INF421, Lecture 1
Lists and Complexity

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Course

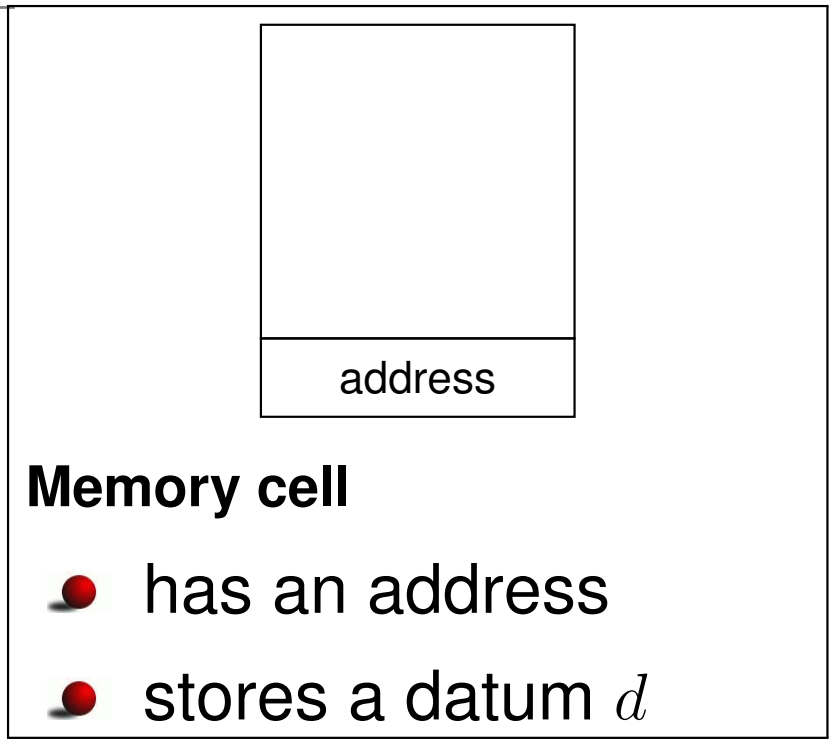
- **Objective**: to teach you some data structures and associated algorithms
- **Evaluation**: TP noté en salle info le 16 septembre, Contrôle à la fin. Note: $\max(CC, \frac{3}{4}CC + \frac{1}{4}TP)$
- **Organization**: fri 26/8, 2/9, 9/9, 16/9, 23/9, 30/9, 7/10, 14/10, 21/10, amphi 1030-12 (Arago), TD 1330-1530, 1545-1745 (SI31,32,33,34)
- **Books**:
  1. Ph. Baptiste & L. Maranget, *Programmation et Algorithmique*, Ecole Polytechnique (Polycopié), 2006
  2. G. Dowek, *Les principes des langages de programmation*, Editions de l'X, 2008
  3. D. Knuth, *The Art of Computer Programming*, Addison-Wesley, 1997
  4. K. Mehlhorn & P. Sanders, *Algorithms and Data Structures*, Springer, 2008
- **Website**: `www.enseignement.polytechnique.fr/informatique/INF421`
- **Contact**: `liberti@lix.polytechnique.fr` (e-mail subject: INF421)

Lecture summary

- Reminders
- Complexity
- Lists

Reminders

Memory



Memory cell

- has an address
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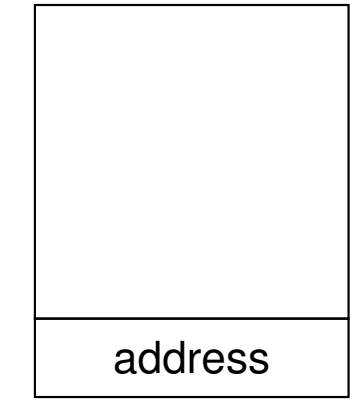

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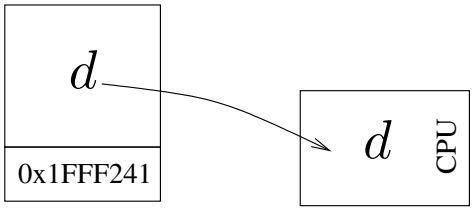

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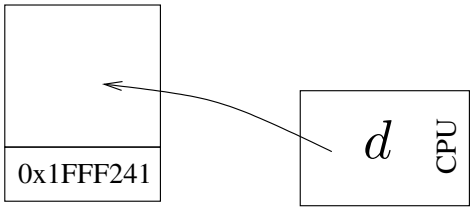

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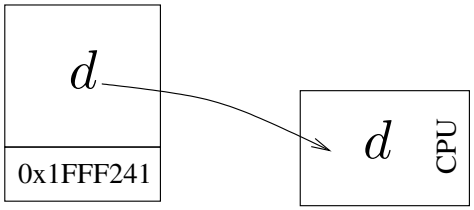

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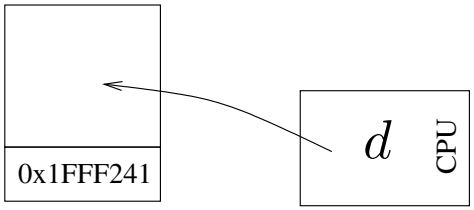

Representation of memory: *a sequence of cells*

d_0	d_1	d_2	d_3	d_4	d_5
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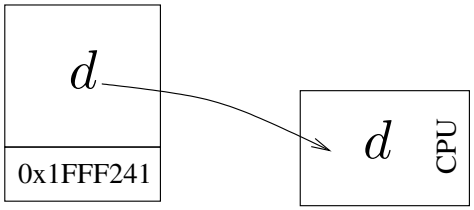

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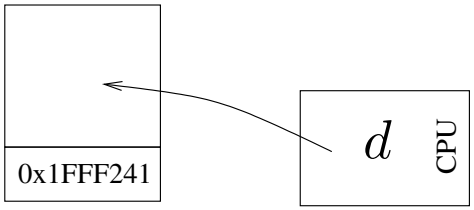

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A function $D : \mathbb{A} \to \mathbb{D}$

$\mathbb{A}$: set of addresses

$\mathbb{D}$: set of data elements

Assumptions

- For theoretical purposes, assume memory is infinite

→ In practice it is finite

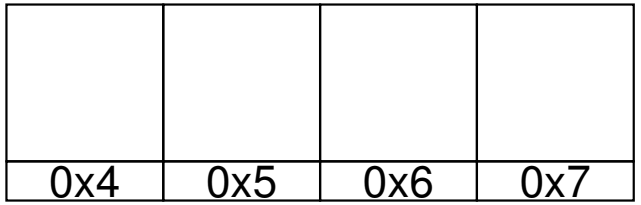
- Each datum can be stored in a single cell

→ Different data elements might have different sizes

Naming memory

A program variable is just a name for a chunk of memory

`x` denotes:



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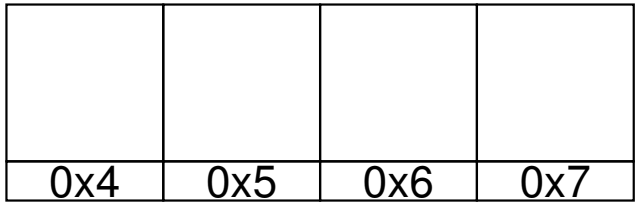

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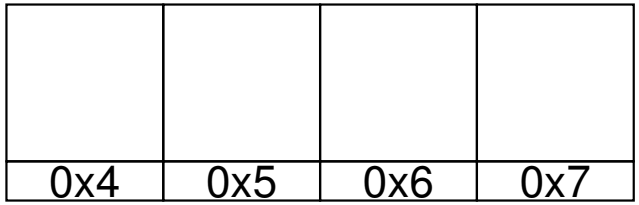

- We simply associate a name to the starting address
- The size of the chunk is given by the name's **type**
- **Basic types**: `int`, `long`, `char`, `float`, `double`
- **Composite types**: Cartesian products of basic types

  if `y.a` $\in$ `int` and `y.b` $\in$ `float` then `y` $\in$ `int` $\times$ `float`

Basic operations

- **Assignment**: write value in memory cell(s) named by variable (i.e. “variable=value”)
- **Arithmetic**: $+, -, \times, \div$ for integer and floating point numbers
- **Test**: evaluate a logical condition: if true, change address of next instruction to be executed
- **Loop**: instead of performing next instruction in memory, jump to an instruction at a given address (more like a “go to”)

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WARNING! *In these slides, I use "$=$" to mean two different things*:

1. in assignments, "variable $=$ value" means "put value in the cell whose address is named by variable"
2. in tests, "variable $=$ value" is TRUE if the cell whose address is named by variable contains value, and FALSE otherwise

in C/C++/Java "$=$" is used for assignments, and "$==$" for tests

Composite operations: programs

Programs are built recursively from basic operations

- If `A`, `B` are ops, then concatenation "`A;B`" is an op

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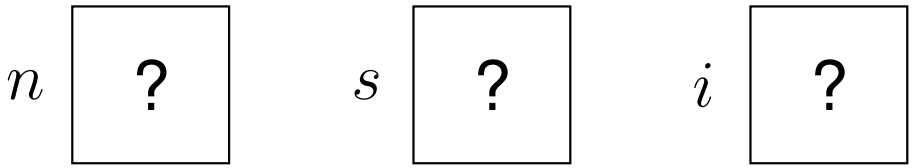
- If `A` is an op and `T` is a test, "`while (T) A`" is an op

  Semantics: `1:(if (T) A else (go to 2)) (go to 1) 2:`

An example

1: input n;
2: `int` $s = 0$;
3: `int` $i = 1$;
4: **while** $(i \leq n)$ **do**
5: $s = s + i$;
6: $i = i + 1$;
7: **end while**
8: output s;



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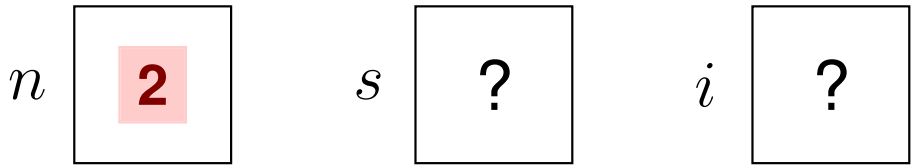

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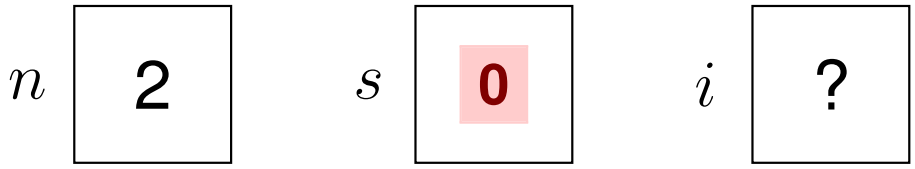

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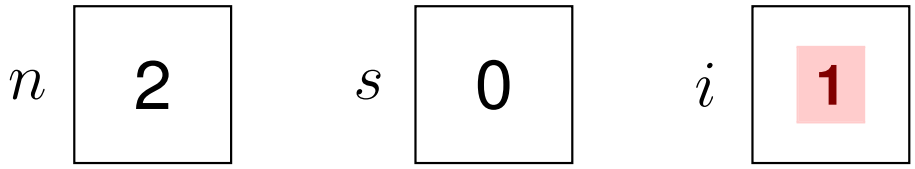

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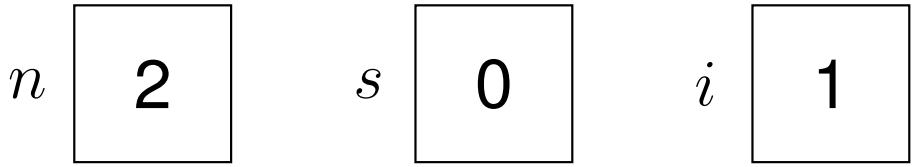

$i \leq n \equiv 1 \leq 2$: true

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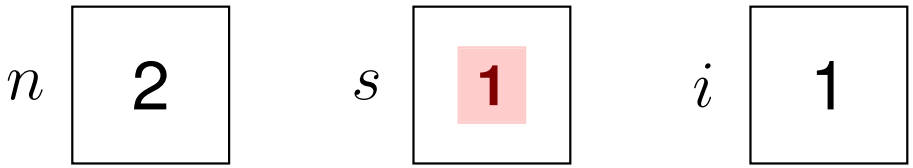

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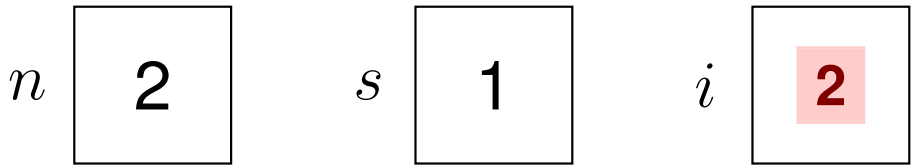

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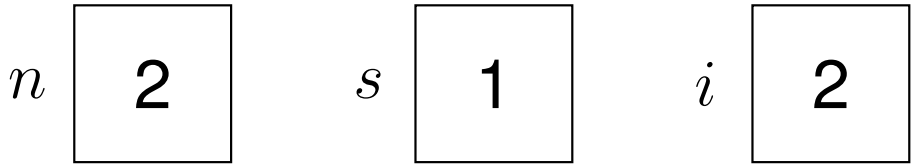

$i \leq n \equiv 2 \leq 2$: true

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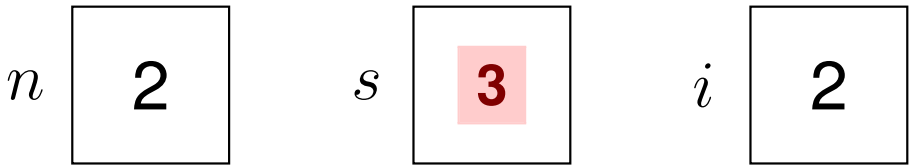

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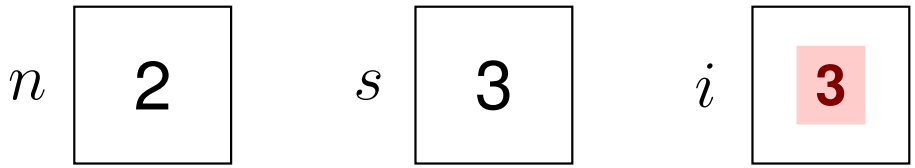

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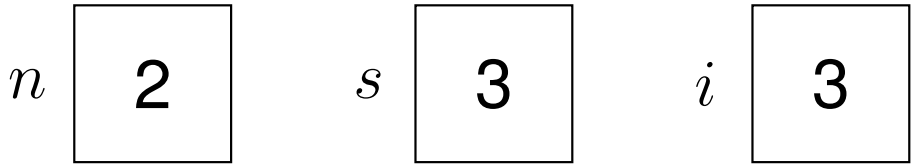

$i \leq n \equiv 3 \leq 2$: false

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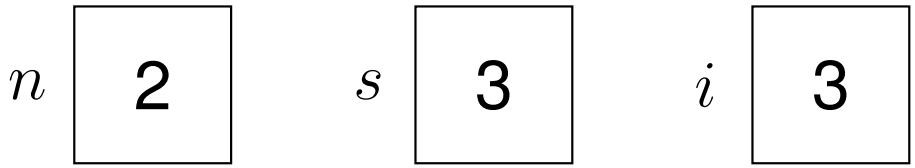

output $s = 3$

Complexity

Complexity

- Several different programs can yield the same result: which is best?
- Evaluate their time (and/or space) complexity
  - **time complexity**: how many "basic operations"
  - **space complexity**: how much memory

  used by the program during execution
- *Worst case*: max values during execution
- *Best case*: min values during execution
- *Average case*: average values during execution

P: a program
t_P: number of basic operations performed by P

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- **Loop**: it's complicated

(depends on how and when loop terminates)

Loop complexity example

The complete loop

Let P be the following program:

1: $i = 0$;
2: **while** $(i < n)$ **do**
3: A;
4: $i = i + 1$;
5: **end while**

- Assume A does not change the value of i
- Body of loop executed n times
- $t_P(n) = 1 + n(t_A + 3)$
- *Why the '3'? Well,* $t_{(i<n)} = 1, t_{(i+1)} = 1, t_{(i=\cdot)} = 1$

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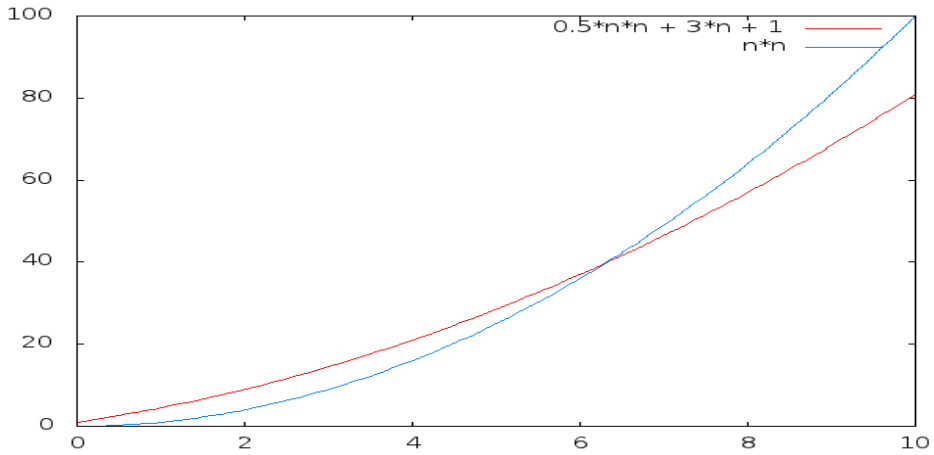

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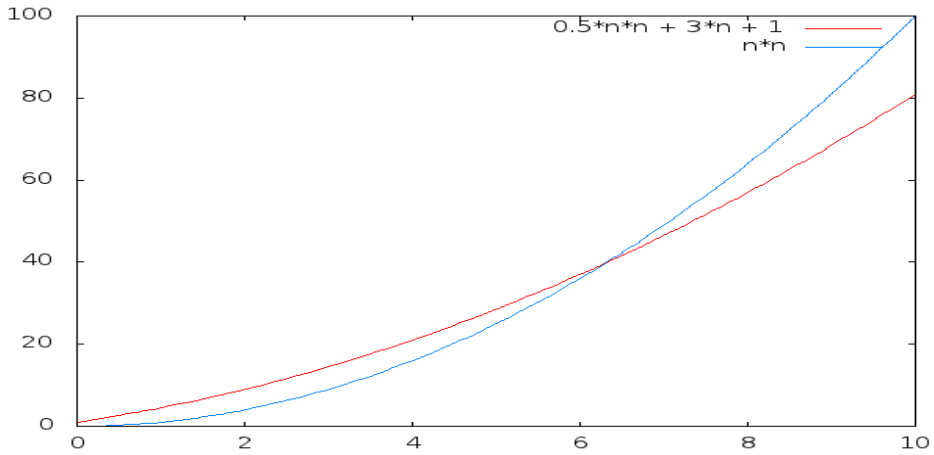

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- A function $f(n)$ is *order of* $g(n)$ (notation: $O(g(n))$) if:

$$\exists c > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ (f(n) \leq cg(n)) \qquad (4)$$

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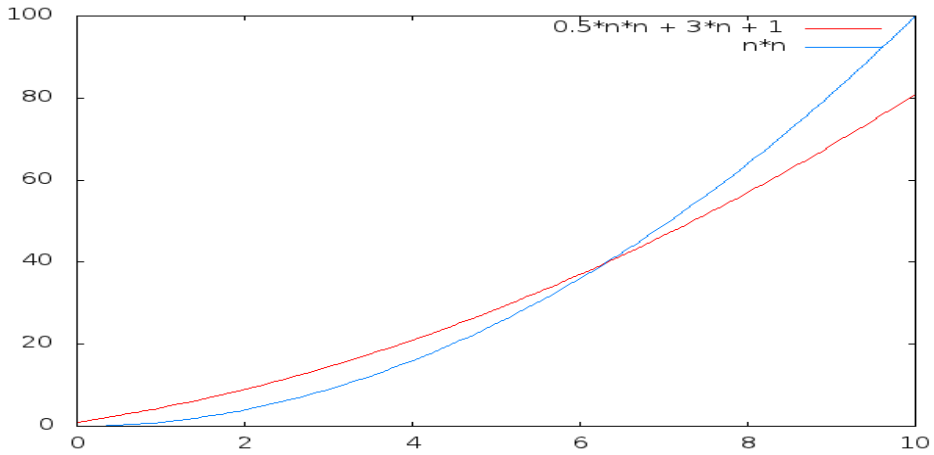

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$$\exists c > 0\ \exists n_0 \in \mathbb{N}\ \forall n > n_0\ (f(n) \leq cg(n)) \qquad (5)$$

- For $\frac{1}{2}n^2 + 3$, $c = 1$ and $n_0 = 2$

Some examples

Functions	*Order*
$an + b$ with a, b constants	$O(n)$
polynomial of degree d' in n	$O(n^d)$ with $d \geq d'$
$n + \log n$	$O(n)$
$n + \sqrt{n}$	$O(n)$
$\log n + \sqrt{n}$	$O(\sqrt{n})$
$n \log n^3$	$O(n \log n)$
$\frac{an+b}{cn+d}$, a, b, c, d constants	$O(1)$

- Make an effort to find the best (most slowly increasing) function $g(n)$ when saying "$f(n)$ is $O(g(n))$"
- E.g. no one would say that $2n + 1$ is $O(n^4)$ (although it's technically true) — rather say $2n + 1$ is $O(n)$

Remark

- The complexity order is an asymptotic description of $t_P(n)$
- If $t_P(n)$ does not depend on n, its order of complexity is $O(1)$ (i.e. constant)
- **Example**: looping 10^{1000} times over an $O(1)$ code still yields an $O(1)$ program
- In other words, n must appear as a parameter of the program for the complexity order to be anything other than constant

Complexity of easy loops

```
1: input n;
2: int s = 0;
3: int i = 1;
4: while (i ≤ n) do
5:    s = s + i;
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```
1: for i = 0; i < n − 1; i = i + 1 do
2:    for j = i + 1; j < n; j = j + 1 do
3:       print i, j;
4:    end for
5: end for
```

- $t(n) = 1 + \underbrace{(5(n-1)+6) + \ldots + (5+6)}_{n-1}$
  $= 1 + 5((n-1) + \ldots + 1) + 6(n-1) = \frac{5}{2}n(n-1) + 6n - 5$
  $= \frac{5}{2}n^2 + \frac{7}{2}n - 5$
- $t(n)$ is $O(n^2)$

Arrays

Like a vector in maths

- A vector $x \in \mathbb{Q}^n$ is an n-tuple $(x_1, \ldots, x_n)$ for some $n \in \mathbb{N}$
- In computers: x is the name for a memory address with n successive cells
- Indexing starts from 0 (last cell is called x_{n-1})

x :	x_0	x_1	x_2	x_3	x_4

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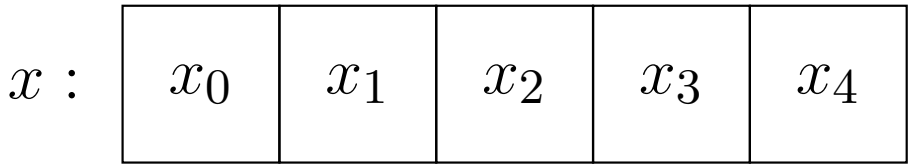


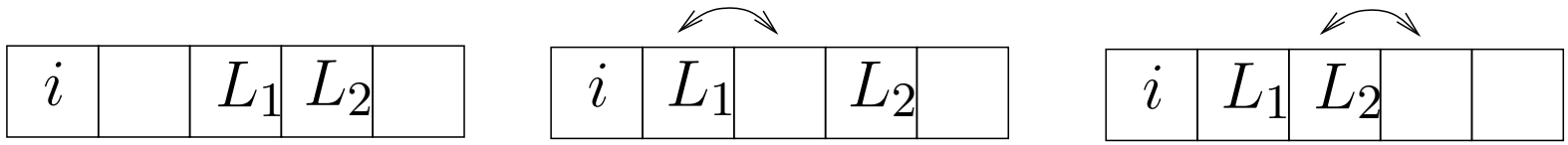
- An array is **allocated** when the memory is reserved
- The size of the array, n, is decided at allocation time
- Usually, the size of the array does not change
- When the array is no longer useful, the reserved memory can be **deallocated** or **freed**

Array operations

For an array of size n:

Operations	*Complexity*
Read value of i-th component	$O(1)$
Write value in i-th component	$O(1)$
Size	$O(1)$
Remove element (cell)	*forget it**
Insert element (cell)	*forget it**
Move subsequence to position i	$O(n)$

Moving subsequence L to position i:
extract (contiguous) subsequence L from the array, and re-insert it after position i and before position $i+1$



*: can actually simulate these operations using pointers

Norm of a vector in $\mathbb{R}^5$

1: input $x \in \mathbb{Q}^5$;
2: `int` $i = 0$;
3: `float` $a = 0$;
4: **while** $(i < 5)$ **do**
5: $a = a + x_i \times x_i$;
6: **end while**
7: $a =$ `sqrt`(a);

- Computes $\sqrt{\sum_{i=0}^{4} x_i^2}$
- Complexity: $O(1)$ (why?)

Incomplete loop

```
1: input x ∈ {0,1}^n;
2: int i = 0;
3: while (i < n ∧ x_i = 1) do
4:    x_i = 0;
5:    i = i + 1;
6: end while
7: if (i < n) then
8:    x_i = 1;
9: end if
10: output x;
```

Input	*Output*
(0,0,0,0)	(1,0,0,0)
(1,1,0,0)	(0,0,1,0)
(0,1,1,0)	(1,1,1,0)
(1,1,1,1)	(0,0,0,0)

- Components of x can only be 0 or 1
- Loop continues over all components as long as their value is 1; at the first 0 component, it stops
- Complexity?

Worst case complexity of incomplete loop

- Among all inputs of the algorithm, find those yielding the worst complexity
- In the case above, $x = (1, 1, \ldots, 1)$ always makes the loop continue to the end, i.e. for n iterations

Thm.

$(1, 1 \ldots, 1)$ is the input yielding worst complexity

Proof

Suppose false, then there is a vector $x \neq (1, \ldots, 1)$ yielding a complexity $t(n) > n$. Since $x \neq (1, \ldots, 1)$, x contains at least one 0 component. Let $j < n$ be the smallest index such that $x_j = 0$: at iteration j the loop breaks, and the complexity is $t(n) = j$, which is smaller than n: contradiction.

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Potential difficulty of this approach: identifying the worst-case inputs and *proving* no other input is worse

Average case complexity of incomplete loop (1/2)

- Average case analysis needs a probability space:
  - assume the event $x_i = b$ is independent of the events $x_j = b$ for all $i \neq j$
  - assume each cell x_i of the array contains 0 or 1 with equal probability $\frac{1}{2}$

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Geometric series $\sum_{k\geq 0} q^k = \frac{1}{1-q}$ for $q \in [0,1)$. Differentiate w.r.t. q, get $\sum_{k\geq 0} kq^{k-1} = \frac{1}{(1-q)^2}$; multiply by q, get $\sum_{k\geq 0} kq^k = \frac{q}{(1-q)^2}$. For $q = \frac{1}{2}$, get $\sum_{k\geq 0} k2^{-k} = (1/2)/(1/4) = 2$.

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Hence, the average complexity is constant $O(1)$

Jagged arrays

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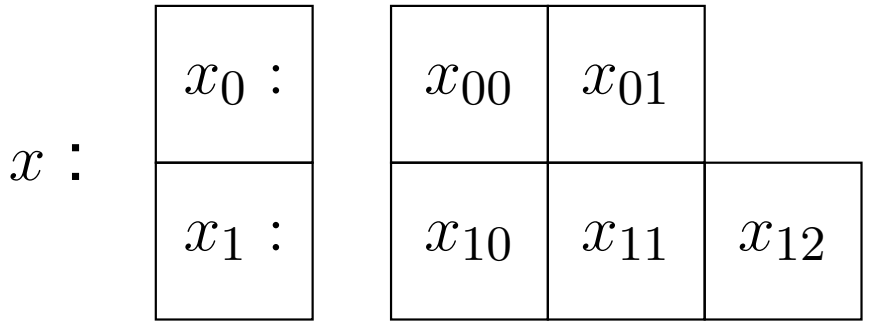
- E.g. $x = ((x_{00}, x_{01}), (x_{10}, x_{11}, x_{12}))$



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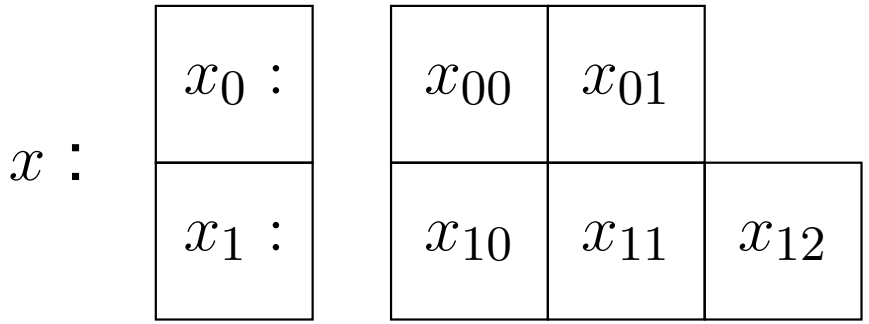

- **Special case**: when all subvector sizes are the same, get a matrix: `int x[][] = new int [2][3];`

$$x = \begin{pmatrix} x_{00} & x_{01} & x_{02} \\ x_{10} & x_{11} & x_{12} \end{pmatrix}$$

Representing relations

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- **Representation**:
  - array of n components
  - the i-th component is the array of v_j related to v_i
- Example: $V = \{1, 2, 3\}$,
  $E = \{(1, 1), (1, 2), (2, 3), (3, 1), (3, 2), (3, 3)\}$

E :

1	1 2
2	3
3	1 2 3

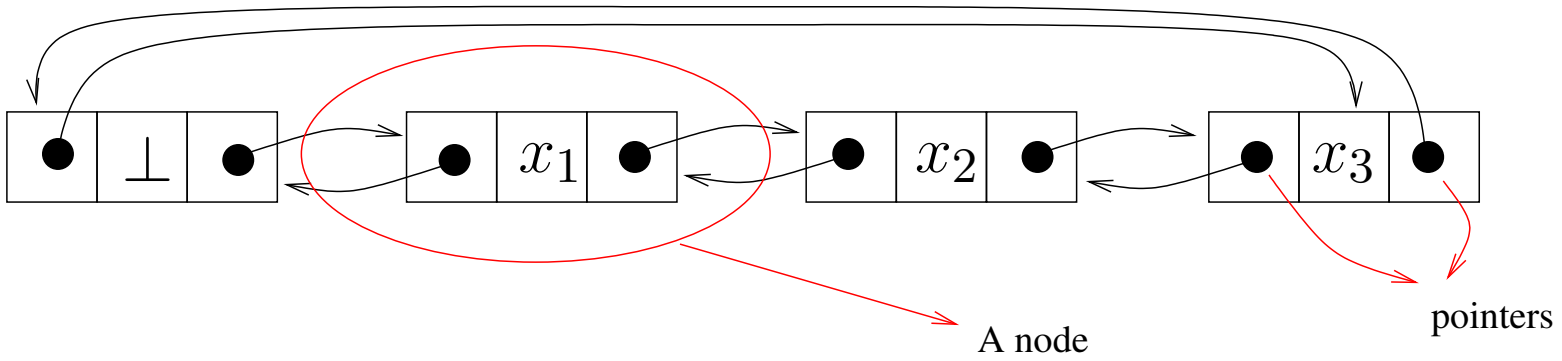
Application: Networks

Array shortcomings

- Essentially fixed size
- Size must be known in advance
- Changing relative positions of elements is inefficient

Lists

Doubly linked list



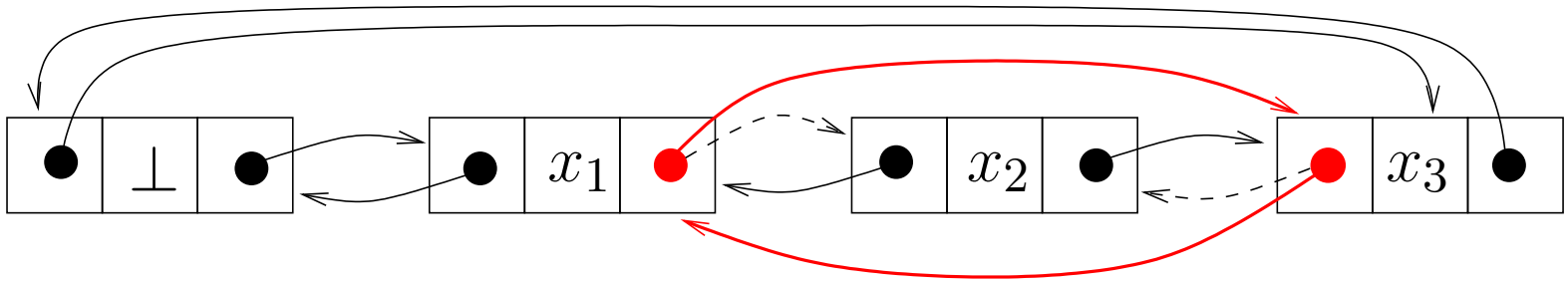
- **Node** N: a list element

N.prev	=	address of previous node in list
N.next	=	address of next node in list
N.datum	=	the data element stored in the node

- **Placeholder node** $\perp$: before the first element, after the last element, no stored data
- *Every node has two pointers, and is pointed to by two nodes*

Remove a node

Remove current node (`this`)



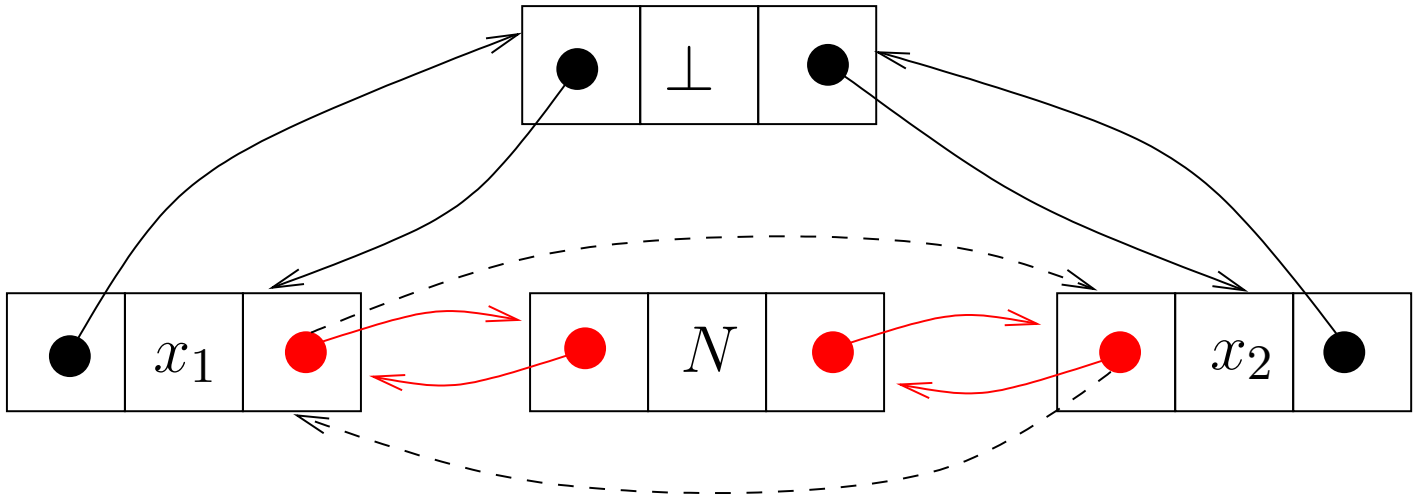
In the example, `this`$= x_2$

```
1: this.prev.next = this.next ;
2: this.next.prev = this.prev ;
```

Worst case complexity: $O(1)$

Insert a node

Insert current node (`this`) after node x_1



In the example, `this`$= N$

1: `this.prev` $= x_1$;
2: `this.next` $= x_1$`.next` ;
3: x_1`.next` $=$ `this` ;
4: `this.next.prev` $=$ `this` ;

Worst case complexity: $O(1)$

Find next

- Given a list L and a node x, find next occurrence of element b
- If $b \in L$ return node where b is stored, else return $\perp$

```
1: while (x.datum ≠ b ∧ x ≠ ⊥) do
2:   x = x.next
3: end while
4: return x
```

Warning: *every test costs 2 basic operations, inefficient*

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```
1: ⊥.datum = b
2: while (x.datum ≠ b) do
3:   x = x.next
4: end while
5: return x
```

Now $t_{\text{test}} = 1$

List operations

For a doubly-linked list of size n:

Operations	*Complexity*
Read/write value of i-th node	$O(n)$
Find next	$O(n)$
Size[a]	$O(n)$
Is it empty?	$O(1)$
Read/write value of first/last node	$O(1)$
Remove element	$O(1)$
Insert element	$O(1)$
Move subsequence to position i	$O(1)$
Pop from front/back	$O(1)$
Push to front/back	$O(1)$
Concatenate	$O(1)$

[a] Some implementations are $O(1)$ by storing and updating size

End of Lecture 1