Section 4

Systematics

Types of MP

Continuous variables:

- LP (linear functions)
- QP (quadratic obj. over affine sets)
- QCP (linear obj. over quadratically def'd sets)
- QCQP (quadr. obj. over quadr. sets)
- cNLP (convex sets, convex obj. fun.)
- SOCP (LP over 2nd ord. cone)
- SDP (LP over PSD cone)
- COP (LP over copositive cone)
- NLP (nonlinear functions)

Types of MP

Mixed-integer variables:

- IP (integer programming), MIP (mixed-integer programming)
- *extensions:* MILP, MIQ, MIQCP, MIQCQP, cMINLP, MINLP
- BLP (LP over $\{0,1\}^n$)
- BQP (QP over $\{0,1\}^n$)

More "exotic" classes:

- MOP (multiple objective functions)
- BLevP (optimization constraints)
- SIP (semi-infinite programming)

Section 5

Linear Programming

Generalities

- *Simplex method*
  - practically fast
  - exploration of polyhedron vertices
  - exponential-time in the worst-case
  - average complexity: polynomial
  - smoothed complexity: polynomial
- *Ellipsoid method*
  - (weakly) polytime
  - mostly used for theoretical purposes
- *Interior-point method* (IPM)
  - practically fast
  - follows a central path
  - (weakly) polytime
  - can be used for many convex MPs, nost just linear

Distribution of oil

An oil distribution company needs to ship a large quantity of crude from the main port to the refining plant, which is unfortunately far from the port, using their pipe networks over the country.

Model the problem of determining the maximum quantity of oil they can hope to ship.

[Hint: what are the decision variables? (etc.)]

Subsection 1

Maximum flow

Network flows

Given a digraph $G = (V, A)$ with an *arc capacity* function $c : A \to \mathbb{R}_+$ and two distinct nodes $s, t \in V$, find the flow from s to t having maximum value

- Given $G = (V, A, c, s, t)$ a *flow* from s to t is a function $f : A \to \mathbb{R}_+$ s.t.:

$$\forall v \in V \smallsetminus \{s,t\} \qquad \sum_{u \in N^-(v)} f_{uv} = \sum_{w \in N^+(v)} f_{vw}$$

$$\forall (u,v) \in A \quad f_{uv} \leq c_{uv}$$

- The *value* of a flow f is given by $\sum_{v \in N^+(s)} f_{sv}$

Defn.: $N^-(v) = \{u \in V \mid (u,v) \in A\}, N^+(v) = \{w \in V \mid (v,w) \in A\}$

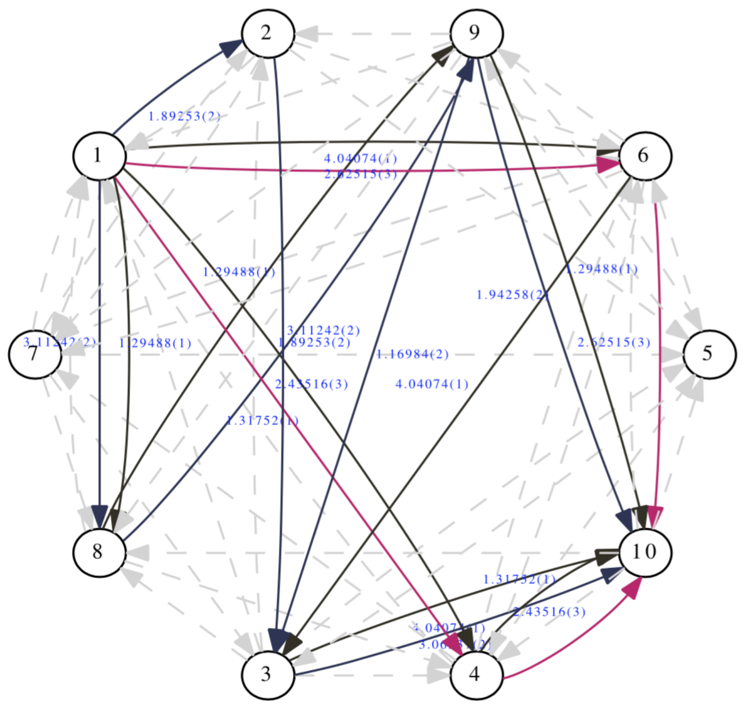
The MAX FLOW problem

$$\left.\begin{array}{rccc} \max & \sum\limits_{v\in N^+(s)} f_{sv} & & \\ \forall v \in V \smallsetminus \{s,t\} & \sum\limits_{u\in N^-(v)} f_{uv} & = & \sum\limits_{w\in N^+(v)} f_{vw} \\ \forall (u,v) \in A & f_{uv} & \in & [0, c_{ij}] \end{array}\right\}$$

- **Constraint matrix is *totally unimodular***
  ⇒ optima have integer components
- **Dual of MAX FLOW is MIN CUT**
  ⇒ optimal value $= 0$ iff network disconnected
- *for these two important results, see MAP557*

Multicommodity flow

- *Many different flows on the same network*
- **Given $N = (V, A, c, s, t)$ where:**
  - $G = (V, A)$ **is a digraph**
  - $c : A \to \mathbb{R}_+$ **is an arc capacity function**
  - $s, t \in V^r$ **s.t.**$\forall k \leq r \; (s_k \neq t_k)$
- **Find a set of flows $\{f^k \mid k \leq r\}$ from s_k to t_k**
  - **having max. total value**
  - **satisfying arc capacity**



LP Formulation

- **Maximize total value:**

$$\max \sum_{k\le r} \sum_{v\in N^+(s_k)} f^k_{s_k v}$$

- **Satisfy flow equations:**

$$\forall k \le r, v \in V \smallsetminus \{s_k, t_k\} \qquad \sum_{u\in N^-(v)} f^k_{uv} = \sum_{w\in N^+(v)} f^k_{vw}$$

- **Satisfy arc capacity:**

$$\forall (u,v) \in A \quad \sum_{k\le r} f^k_{uv} \le c_{uv}$$

- **They are bounded:**

$$\forall k \le r, (u,v) \in A \quad f^k_{uv} \in [0, c_{uv}]$$

Minimum cost flows

- **Flow equations define connected subgraphs:**

  G connected $\Rightarrow \forall u \neq v \in V(G)$ *a unit of flow entering* u *will exit* u *as long as "demand"*$= 0$ *at intermediate nodes. Conversely: if there is a flow from* u *to* v *then* G *must be connected*

- **E.g. a SP $s \to t$ is the connected subgraph of minimum cost containing s, t:**

$$\left.\begin{array}{rrcl}
\min\limits_{x:A\to\mathbb{R}} & \sum\limits_{(u,v)\in A} c_{uv}x_{uv} & & \\
\forall v \in V & \sum\limits_{(u,v)\in A} x_{uv} - \sum\limits_{(v,u)\in A} x_{vu} & = & \left\{\begin{array}{rl} -1 & u = s \\ 1 & u = t \\ 0 & \textbf{othw.} \end{array}\right. \\
\forall (u,v) \in A & x_{uv} & \in & [0,1]
\end{array}\right\} [\text{SP}]$$

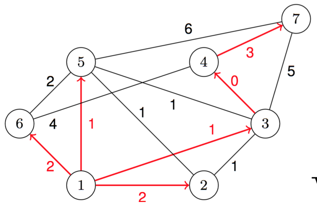
Test this with AMPL

Flattening the formulation

- Every MP involving linear forms only can be written in the form

$$\left.\begin{array}{rcl} \min_x & \gamma^\top x & \\ & Ax & \leq & \beta \\ & x & \in & X \end{array}\right\} [P]$$

- $\gamma, x \in \mathbb{R}^n$, $\beta \in \mathbb{R}^m$, A is $m \times n$, X is the set where variables range

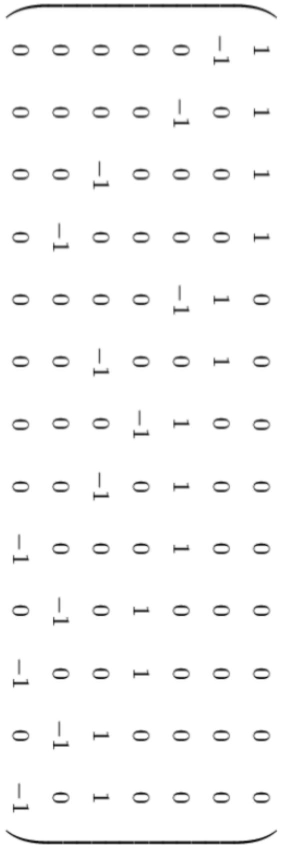
- For P2PSP on  with $s = 1$ and $t = 7$ we have:
  - $\gamma = (2, 1, 1, 2, 1, 1, 0, 1, 5, 4, 3, 2, 6)$,
    $\beta = (1, 0, 0, 0, 0, 0, -1)$, $X = [0, 1]^{13}$
  - $A =$

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

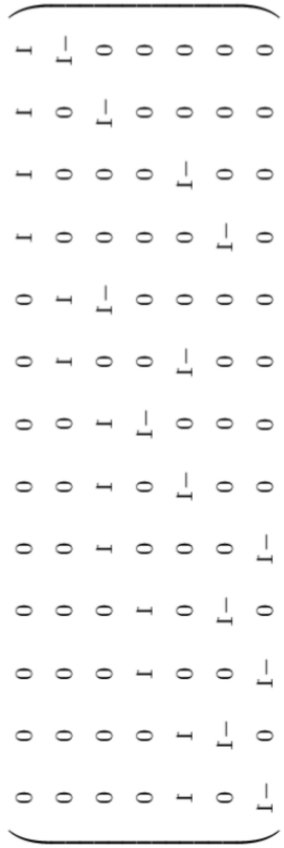
Transpose

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & -1 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & -1 & 0 & -1 \end{pmatrix}$$

(turn) ⟶



(reflect) ⟶



A dual view

- **Let** $A^\top = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & -1 \end{pmatrix}$
- **Turn rows into columns (constraints into variables)**
- **…and columns into rows (variables into constraints)**

LP Dual

- For each constraint define a variable y_i ($i \leq 7$)
- The LP dual is

$$\left.\begin{array}{rrcl} \max_y & -y\beta & & \\ & yA & \leq & \gamma \end{array}\right\} [D]$$

- In the case of the SP formulation, the dual is:

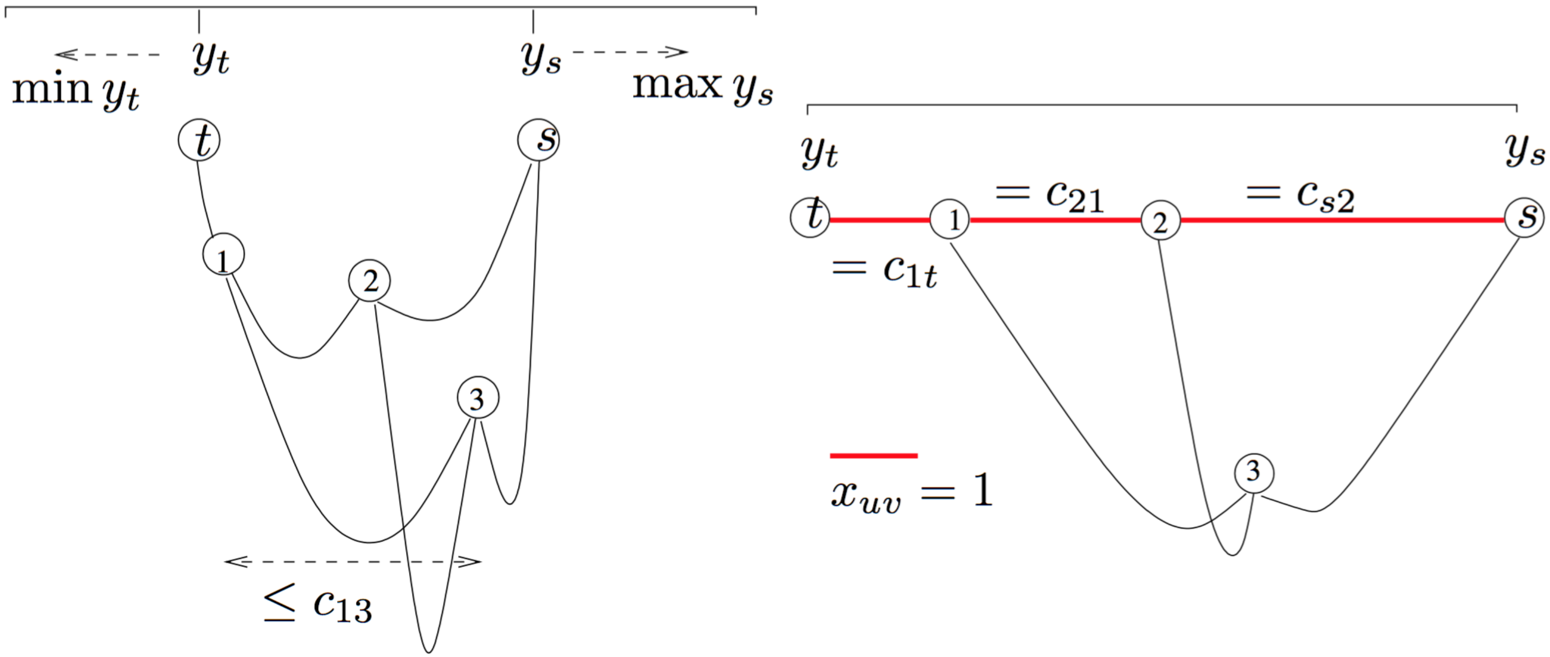
$$\left.\begin{array}{rrcl} \max_y & y_t - y_s & & \\ \forall (u,v) \in A & y_v - y_u & \leq & c_{uv} \end{array}\right\} [D_{\mathbf{SP}}]$$

- For the P2PSP formulation, dual gives same optimal value as the “primal” (*test with AMPL*)

How the hell is this an SP formulation?

A mechanical algorithm

- Weighted arcs = strings as long as the weights
- Nodes = knots
- Pull nodes s, t as far as you can
- At maximum pull, strings corresponding to arcs (u, v) in SP have horizontal projections whose length is exactly c_{uv}



Telecom

An internet provider used historical data to estimate a traffic matrix $T = (T_{ij})$, such that T_{ij} is the typical demand between two nodes i, j of its network digraph $G = (V, A)$. It has a contract with the backbone provider that limits the capacity (in Gbs) on each arc $(i, j) \in A$ to c_{ij}; the same contract also regulates the cost per Gbs, set to γ_{ij}

Model the problem of finding the feasible multiflow of minimum cost that satisfies each demand between source and destination.

Logistics

A truck-based transportation company needs to plan the routes for the incoming week. The demands are given as a list $((s_k, t_k, d_k) \mid k \leq r)$ *where* d_k *trucks have to be dispatched from node* s_k *to node* t_k*. The capacities* c_{uv} *on the arcs* $(u, v) \in A$ *are estimated using traffic data, and the operations cost are estimated to 100$ per Km.*

1. *Model the problem, assuming the company has enough trucks to cover every demand*
2. *Adjust the problem to the situation where the company has sufficient trucks to satisfy half of the total demand, and has to rent the others: the operations costs for the rented trucks are 200$ per Km.*
3. *Suggest a way to efficiently compute a lower bound on the total cost.*

Air courier

The air branch of a shipping company uses a fleet of Boeing 777s and 747s cargo to serve the EMEA demands. A 777 can carry 653 m^3 in volume and 103 tonnes (t) in weight. A 747 can carry 854.5 m^3 and 134.2 t. Each freighter is dedicated to a single segment (origin to destination airport and back once a day: both flights happen within the same 24 hours). The demand matrix is extremely fine-grained, and consists of all order IDs (packages) for the week, with origin and destination airports, weight and volume. The networks consists of airports, linked by the segments that are actually flown. The per-mile cost of flying is a linearly increasing function of the loaded weight (the two functions are different for 777 and 747). Flights can leave empty (in which case the company subcontracts the flight); company policy states that, if loaded, the loaded volume has to fill at least half the capacity. Model the corresponding variant of the multicommodity flow problem.