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The Logic BV and Quantum Causality

Rick Blute
University of Ottawa
http://aix1.uottawa.ca/~rblute/

Prakash Panangaden
McGill University
http://www.cs.mcgill.ca/~prakash/

Lutz Straßburger
INRIA Saclay–Île-de-France
http://www.lix.polytechnique.fr/~lutz/

Abstract

We describe how a logic with commutative and non-commutative connectives can be used for capturing the essence of discrete quantum causal propagation.

1 Causal graphs and locative slices

In this note we describe how the kinematics of quantum causal evolution can be captured by the logic BV [2]. The setting is discrete quantum mechanics. We imagine a finite "web" of spacetime points. They are viewed as vertices in a directed acyclic graph (DAG); the edges of the DAG represent causal links mediated by the propagation of matter [1]. The fact that the graph is acyclic captures a basic causality requirement: there are no closed causal trajectories. The DAG represents a discrete approximation to the spacetime on which a quantum system evolves. The graph is technically a dangling graph; there is a set of half edges—in addition to the ordinary edges—divided into two disjoint subsets: the incoming edges and the outgoing edges. An incoming edge has no initial point but has a terminal point, and dually for outgoing edges.

A *pre-causal graph* $\mathcal{G}$ consists of a quadruple $(\mathcal{V}_\mathcal{G}, \mathcal{B}_\mathcal{G}, \mathcal{I}_\mathcal{G}, \mathcal{O}_\mathcal{G})$, where $\mathcal{V}_\mathcal{G}$ is a set of *vertices*, $\mathcal{B}_\mathcal{G}$ is a set of directed *(binary) edges*, $\mathcal{I}_\mathcal{G}$ is a set of *incoming edges*, and $\mathcal{O}_\mathcal{G}$ is a set of *outgoing edges*, such that $\mathcal{V}_\mathcal{G}$, $\mathcal{B}_\mathcal{G}$, $\mathcal{I}_\mathcal{G}$, and $\mathcal{O}_\mathcal{G}$ are pairwise disjoint (and finite), and two functions $\mathsf{source}_\mathcal{G}\colon \mathcal{B}_\mathcal{G} \cup \mathcal{O}_\mathcal{G} \to \mathcal{V}_\mathcal{G}$ and $\mathsf{target}_\mathcal{G}\colon \mathcal{B}_\mathcal{G} \cup \mathcal{I}_\mathcal{G} \to \mathcal{V}_\mathcal{G}$, called *source* and *target*, respectively. The elements of the set $\mathcal{E}_\mathcal{G} = \mathcal{B}_\mathcal{G} \cup \mathcal{I}_\mathcal{G} \cup \mathcal{O}_\mathcal{G}$ are called *edges*. On this set we define the *precedence relation* $\prec_\mathcal{G} \subseteq \mathcal{E}_\mathcal{G} \times \mathcal{E}_\mathcal{G}$ as $e_1 \prec e_2$ iff $\mathsf{target}(e_1) = \mathsf{source}(e_2)$ The *ancestor relation* $<_\mathcal{G}$ is the transitive closure of $\prec_\mathcal{G}$. If $<_\mathcal{G}$ is irreflexive we say that $\mathcal{G}$ is a *causal graph*. A *slice* $\mathcal{S}$ in a causal graph $\mathcal{G}$ is an anti-chain in $\leq_\mathcal{G}$. A slice does not need to be maximal.

To each edge in a causal graph we can associate a Hilbert space $\mathcal{H}$ and a density matrix ρ associated with the subsystem on that edge. At each vertex we imagine that we have an interaction, which may be any one of the following: subsystems come together, a subsystem breaks into pieces, a subsystem is subject to a unitary transformation, a subsystem is subject to a measurement, or a subsystem is partly discarded. When subsystems come together we form the tensor product of their state spaces. If they have no interaction we form the tensor product of their density matrices, otherwise we have a unitary operator acting on the combined density matrices. When a system breaks apart we can have a single density matrix for all the pieces; if, however, we wish to separate the density matrices of the individual components we compute partial traces, which has the effect of removing information about nonlocal correlations. In fact, the presence of non-local correlations is what distinguishes this from Petri nets.

Density matrices can be associated with any slice. If we keep all the data associated with maximal slices then we cannot guarantee that information does not propagate between acausal paths. One solution to guaranteeing causal propagation is to only propagate along the individual edges. In this scheme we would only allow the operators (completely positive maps) at the vertices to act on density matrices associated with single edges. This would indeed guarantee causal propagation but would kill all nonlocal correlations. The solution to the problem of ensuring causal evolution while preserving important non-local correlations is to work with *locative slices*, which are defined below.

Let $\mathcal{G}$ be be a causal graph, and let $\mathcal{S} \subseteq \mathcal{E}$ and let $v \in \mathcal{V}$ be a vertex such that $\mathsf{target}^{-1}(v) \subseteq \mathcal{S}$. Then the set $\mathcal{S}' = \mathcal{S} \setminus \mathsf{target}^{-1}(v) \cup \mathsf{source}^{-1}(v)$ is called the *propagation of* $\mathcal{S}$ *through* v. Clearly, if $\mathcal{S}$ is a slice, then the propagation of $\mathcal{S}$ through v is also a slice. We say a slice $\mathcal{S}'$ is *reachable* from a slice $\mathcal{S}$ if there are an $n \geq 0$ and slices $\mathcal{S}_0, \ldots, \mathcal{S}_n \subseteq \mathcal{E}$ and vertices $v_1, \ldots, v_n \in V$ such that for all $i \in \{1, \ldots, n\}$ we have that $\mathcal{S}_i$ is the propagation of $\mathcal{S}_{i-1}$ through v_i, and $\mathcal{S} = \mathcal{S}_0$ and $\mathcal{S}' = \mathcal{S}_n$. A slice $\mathcal{S}$ in a causal graph $\mathcal{G}$ is called *locative*, if it is reachable from a slice $\mathcal{I}' \subseteq \mathcal{I}$.

The point is that if $\mathcal{S}$ is locative then its density matrix can be computed without ever comuting partial traces: no information is lost.

2 A Logic for Causal Propagation

After having discussed the "physics" of discrete quantum causal propagation, we will no describe a logic capturing the essence of the concept of "locative" slice. The basic idea is to have a propositional logic where the edge of the causal graph are represented by atoms. The vertices are represented by axioms and the locative slices correspond to derivable formulas. The approach taken in [1] was to use as key unit of deduction a sequent $a_1, \ldots, a_k \vdash b_1, \ldots, b_l$ meaning that the slice $\{b_1, \ldots, b_l\}$ is reachable from $\{a_1, \ldots, a_k\}$. However, this approach was not able to entirely capture the notion of locative slices, because correlations develop dynamically as the system evolves, or equivalently, as the deduction proceeds. The solution taken in [1] was to let axioms evolve dynamically.

The deep reason behind the problems of [1] was that the underlying logic was multiplicative linear logic (MLL): The sequent above represents the formula $a_1 \otimes \cdots \otimes a_k \multimap b_1 \parr \cdots \parr b_l$ or eqvivalently $a_1^\perp \parr \cdots \parr a_k^\perp \parr b_1 \parr \cdots \parr b_l$, i.e., the logic is not able see the aspect of *time* in the causality. For this reason we propose to use the logic BV, which is essentially MLL (with mix) enhanced by a third binary connective $\triangleleft$ (called *seq* or *before*) which is associative and non-commutative and self-dual, i.e., the negation of $A \triangleleft B$ is $A^\perp \triangleleft B^\perp$. It is this non-commutative connective, which allows us to properly capture quantum causality.

A vertex $v \in \mathcal{V}_\mathcal{G}$ in a causal graph $\mathcal{G}$ is now encoded by the formula $V = (a_1^\perp \otimes \cdots \otimes a_k^\perp) \triangleleft [b_1 \parr \cdots \parr b_l]$, where $\{a_1, \ldots, a_k\} = \mathsf{target}^{-1}(v)$ is the set of edges having their target in v, and $\{b_1, \ldots, b_l\} = \mathsf{source}^{-1}(v)$ is the set of edges having their source in v. For a slice $\mathcal{S} = \{e_1, \ldots, e_n\} \subseteq \mathcal{E}_\mathcal{G}$ we define its encoding to be the formula $S = e_1 \parr \cdots \parr e_n$.

Let us now define the rules of the game. Since there is no sequent system for BV [5], we give a system in the calculus of structures, using deep inference. The inference rules are:

$$\mathsf{ai}\!\downarrow \frac{F\{\circ\}}{F\{a \parr a^\perp\}} \qquad \mathsf{s}\frac{F\{A \otimes [B \parr C]\}}{F\{(A \otimes B) \parr C\}} \qquad \mathsf{ai}\!\uparrow \frac{F\{a \otimes a^\perp\}}{F\{\circ\}}$$

$$\mathsf{q}\!\downarrow \frac{F\{[A \parr C] \triangleleft [B \parr D]\}}{F\{\langle A \triangleleft B\rangle \parr \langle C \triangleleft D\rangle\}} \qquad \mathsf{q}\!\uparrow \frac{F\{\langle A \triangleleft B\rangle \otimes \langle C \triangleleft D\rangle\}}{F\{(A \otimes C) \triangleleft (B \otimes D)\}}$$

They have to be read as ordinary rewrite rules acting on the formulas inside arbitrary contexts $F\{\ \}$. Note that we push negation via DeMorgan equalities to the atoms, and thus, all contexts are positive. The letters A, B, C, D stand for arbitrary formulas and a is an arbitrary atom. The rewriting is done modulo the associativity of all three connectives $\parr$, $\triangleleft$, and $\otimes$, the commutativity of the two connectives $\parr$ and $\otimes$, and the unit laws for $\circ$, which is unit to all three connectives, i.e., $A = A \parr \circ = A \otimes \circ = A \triangleleft \circ = \circ \triangleleft A$.

The set of rules $\{\mathsf{ai}\!\downarrow, \mathsf{ai}\!\uparrow, \mathsf{s}, \mathsf{q}\!\downarrow, \mathsf{q}\!\uparrow\}$ is called SBV, and the set $\{\mathsf{ai}\!\downarrow, \mathsf{s}, \mathsf{q}\!\downarrow\}$ is called BV [2]. We write $\begin{array}{c}P\\ \Big\| \,\mathsf{SBV}\\ Q\end{array}$ to denote a derivation from P to Q using SBV. The system BV obeys cut elimination, i.e., the general cut rule (the rule ai↑ without the restriction to atoms) is admissible for BV. As a corollary, we have the following:

2.1 Theorem $\begin{array}{c}A\\ \Big\| \,\mathsf{SBV}\\ B\end{array}$ *if and only if* $\begin{array}{c}\circ\\ \Big\| \,\mathsf{BV}\\ A^\perp \parr B\end{array}$.

Let us now come back to causal graphs. Recall the encoding of vertices and slices described above. We can now formulate our main result:

2.2 Theorem *Let $\mathcal{G}$ be a causal graph. A subset $\mathcal{S} \subseteq \mathcal{E}_\mathcal{G}$ is a locative slice if and only if there is a derivation*

$$\begin{array}{c}I \otimes V_1 \otimes \ldots \otimes V_n\\ \Big\| \,\mathsf{SBV}\\ S\end{array},$$

where S is the encoding of $\mathcal{S}$, and I is the encoding of a subset of $\mathcal{I}_\mathcal{G}$, and $V_1, \ldots, V_n$ encode vertices $v_1, \ldots, v_n \in \mathcal{V}_\mathcal{G}$.

The "only if" direction is easy. We simply simulate in SBV the propagation of slices in the DAG. The "if" direction is a bit tricky. We make crucial use of Theorem 2.1, and the following property of BV:

2.3 Proposition *For any pairwise distinct atoms $a_1, a_2, \ldots, a_n$ the formula*

$$\langle a_1 \triangleleft a_2^\perp\rangle \parr \langle a_2 \triangleleft a_3^\perp\rangle \parr \cdots \parr \langle a_n \triangleleft a_1^\perp\rangle$$

is not provable in BV.

For more details on this property, the reader is referred to [4]. In [3] an extension of BV by the exponentials ! and ? of linear logic is proposed, which can be used to provide a more general version of Theorem 2.2.

References

[1] R. Blute, I. Ivanov, and P. Panangaden. Discrete quantum causal dynamics. *International Journal of Theoretical Physics*, 42:2025–2041, 2003.

[2] A. Guglielmi. A system of interaction and structure. *ACM Transactions on Computational Logic*, 8(1), 2007.

[3] A. Guglielmi and L. Straßburger. A non-commutative extension of MELL. In M. Baaz and A. Voronkov, editors, *Logic for Programming, Artificial Intelligence, and Reasoning, LPAR 2002*, volume 2514 of *LNAI*, pages 231–246. Springer-Verlag, 2002.

[4] L. Straßburger. *Linear Logic and Noncommutativity in the Calculus of Structures*. PhD thesis, Technische Universität Dresden, 2003.

[5] A. F. Tiu. A system of interaction and structure II: The need for deep inference. *Logical Methods in Computer Science*, 2(2):1–24, 2006.