullet \wedge \dots \wedge \bar{A}_n^\bullet \to B^\bullet \vee C^\bullet$ of $[\![P \rhd \Gamma]\!]^\bullet$. In case there is more than one $\vee$-root, then the result is obviously independent from the order in which the $\vee$s are attached.

4. If one of the roots in the prenet is a $\wedge$, say $A_1 = B \wedge C$, then we can proceed as in the previous case since $\wedge$ and $\vee$ are isomorphic (since $\mathscr{K}$ is purely graphical).

5. If the prenet can be split into two subnets, i.e., $P = P' \oplus P''$ with $P' \rhd A_1, \dots, A_j$ and $P'' \rhd A_{j+1}, \dots, A_n$, then by induction hypothesis we have two equivariant families $[\![P' \rhd A_1, \dots, A_j]\!]^\bullet$ and $[\![P'' \rhd A_{j+1}, \dots, A_n]\!]^\bullet$. Their disjoint sum yields $[\![P \rhd \Gamma]\!]^\bullet$. That this is well-defined follows from Proposition 2.3.10, and that it is independent from the order in which the subnets are put together (in case $P \rhd \Gamma$ splits into more than two parts) follows from the remark after Proposition 2.3.13.

6. If there is a formula in Γ, whose leaves are all celibate, without loss of generality, assume it is A_1, then we can look at $P|_{\Gamma'} \rhd \Gamma'$, where $\Gamma' = A_2, \dots, A_n$. This means that by induction hypothesis there is an equivariant family $[\![P|_{\Gamma'} \rhd \Gamma']\!]^\bullet$. We can compose an arbitrary member of it with $\amalg^{A_1^\bullet}$ to get a member of $[\![P \rhd \Gamma]\!]^\bullet$. That this is indeed independent from the choice of the member of $[\![P|_{\Gamma'} \rhd \Gamma']\!]^\bullet$ follows immediately from (9). That we get the same result if we first decompose A_1 according to cases 3 and 4 and then apply $\amalg$ follows from the fact that $\amalg$ is a quasientropy. Note: at present this case is redundant, and could be simply removed from the proof. But it becomes important when we deal with proof nets instead of prenets.

7. If one of the A_i is a polygamous atom, say $A_1 = a_1$, then let $k = |\mathscr{S}(a_1)|$. Say $\mathscr{S}(a_1) = \{v_1, \dots, v_k\}$, where each v_j is labelled by $\bar{a}$. Let Γ' be $a_1, \dots, a_k, A_2, \dots, A_n$ (i.e., it is Γ where a is replaced by k copies of a) and let $P' = P|_{A_2,\dots,A_n} + Q$ be the linking for Γ', where $Q = \{ a_j \frown v_j \mid j \in \{1, \dots, k\}\}$. Then $P' \rhd \Gamma'$ is of smaller size than $P \rhd \Gamma$. Hence, we can apply the induction hypothesis to get the equivariant family $[\![P' \rhd \Gamma']\!]^\bullet$ with member $\bar{A}_2^\bullet \wedge \dots \wedge \bar{A}_n^\bullet \to a^\bullet \vee \dots \vee a^\bullet$. If we compose this with $\nabla^k_{a^\bullet} : a^\bullet \vee \dots \vee a^\bullet \to a^\bullet$, we get the desired equivariant family $[\![P \rhd \Gamma]\!]^\bullet$.

3.4.4 Equivariant family construction (with cuts) We now extend this construction to prenets with cuts. Consider a prenet $P \rhd \Gamma$ with $\Gamma = A_1, \dots, A_n, B_1 \between \bar{B}_1, \dots, B_m \between \bar{B}_m$ (for some $n \geq 1, m \geq 0$), where $A_1, \dots, A_n$ are the formulas in Γ that are not cuts, and $B_1 \between \bar{B}_1, \dots, B_m \between \bar{B}_m$ are the cuts. We construct a uniquely defined equivariant family $[\![P \rhd \Gamma]\!]^\bullet$ of arrows over $A_1, \dots, A_n$. This is done by first applying our construction to the prenet $P \rhd \Gamma'$ with $\Gamma' = A_1, \dots, A_n, B_1 \wedge \bar{B}_1, \dots, B_m \wedge \bar{B}_m$, where all cuts are replaced by $\wedge$-formulas. This yields the equivariant family $[\![P \rhd \Gamma']\!]^\bullet$ with the representative

$$g \colon \bar{A}_1^\bullet \wedge \dots \wedge \bar{A}_n^\bullet \longrightarrow (B_1^\bullet \wedge \bar{B}_1^\bullet) \vee \dots \vee (B_m^\bullet \wedge \bar{B}_m^\bullet)$$

Look at $\check{1}_{B_j^\bullet} : B_j^\bullet \wedge \bar{B}_j^\bullet \dashrightarrow \text{ff}$, the coname of the identity for $B_j^\bullet$ in $\mathscr{K}$, which exists for every $B_j^\bullet$. By taking the $\vee$ of the family $(\check{1}_{B_j^\bullet})$, we construct (see Section 2.1)

$$h \colon (B_1^\bullet \wedge \bar{B}_1^\bullet) \vee \dots \vee (B_m^\bullet \wedge \bar{B}_m^\bullet) \dashrightarrow \text{ff}$$

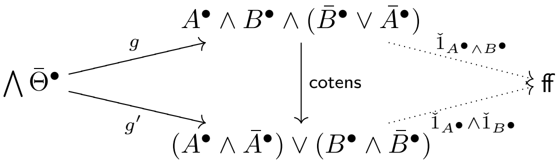
and by composition we get $\bar{A}_1^\bullet \wedge \dots \wedge \bar{A}_n^\bullet \dashrightarrow \text{ff}$, which represents $[\![P \rhd \Gamma]\!]^\bullet$ that we want to define.

The important fact about this construction is that it is preserved by cut elimination:

3.4.5 Lemma *Let $P \rhd \Gamma$ be a prenet, and $P' \rhd \Gamma'$ be the result of applying the cut elimination procedure to it. Then $[\![P \rhd \Gamma]\!]^\bullet$ and $[\![P' \rhd \Gamma']\!]^\bullet$ are same equivariant family in $\mathscr{K}$.*

Proof: Let π denote $P \rhd \Gamma$ and π' denote $P' \rhd \Gamma'$. It is sufficient to show the case where π' is obtained from π by a single cut elimination step. Then the lemma follows by induction on the length of the cut reduction. There are two cases:

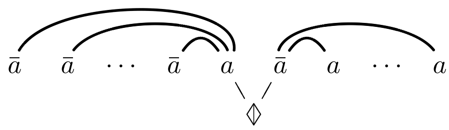
First, a compound cut has been reduced, i.e., $P' = P$ and $\Gamma = (A \wedge B) \between (\bar{B} \vee \bar{A}), \Theta$ and $\Gamma = A \between \bar{A}, B \between \bar{B}, \Theta$. Consider the following diagram in $\mathscr{K}$:



(29)

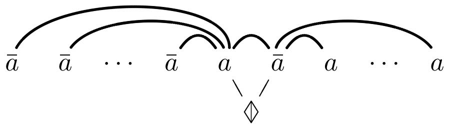
The upper path represents $[\![\pi]\!]^\bullet$ and the lower path represents $[\![\pi']\!]^\bullet$. The left triangle commutes because of graphicality (in $\mathscr{K}^\oplus$, cotens is an iso), and the right triangle commutes by (3).

The second case is where an atomic cut has been reduced. We will proceed by induction on the size of π. Consider first the case where π is connected and contains no binary connectives except for the cut to be reduced, and where all leaves not belonging to the cut are monogamous. Then π must be of the following shape (we first assume that a is not a constant)



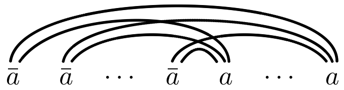
(30)

or



(31)

In both cases the reduct π' is:



(32)

It can easily be seen that a representative of $[\![\pi']\!]^\bullet$ is given by

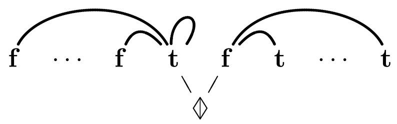
$$a^{\wedge m} \xrightarrow{\Delta_a^n \wedge \cdots \wedge \Delta_a^n} a^{\wedge mn} \xrightarrow{\mathsf{mix}} a^{\vee mn} \xrightarrow{\nabla_a^m \vee \cdots \vee \nabla_a^m} a^{\vee n}\ , \quad (33)$$

where m is the number of $\bar{a}$ in (32) and n the number of a. By applying (8) and (28), we can see that (33) is the same as

$$a^{\wedge m} \xrightarrow{\mathsf{mix}} a^{\vee m} \xrightarrow{\nabla_a^m} a \xrightarrow{\Delta_a^n} a^{\wedge n} \xrightarrow{\mathsf{mix}} a^{\vee n}\ . \quad (34)$$

It follows immediately from the construction in 3.4.3 and 3.4.4 that this is a representative of $[\![\pi]\!]^\bullet$ if π is (30). (See e.g., [12] for a more detailed treatment.) For showing that (34) also represents $[\![\pi]\!]^\bullet$ if π is (31), we use the loop-killing property of $\mathscr{K}$.

Let us now assume a is a constant, say $a = \mathbf{t}$. Then the situation is exactly the same, with the difference that there might be a "$\mathbf{t}$-loop":

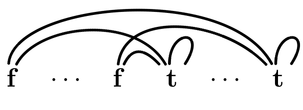


(35)

This represents the map

$$\mathbf{t}^{\wedge m} \xrightarrow{\mathbf{t}^{\wedge m} \wedge \hat{\mathbf{t}}} \mathbf{t}^{\wedge m'} \xrightarrow{\mathsf{mix}} \mathbf{t}^{\vee m'} \xrightarrow{\nabla_{\mathbf{t}}^{m'}} \mathbf{t} \xrightarrow{\Delta_{\mathbf{t}}^n} \mathbf{t}^{\wedge n} \xrightarrow{\mathsf{mix}} \mathbf{t}^{\vee n}\ . \quad (36)$$

where $m' = m + 1$. If we eliminate the cut from (35), we get



(37)

A representative of the equivariant family obtained from this is

$$\mathbf{t}^{\wedge m} \xrightarrow{\Delta_{\mathbf{t}}^n \wedge \cdots \wedge \Delta_{\mathbf{t}}^n} \mathbf{t}^{\wedge mn} \xrightarrow{\mathbf{t}^{\wedge mn} \wedge \hat{\mathbf{t}} \wedge \cdots \wedge \hat{\mathbf{t}}} \mathbf{t}^{\wedge m'n} \xrightarrow{\mathsf{mix}} \mathbf{t}^{\vee m'n} \xrightarrow{\nabla_{\mathbf{t}}^{m'} \vee \cdots \vee \nabla_{\mathbf{t}}^{m'}} \mathbf{t}^{\vee n}\ . \quad (38)$$

As before we have that (36) and (38) are the same, this time also using (11).

Let us now consider the general case in which the size of π is larger. Since the cut is atomic, any step in 3.4.3 which can be applied to π without touching the cut can also be applied to π'. Therefore we can proceed by induction. □

An immediate consequence of the equivariant-family-construction is

3.4.6 Theorem $\mathbf{Pre}^\flat(\mathscr{A})$, *resp.* $\mathbf{Pre}^\sharp(\mathscr{A})$, *is the free purely graphical, loop-killing, and* Δ-∇*-strong* $\mathsf{K}^\flat$*-autonomous category, resp.* $\mathsf{K}^\sharp$*-autonomous category, generated from* $\mathscr{A}$.

Proof: We have to show that there is a unique K-autonomous functor $G\colon \mathbf{Pre}(\mathscr{A}) \to \mathscr{K}$ with $G^\circ = \mathrm{Obj}(G) \circ \eta_{\mathscr{A}}$, where $\mathrm{Obj}(G)$ is the restriction of G on objects and $\eta_{\mathscr{A}}\colon \mathscr{A} \to \mathrm{Obj}(\mathbf{Pre}(\mathscr{A}))$ maps every $a \in \mathscr{A}$ to itself, seen as formula. We let $G(A) = A^\bullet$ for objects, and given a map $f\colon A \to B$ in $\mathbf{Pre}(\mathscr{A})$, i.e., a prenet $P \rhd \bar{A}, B$, we let $G(f)\colon A^\bullet \to B^\bullet$ be the corresponding member of $[\![P \rhd \bar{A}, B]\!]^\bullet$. That this is indeed a functor (i.e., preserves identies and composition) follows from Lemma 3.4.5. By the construction of the equivariant families it now follows that is is a $\mathsf{K}^\flat$-autonomous functor, respectively a $\mathsf{K}^\sharp$-autonomous functor. □

3.5. From prenets to proof nets

In this section we will consider those prenets, that come from actual proofs—the proof nets.

3.5.1 Definition A *conjunctive resolution* of a prenet $P \rhd \Gamma$ is a sub-prenet $P|_{\Gamma'} \rhd \Gamma'$ where Γ' has been obtained by deleting one child subformula for every conjunction node and every cut node of Γ (i.e., in $P|_{\Gamma'} \rhd \Gamma'$ every $\wedge$-node and every $\between$-node is unary).

3.5.2 Definition A prenet $P \rhd \Gamma$ is said to be *correct* if for every one of its conjunctive resolutions $P|_{\Gamma'} \rhd \Gamma'$ the graph $P|_{\Gamma'}$ has at least one edge. A *proof net* is a correct prenet.

The examples in (27) and in the proof of Lemma 3.4.5 are proof nets.

3.5.3 Theorem *The cut reduction relation $\to$ preserves correctness [13].*

Proof: There are two cases to consider: First, a compound cut is reduced. Let $P \rhd (A \wedge B) \between (\bar{B} \vee \bar{A}), \Gamma$ be correct. Then, by definition, $P|_{A,\Gamma} \rhd A, \Gamma$ and $P|_{B,\Gamma} \rhd B, \Gamma$ and $P|_{\bar{B},\bar{A},\Gamma} \rhd \bar{B}, \bar{A}, \Gamma$ are also correct. Hence, also $P|_{A,B,\Gamma} \rhd A, B, \Gamma$ and $P|_{A,\bar{B},\Gamma} \rhd A, \bar{B}, \Gamma$ and $P|_{\bar{A},B,\Gamma} \rhd \bar{A}, B, \Gamma$ are correct. Therefore, we immediately get that $P \rhd A \between \bar{A}, B \between \bar{B}, \Gamma$ is correct. The second case to consider is the reduction of an atomic cut. Assume that $P \rhd a_u \between \bar{a}_v, \Gamma$ is correct and the reduct $P' \rhd \Gamma$ is not, i.e., we have a conjunctive resolution Γ' where $P'|_{\Gamma'}$ has no edges. This means that $P|_{\Gamma'}$ also has no edges. By correctness of the first net, we must have that $P|_{a_u,\Gamma'}$ as well as $P|_{\bar{a}_v,\Gamma'}$ cannot be edge-free. There are now three possibilities. Either we have $i, j \in \mathscr{L}(\Gamma')$ with $\overset{\frown}{i\ a_u}, \overset{\frown}{\bar{a}_v\ j} \in P$, or we have $h \in \mathscr{L}(\Gamma')$ with $\overset{\frown}{a_u\ a_u}, \overset{\frown}{\bar{a}_v\ h} \in P$ (i.e., a must be $\mathbf{t}$), or we have $h \in \mathscr{L}(\Gamma')$ with $\overset{\frown}{\bar{a}_v\ \bar{a}_v}, \overset{\frown}{a_u\ h} \in P$ (i.e., $a = \mathbf{f}$). But then, in the first case we have $\overset{\frown}{i\ j} \in P'$, and in the other two cases $\overset{\frown}{h\ h} \in P'$. Contradiction. □

Observe that the identity nets, as well as the nets defining Δ, Π, assoc, twist, and switch are all correct. The only net that is not correct is the one representing mix^{-1}. Therefore we immediately have that also the two conclusion proof nets form a graphical K-autonomous category, which is Δ-∇-strong and loop-killing. But it is no longer purely graphical. We call this category $\mathbf{Net}^\flat(\mathscr{A})$, resp. $\mathbf{Net}^\sharp(\mathscr{A})$. It is a subcategory of $\mathbf{Pre}^\flat(\mathscr{A})$, resp. $\mathbf{Pre}^\sharp(\mathscr{A})$ that shares the same objects. We now have:

3.5.4 Theorem $\mathbf{Net}^\flat(\mathscr{A})$*, resp.* $\mathbf{Net}^\sharp(\mathscr{A})$*, is the free graphical, loop-killing, and* Δ-∇*-strong* $\mathsf{K}^\flat$*-autonomous category, resp.* $\mathsf{K}^\sharp$*-autonomous category, generated from* $\mathscr{A}$.

Proof: The proof is almost the same as for prenets. The only thing that we have to show is that the construction of the equivariant families can be done on proof nets without using mix^{-1}. Let us inspect the cases in 3.4.3. Most importantly, case 0 no longer exists. It is easy to see, that cases 1 and 2 can be used without change. Furthermore, cases 3, 6 and 7 remain valid (case 6 now being necessary) because the net to which the induction hypothesis is applied is correct, provided the $P \rhd \Gamma$ from which we started was already correct. Hence, only cases 4 and 5 are problematic. We modify them as follows. Let $\mathscr{K}$ be an arbitrary graphical, loop-killing, and Δ-∇-strong K-autonomous category, and let $P \rhd \Gamma$ be correct. Everything else is as in Section 3.4:

4. If one of the roots in the net is a $\wedge$, say $A_1 = B \wedge C$, then let $\Theta = A_2, \ldots, A_n$ and $P_1 = P|_{B,\Theta}$ and $P_2 = P|_{C,\Theta}$ and $P_3 = P|_{B,C,\Theta}$. If $P \rhd \Gamma$ is correct, then all three nets $P_1 \rhd B, \Theta$ and $P_2 \rhd C, \Theta$ and $P_3 \rhd B, C, \Theta$ are also correct (and of smaller size than $P \rhd \Gamma$). Hence, by induction hypothesis we have a unique equivariant family in $\mathscr{K}$ for each of them, with representatives $g_1\colon \bigwedge \bar{\Theta}^\bullet \to B^\bullet$ and $g_2\colon \bigwedge \bar{\Theta}^\bullet \to C^\bullet$ and $g_3\colon \bigwedge \bar{\Theta}^\bullet \to C^\bullet \vee B^\bullet$, respectively. From these we construct the map g by
$$\begin{array}{rcl} \bigwedge \bar{\Theta}^\bullet & \xrightarrow{\Delta^3_{\bigwedge \bar{\Theta}^\bullet}} & \bigwedge \bar{\Theta}^\bullet \wedge \bigwedge \bar{\Theta}^\bullet \wedge \bigwedge \bar{\Theta}^\bullet \\ & \xrightarrow{g_1 \wedge g_3 \wedge g_2} & B^\bullet \wedge (C^\bullet \vee B^\bullet) \wedge C^\bullet \\ & \xrightarrow{\mathsf{cotens}} & (B^\bullet \wedge C^\bullet) \vee (B^\bullet \wedge C^\bullet) \\ & \xrightarrow{\nabla_{B^\bullet \wedge C^\bullet}} & B^\bullet \wedge C^\bullet \;. \end{array} \tag{39}$$
We let $[\![P \rhd \Gamma]\!]^\bullet = [\![g]\!]$.
5. We apply that case only if the net can be split into two subnets (i.e., $P = P' \oplus P''$ such that $P' \rhd A_1, \ldots, A_j$ and $P'' \rhd A_{j+1}, \ldots, A_n$ for some $1 \leq j < n$) which are *both correct*. Then we can apply the induction hypothesis and proceed as in 3.4.3.

It follows from graphicality (and Theorem 2.4.15), that this construction yields the same map as the construction in 3.4.3 (but here it only works if $P \rhd \Gamma$ is correct). By Theorem 3.5.3 and Lemma 3.4.5 it follows that the constructed $G\colon \mathbf{Net}(\mathscr{A}) \to \mathscr{K}$ is indeed a functor. □

4. Conclusions and Future Work

There are not enough examples yet for anybody to be able to give a definitive answer the question "what is a Boolean category?". The final axiomatization will be the product of a succession of refinements. But we believe we have made a significant progress in that quest: the axioms for a K-autonomous category are general and easy to verify; they should inspire new semantics. The conditions of graphicality and Δ-∇-strength build a bridge for denotational semantics and the Geometry of Interaction; they also show that the world is very big and that our category of proof-nets is still at the degenerate end of the spectrum. From Theorem 2.4.15 we learned that things like W-weighted nets [13] have limitations if we want to construct Boolean categories that are not idempotent. In the near future, we intend to work on

- finding Boolean categories that are not idempotent.
- incorporating Hyland's recent work [10] in that framework.
- the study of the Kleisli categories associated with comonoids of the form (X, Δ_X, Π_X). As Lambek has pointed out a long time ago, this corresponds to theories that are no longer pure, but where X has been added as an axiom. We can now try to relate the complexity of X to the structure of that category, and ask questions like "when does such a category of have cut-elimination?".
- extension to first-order logic.

References

[1] R. Blute. Linear logic, coherence and dinaturality. *Theoretical Computer Science*, 115:3–41, 1993.

[2] K. Brünnler and A. F. Tiu. A local system for classical logic. In R. Nieuwenhuis and A. Voronkov, editors, *LPAR 2001*, volume 2250 of *Lecture Notes in Artificial Intelligence*, pages 347–361. Springer-Verlag, 2001.

[3] J. Cockett and R. Seely. Weakly distributive categories. *Journal of Pure and Applied Algebra*, 114:133–173, 1997.

[4] K. Došen and Z. Petrić. *Proof-Theoretical Coherence*. KCL Publications, London, 2004.

[5] C. Führmann and D. Pym. On the geometry of interaction for classical logic (extended abstract). In *19th IEEE Symposium on Logic in Computer Science (LICS 2004)*, pages 211–220, 2004.

[6] C. Führmann and D. Pym. Order-enriched categorical models of the classical sequent calculus. 2004.

[7] J.-Y. Girard. A new constructive logic: Classical logic. *Mathematical Structures in Computer Science*, 1:255–296, 1991.

[8] A. Guglielmi. A system of interaction and structure, 2002. To appear in *ACM Transactions on Computational Logic*. On the web at: http://www.ki.inf.tu-dresden.de/~guglielm/Research/Gug/Gug.pdf.

[9] D. Hughes and R. van Glabbeek. Proof nets for unit-free multiplicative-additive linear logic. In *18th IEEE Symposium on Logic in Computer Science (LICS 2003)*, 2003.

[10] J. M. E. Hyland. Abstract interpretation of proofs: Classical propositional calculus. In J. Marcinkowski and A. Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *LNCS*, pages 6–21. Springer-Verlag, 2004.

[11] C. Kassel. *Quantum Groups*. Graduate Texts in Mathematics. Springer-Verlag, 1995.

[12] F. Lamarche and L. Straßburger. From proof nets to the free *-autonomous category, 2004. Submitted. Available on the web at http://ps.uni-sb.de/~lutz/papers/freestaraut.pdf.

[13] F. Lamarche and L. Straßburger. Naming proofs in classical propositional logic, 2005. To appear in *Proceedings of TLCA'05*. Available on the web at http://ps.uni-sb.de/~lutz/papers/namingproofsCL.pdf.

[14] J. Lambek and P. J. Scott. *Introduction to higher order categorical logic*, volume 7 of *Cambridge studies in advanced mathematics*. Cambridge University Press, 1986.

[15] S. Mac Lane. Natural associativity and commutativity. *Rice University Studies*, 49:28–46, 1963.

[16] S. Mac Lane. *Categories for the Working Mathematician*. Number 5 in Graduate Texts in Mathematics. Springer-Verlag, 1971.

[17] M. Parigot. $\lambda\mu$-calculus: An algorithmic interpretation of classical natural deduction. In *Logic Programming and Automated Reasoning, LPAR 1992*, volume 624 of *LNAI*, pages 190–201. Springer-Verlag, 1992.

[18] P. Selinger. Control categories and duality: on the categorical semantics of the lambda-mu calculus. *Mathematical Structures in Computer Science*, 11:207–260, 2001.

[19] L. Straßburger and F. Lamarche. On proof nets for multiplicative linear logic with units. In J. Marcinkowski and A. Tarlecki, editors, *Computer Science Logic, CSL 2004*, volume 3210 of *LNCS*, pages 145–159. Springer-Verlag, 2004.

[20] T. Streicher and B. Reus. Classical logic, continuation semantics and abstract machines. *Journal of Functional Programming*, 8(6):543–572, 1998.