INF421, Lecture 7
Sorting

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Course

- **Objective**: teach notions AND develop intelligence
- **Evaluation**: TP noté en salle info, Contrôle à la fin. Note: $\max(CC, \frac{3}{4}CC + \frac{1}{4}TP)$
- **Organization**: fri 31/8, 7/9, 14/9, 21/9, 28/9, 5/10, 12/10, 19/10, 26/10, amphi 1030-12 (Arago), TD 1330-1530, 1545-1745 (SI:30-34)
- **Books**:
  1. K. Mehlhorn & P. Sanders, *Algorithms and Data Structures*, Springer, 2008
  2. D. Knuth, *The Art of Computer Programming*, Addison-Wesley, 1997
  3. G. Dowek, *Les principes des langages de programmation*, Editions de l'X, 2008
  4. Ph. Baptiste & L. Maranget, *Programmation et Algorithmique*, Ecole Polytechnique (Polycopié), 2006
- **Website**: `www.enseignement.polytechnique.fr/informatique/INF421`
- **Blog**: `inf421.wordpress.com`
- **Contact**: `liberti@lix.polytechnique.fr` (e-mail subject: INF421)

Lecture summary

- Sorting complexity in general
- Mergesort
- Quicksort
- Two-way partition

The minimal knowledge

- $\texttt{mergeSort}(s_1, \ldots, s_n)$

  $m = \lfloor \frac{n}{2} \rfloor$;
  $s' = \texttt{mergeSort}(s_1, \ldots, s_m)$;
  $s'' = \texttt{mergeSort}(s_{m+1}, \ldots, s_n)$;
  merge s', s'' such that result $\bar{s}$ is sorted;
  return $\bar{s}$;

  Split in half, recurse on shorter subsequences, then do some work to reassemble them

- $\texttt{quickSort}(s_1, \ldots, s_n)$

  choose a $k \leq n$;
  $s' = (s_i \mid i \neq k \wedge s_i < s_k)$;
  $s'' = (s_i \mid i \neq k \wedge s_i \geq s_k)$;
  return $(\texttt{quickSort}(s'), s_k, \texttt{quickSort}(s''))$;

  Choose a value s_k, split s.t. left subseq. has values $< s_k$, right subseq. has values $\geq s_k$, recurse on subseq.

- $\texttt{twoWaySort}(s_1, \ldots, s_n) \in \{0, 1\}^n$

  $i = 1; j = n$
  while $i \leq j$ **do**
  if $s_i = 0$ **them** $i \leftarrow i + 1$
  else if $s_j = 1$ **then** $j \leftarrow j - 1$
  else swap s_i, s_j; i++; j−− **endif**
  end while

  Only applies to binary sequences. Move i to leftmost 1 and j to rightmost 0. These are out of place, so swap them; continue until i, j meet

The sorting problem

- Consider the following problem:

  SORTING PROBLEM (SP). Given a sequence $s = (s_1, \ldots, s_n)$, find a permutation $\pi \in S_n$ of n symbols with the following property:

  $$\forall 1 \le i < j \le n \; (s_{\pi(i)} \le s_{\pi(j)}),$$

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Type of s influences efficiency: the more generic, the less efficient. E.g. `mergeSort` and `quickSort` OK for all types; `twoWaySort` only OK for boolean

Problem complexity

- Algorithmic complexity : worst-case run time over all inputs
- Problem complexity : worst case run time of most efficient algorithm for problem
- Usually: upper bound on problem complexity

Given problem P, find an $O(f)$ algorithm, say complexity of P is no worse than $O(f)$

- Lower bounds?

Given problem P, show that no algorithm for P can ever be better than $\Omega(f)$

- Seems to require listing all possible algorithms for P

An ill-defined question?

Comparisons

- Sorting algorithms are **comparison-based**

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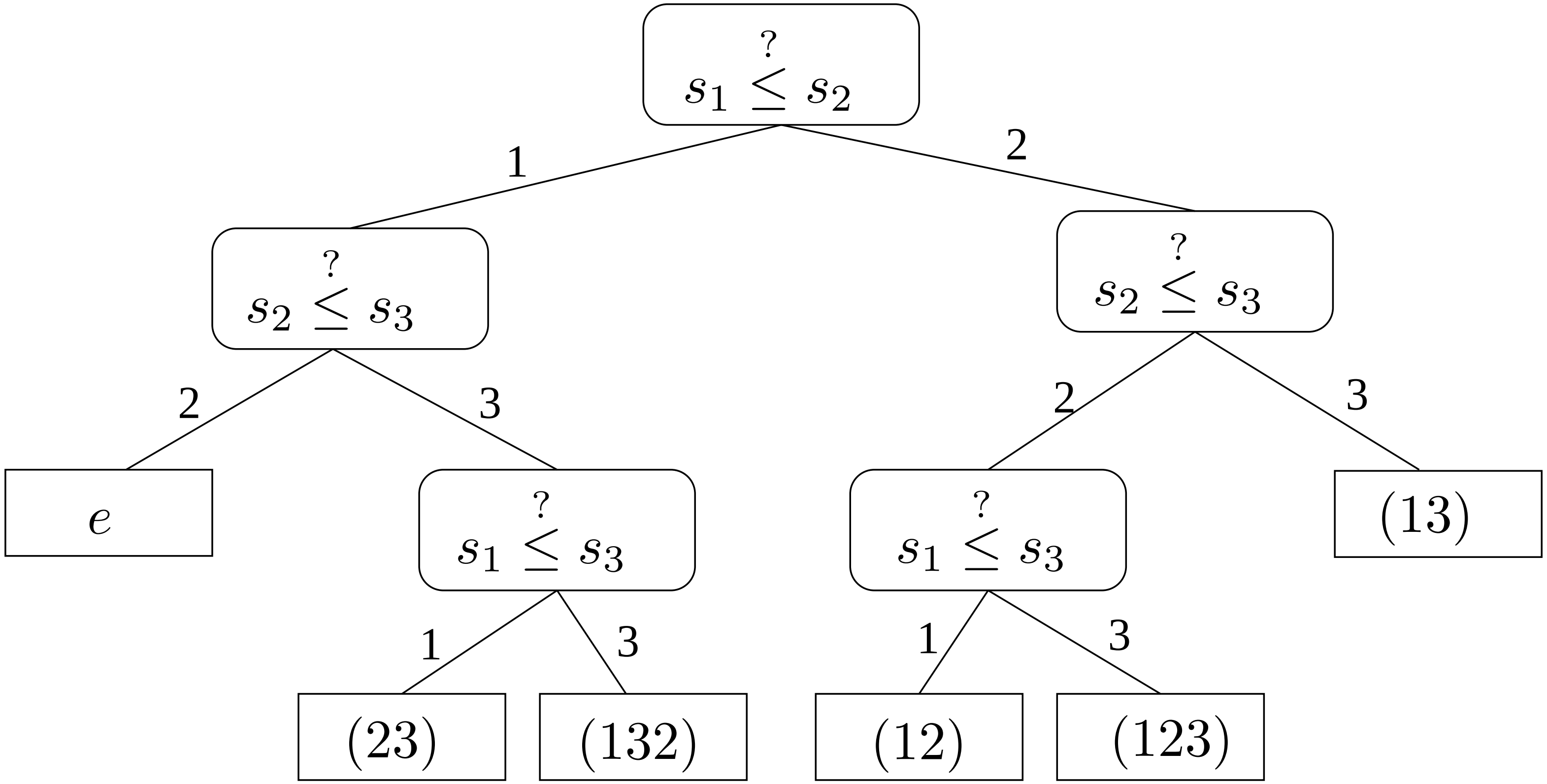
- E.g. sorting tree to order s_1, s_2, s_3:



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- **Also**: best possible CB sorting algorithm $=$ best possible sorting tree

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- By Stirling's approx., $\log n! = n \log n - \frac{1}{\ln 2} n + O(\log n)$ $\Box$
- $\Rightarrow$ B_n is bounded below by a function proportional to $n \log n$ (we say B_n is $\Omega(n \log n)$)

Today's magic result: first part

Complexity of sorting:

$$\Omega(n \log n)$$

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- Both are $O(n^2)$; insertion sort is fast for small $|s|$

Mergesort

Divide-and-conquer

- Let $s = (5, 3, 6, 2, 1, 9, 4, 3)$

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- Split s midway: $s' = (5, 3, 6, 2)$, $s'' = (1, 9, 4, 3)$
- Sort s', s'': $|s'| < |s|$ and $|s''| < |s|$ $\Rightarrow$ use recursion
- **Base case**: *If* $|s| \leq 1$ *then* s *already sorted by definition*
- Get $s' = (2, 3, 5, 6)$ and $s'' = (1, 3, 4, 9)$
- Merge s', s'' into a sorted sequence $\bar{s}$:

$$\begin{matrix} (2,3,5,6) \\ (1,3,4,\boxed{9}) \end{matrix} \rightarrow (1, 2, 3, 3, 4, 5, 6)$$

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- Return $\bar{s}$

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  - Compare these two elements: $O(1)$
  - There are $|s|$ elements to process: $O(n)$
- You can implement this using lists: if s' is empty return s'', if s'' is empty return s', and otherwise compare the first elements of both and choose smallest

Recursive algorithm

- $\texttt{mergeSort}(s)$ {

```
if |s| ≤ 1 then
    return s;
else
    m = ⌊|s|/2⌋;
    s' = mergeSort(e_1, ..., e_m);
    s'' = mergeSort(e_{m+1}, ..., e_n);
    return merge(s', s'');
end if
}
```

By INF311, `mergeSort` has worst-case complexity $O(n \log n)$

Today's magic result: second part

Complexity of sorting:

$$\Theta(n \log n)$$

A function is $\Theta(g(n))$ if it is both $O(g(n))$ and $\Omega(g(n))$

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Notice: in `mergeSort`, we recurse *first*, then work on subsequences *afterwards*. In `quickSort`, we work on subsequences *first*, then recurse on them *afterwards*

Partition

- `partition`(s): produces two subsequences s', s'' of $(s_2, \ldots, s_n)$ such that:
  - $s' = (s_i \mid i \neq 1 \wedge \ s_i < s_1)$
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- There are $|s| - 1$ elements to process: $O(n)$
- You can implement this using arrays; moreover, if you use a `swap` function such that, given i, j, swaps s_i with s_j in s, you don't even need to create any new temporary array: *you can update s "in place"*

Recursive algorithm

- $\texttt{quickSort}(s)$ {

  if $|s| \leq 1$ **then**
      return ;
  else
      $(s', s'') = \texttt{partition}(s)$;
      $\texttt{quickSort}(s')$;
      $\texttt{quickSort}(s'')$;
      $s \leftarrow (s', s_1, s'')$;
  end if

  }

Complexity

Worst-case complexity: $O(n^2)$

Average-case complexity: $O(n \log n)$

Very fast in practice

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- Each partitioning call takes $O(n)$
- Get $O(n^2)$

2-Way partitioning

Definition by example

Input: $(1, 0, 0, 1, 1, 0, 0, 0, 1, 1)$
Desired output: $(0, 0, 0, 0, 0, 1, 1, 1, 1, 1)$

Iterating swaps

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$$(1, 0, 0, 1, 1, 0, 0, 0, 1, 1) \rightarrow (\mathbf{0}, 0, 0, 1, 1, 0, 0, \mathbf{1}, 1, 1) \rightarrow$$
$$(0, 0, 0, \mathbf{0}, 1, 0, \mathbf{1}, 1, 1, 1) \rightarrow (0, 0, 0, 0, \mathbf{0}, \mathbf{1}, 1, 1, 1, 1)$$

The algorithm

$i = 0$; $j = n - 1$;
while $i < j$ **do**
  if $s_i = 0$ **then**
    $i \leftarrow i + 1$;
  else if $s_j = 1$ **then**
    $j \leftarrow j - 1$;
  else
    $\mathtt{swap}(s, i, j)$;
    $i \leftarrow i + 1$;
    $j \leftarrow j - 1$;
  end if
end while

Worst-case complexity

- Occurs with input $(1, \ldots, 1, 0, \ldots, 0)$ where number of 1's are around the same as the number of 0's
- Requires $\lfloor \frac{n}{2} \rfloor$ swaps
- Worst-case $O(n)$

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  - no prior knowledge on the type of input *("general input")*
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- Only apparent: the initial theorem was under the following assumptions:
  - no prior knowledge on the type of input *("general input")*
  - only **comparison-based algorithms** are concerned
- Neither assumption is true for 2-way partitioning
  - we know that the input sequence is of binary type
  - the algorithm never uses a comparison

Appendix
[P. Cameron, *Combinatorics*]

Quicksort: average complexity 1/10

- Let $n = |s|$
- Let q_n be the average number of comparisons made by `quickSort` to sort an n-sequence
- `partition`(s) involves $n-1$ comparisons
- Assume the pivot $p = s_1$ is the k-th smallest element of s
- Then, recursion takes $q_{k-1} + q_{n-k}$ comparisons on average
- Average this over the n values that k can take
- This implies:

$$q_n = n - 1 + \frac{1}{n}\sum_{k=1}^{n}(q_{k-1} + q_{n-k}) \qquad (1)$$

Quicksort: average complexity 2/10

- Notice that in the sum $\sum_{k=1}^{n}(q_{k-1}+q_{n-k})$, each q_k occurs twice

k	q_{k-1}	q_{n-k}
1	q_0	q_{n-1}
2	q_1	q_{n-2}
$\vdots$	$\vdots$	$\vdots$
$n-1$	q_{n-2}	q_1
n	q_{n-1}	q_0

- Hence we can write:

$$q_n = n - 1 + \frac{2}{n}\sum_{k=0}^{n-1} q_k \tag{2}$$

Quicksort: average complexity 3/10

- Equation (2) is a *recurrence relation*
- A *solution* of a recurrence relation is a closed-form expression for q_n which does not include the symbol q_k for any integer $k \geq 0$
- One solution method consists in writing the solution as the infinite sequence $(q_0, q_1, q_2, \ldots, q_n, \ldots)$ as a *formal power series*:

$$Q(t) = \sum_{n \geq 0} q_n t^n \tag{3}$$

- If $Q(t)$ is known, then the value for each q_n can also be obtained:

  Differentiate $Q(t)$ n *times with respect to* t, *set* $t = 0$, *and divide the result by* $n!$

Quicksort: average complexity 4/10

- Multiply each side of the recurrence relation (2) by nt^n and sum over all $n \geq 0$, get:

$$\sum_{n\geq 0} nq_n t^n = \sum_{n\geq 0} n(n-1)t^n + 2\sum_{n\geq 0}\left(\sum_{k=0}^{n-1} q_k\right) t^n \qquad (4)$$

- We now replace each of these three terms so as to be able to derive a more convenient expression for $Q(t)$

Quicksort: average complexity 5/10

- Differentiate $Q(t)$ with respect to t and multiply by t to get an expression for the first term:

$$t\frac{dQ(t)}{dt} = t\sum_{n\geq 0} nq_n t^{n-1} = \sum_{n\geq 0} nq_n t^n, \tag{5}$$

- For the second term: by lecture 1, $\sum_{n\geq 0} t^n = \frac{1}{1-t}$
- Differentiate this equation twice with respect to t, we get:

$$\sum_{n\geq 0} n(n-1)t^{n-2} = \frac{2}{(1-t)^3} \tag{6}$$

- Now multiply both members by t^2 to get an expression for the second term:

$$\sum_{n\geq 0} n(n-1)t^n = \frac{2t^2}{(1-t)^3} \tag{7}$$

Quicksort: average complexity 6/10

- Now for the third: the n-th term of the sum $\sum_{n\geq 0}(\sum_{k=0}^{n-1} q_k)t^n$ can be written as

$$\sum_{k=0}^{n-1} t^{n-k}(q_k t^k)$$

- Hence, the whole sum over n can be written as the following product (convince yourself that this is true):

$$(t + t^2 + t^3 + \ldots)(q_0 + q_1 t + q_2 t^2 + q_3 t^3 + \ldots)$$

- The first factor is $\sum_{n\geq 0} t^n = \frac{1}{1-t}$, and the second is simply the expression for $Q(t)$
- Hence, the third term is $\frac{2tQ(t)}{1-t}$

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- Putting it all together, we obtain a first-order differential equation for $Q(t)$:

$$tQ'(t) = \frac{2t^2}{(1-t)^3} + \frac{2t}{1-t}Q(t) \quad (8)$$

- Remark that if we differentiate the expression $(1-t)^2Q(t)$ (*which I pulled out of a hat*) w.r.t. t, we get:

$$\frac{d}{dt}((1-t)^2Q(t)) = (1-t)^2Q'(t) - 2(1-t)Q(t) \quad (9)$$

- We rearrange the terms of Eq. (8) to get:

$$tQ'(t) - \frac{2t}{1-t}Q(t) = \frac{2t^2}{(1-t)^3} \quad (10)$$

- We multiply Eq. (10) through by $\frac{(1-t)^2}{t}$ and get:

$$(1-t)^2Q'(t) - 2(1-t)Q(t) = \frac{2t}{1-t} \quad (11)$$

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- The RHS of Eq. (9) is the same as the LHS of Eq. (11), hence we can rewrite Eq. 9 as:

$$\frac{d}{dt}((1-t)^2 Q(t)) = \frac{2t}{1-t} \tag{12}$$

- Now, straightforward integration w.r.t. t yields:

$$Q(t) = \frac{-2(t + \log(1-t))}{(1-t)^2} \tag{13}$$

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- The next step consists in writing the power series for $\log$ and $1/(1-t)^2$, rearrange them in a product, and read off the coefficient q_n of the term in t^n. Without going into details, this yields:

$$q_n = 2(n+1)\sum_{k=1}^{n}\frac{1}{k} - 4n \tag{14}$$

for all $n \geq 0$

- For all $n \geq 0$, the term $\sum_{k=1}^{n}\frac{1}{k}$ is an approximation of:

$$\int_1^n \frac{1}{x}dx = \log(n) + O(1) \tag{15}$$

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- Finally, we get an asymptotic expression for q_n:

$$\forall n \geq 0 \quad q_n = 2n\log(n) + O(n) \tag{16}$$

- This shows that the average number of comparisons taken by `quickSort` is $O(n \log n)$

End of Lecture 4