Reformulation-Linearization Methods for Global Optimization

Hanif D. Sherali[1], Leo Liberti[2]

[1] *Virginia Polytechnic Institute and State University, Department of Industrial and Systems Engineering (0118), 250 Durham Hall, Blacksburg, VA 24061, USA*
Email: `hanifs@vt.edu`

[2] *LIX, École Polytechnique, F-91128 Palaiseau, France*
Email: `liberti@lix.polytechnique.fr`

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Discrete and continuous nonconvex programming problems arise in a host of practical applications in the context of production planning and control, location-allocation, distribution, economics and game theory, quantum chemistry, and process and engineering design situations. Several recent advances have been made in the development of branch-and-cut type algorithms for mixed-integer linear and nonlinear programming problems, as well as polyhedral outer-approximation methods for continuous nonconvex programming problems. At the heart of these approaches is a sequence of linear (or convex) programming relaxations that drive the solution process, and the success of such algorithms is strongly tied in with the strength or tightness of these relaxations.

The Reformulation-Linearization-Technique (RLT) is a method that generates such tight linear programming relaxations for not only constructing exact solution algorithms, but also to design powerful heuristic procedures for large classes of discrete combinatorial and continuous nonconvex programming problems. Its development originated in [4, 5, 6], initially focusing on 0-1 and mixed 0-1 linear and polynomial programs [21, 22], and later branching into the more general family of continuous, nonconvex polynomial programming problems [18, 45, 49]. For the family of mixed 0-1 linear (and polynomial) programs in n 0-1 variables, the RLT generates an n-level hierarchy, with the n-th level providing an explicit algebraic characterization of the convex hull of feasible solutions [21, 22]. The RLT essentially consists of two steps — a reformulation step in which certain additional nonlinear valid inequalities are automatically generated, and a linearization step in which each product term is replaced by a single continuous variable. The level of the hierarchy directly corresponds to the degree of the polynomial terms produced during the reformulation stage. Hence, in the reformulation phase, given a value of the level $d \in \{1, \ldots, n\}$, the RLT constructs various polynomial factors of degree d comprised of the product of some d binary variables x_j or their complements $(1 - x_j)$. These factors are then used to multiply each of the defining constraints in the problem (including the variable bounding restrictions), to create a (nonlinear) polynomial mixed-integer zero-one programming problem. Suitable additional constraint-factor products can be used to further enhance the procedure. In general, for a variable restricted to lie in the interval $[l_j, u_j]$, the nonnegative expressions $(x_j - l_j)$ and $(u_j - x_j)$ are referred to as *bound-factors*, and for a structural inequality $\alpha x \geq \beta$, for example, the expression $(\alpha x - \beta)$ is referred to as a *constraint-factor*; implied product constraints can be generated using either bound-factors or constraint-factors. After using the relationship $x_j^2 = x_j$ for each binary variable x_j, $j \in \{1, \ldots, n\}$, which in effect accounts for the tightening of the relaxation, the linearization phase substitutes a single variable w_J (respectively, v_{Jk}), in place of each nonlinear term of the type $\prod_{j \in J} x_j$ (respectively, $y_k \prod_{j \in J} x_j$), where y represents the set of continuous variables. Hence, relaxing integrality, the nonlinear polynomial problem is linearized into a higher dimensional polyhedral set X_d defined in terms of the original variables (x, y) and the new variables (w, v). Denoting the projection of X_d onto the space of the original (x, y)-variables as X_{P_d}, it is shown that as d varies from 1 to n, we get,

$$X_{P_0} \supseteq X_{P_1} \supseteq X_{P_2} \supseteq \ldots \supseteq X_{P_n} = \text{conv}(X),$$

where X_{P_0} is the ordinary linear programming relaxation, and $\text{conv}(X)$ represents the convex hull of the original feasible region X. An extension of this development to the case of general integer/discrete variables is presented in [25, 7], where the bound-factors are replaced by suitable Lagrange interpolating polynomials, and a further extension to 0-1 mixed-integer as well as general mixed-discrete semi-infinite and bounded convex programming problems is presented in [26] (see also [50]). Lovasz and Shrijver [16] and Boros et al. [9] have also independently developed various concepts related to the RLT process. This RLT process has also been extended and enhanced in [27] through the use of more generalized constraint-factors that imply the bounding restrictions $0 \leq x_j \leq 1$ for $j \in \{1, \ldots, n\}$. A similar hierarchy of relaxations leading to the convex hull representation is obtained based on the use of these generalized factors in the reformulation phase, in lieu of simply the bound-factors x_j and $(1 - x_j)$, for $j \in \{1, \ldots, n\}$. In addition, this hierarchy embeds within its construction stronger logical implications than only $x_j^2 = x_j$, $\forall j \in \{1, \ldots, n\}$. For example, consider an RLT constraint that has been generated by taking the product of some nonnegative polynomial factor F with a defining constraint $\gamma^\top x \geq \delta$ to yield $[F(\gamma^\top x - \delta)]_L \geq 0$, where $[\cdot]_L$ denotes the linearization of the polynomial expression $[\cdot]$ under the RLT substitution process. Then, this constraint can be tightened by deriving a stronger valid inequality of the type $\hat{\gamma}^\top x \geq \hat{\delta}$ under the condition that $F > 0$, and then imposing the RLT constraint $[F(\hat{\gamma}^\top x - \hat{\delta})]_L \geq 0$, which is valid whenever $F = 0$ or $F > 0$. The resulting overall RLT process is shown in [27] to not only subsume the previous development, but also provide the opportunity to exploit frequently-arising special structures such as generalized/variable upper bounds, covering, partitioning, and packing constraints, as well as sparsity.

The hierarchy of higher-dimensional representations produced in this manner markedly strengthens the usual continuous relaxation, as is evidenced not only by the fact that the convex hull representation is obtained at the highest level, but that in computational studies on many classes of problems, even the first level representation helps design algorithms that significantly dominate existing procedures in the literature [4, 6, 20, 27, 30, 41]. Based on a special case of the RLT process that employs the bound-factors for only a single variable at a time, Balas *et al.* [8] describe a *lift-and-project* cutting plane algorithm that is shown to produce encouraging results. The theoretical implications of this hierarchy are noteworthy; the resulting representations subsume and unify many published linearization methods for nonlinear 0-1 programs, and the algebraic representation available at level n promotes new methods for identifying and characterizing facets and valid linear inequalities in the original variable space, as well as for providing information that directly bridges the gap between discrete and continuous sets [3, 38, 40]. Indeed, since the level-n formulation characterizes the convex hull, all valid inequalities in the original variable space must be obtainable via a suitable projection; thus such a projection operation serves as an all-encompassing tool for generating valid inequalities. References [38, 40] provide discussions on generating facets and tight valid inequalities for several classes of problems. Reference [3] discusses persistency issues for certain constrained and unconstrained pseudo-Boolean programming problems whereby variables that take on 0-1 values at an optimum to an RLT relaxation would persist to take on these same values at an optimum to the original problem. References [39, 2, 1, 34, 42, 13], respectively discuss the use of RLT to generate improved model representations for the set partitioning, quadratic assignment, traveling salesman problems, and to 0-1 mixed-integer programs subject to various assignment constraints.

Although the Reformulation-Linearization Technique was originally designed to employ factors involving 0-1 variables in order to generate 0-1 (mixed-integer) polynomial programming problems that are subsequently re-linearized, the approach has also been extended to solve continuous, bounded variable polynomial programming problems. Problems of this type involve the optimization of a polynomial objective function subject to polynomial constraints in a set of continuous, bounded variables, and arise in numerous applications in engineering design, production, location, and distribution problems. Reference [45] prescribes an RLT process that employs suitable polynomial-factors (bound-factors based on bounding restrictions $l_j \leq x_j \leq u_j$, $j \in \{1, \ldots, n\}$, as well as constraint factors) to generate additional polynomial constraints through a multiplication process, which upon linearization through variable redefinitions as above, produces a linear programming relaxation. The resulting relaxation is used in concert with a suitable designed partitioning technique that attempts to reduce the error between the original nonlinear and their resulting linearized terms, in order to develop an algorithm that is proven to converge to a global optimum for this problem. Special classes of polynomial constraints based on grid factors, Lagrange interpolating polynomials, and mean value theorem constraints can be generated to further tighten these RLT relaxations [48]. In some cases (e.g., see [47]), it is beneficial to retain certain simple convex constraints in the relaxation, resulting in a more general *Reformulation-Linearization/Convexification Technique*. Additionally, Sherali and Fraticelli [35] have proposed a class of semidefinite cuts based on semidefinite relaxation enhancements that can be used to significantly tighten RLT representations. While RLT essentially operates on polynomial functions having integral exponents, many engineering design applications lead to polynomial programs having general rational exponents. For such problems, a global optimization technique has been designed [18] by introducing a new level of approximation at the reformulation step, and accordingly, redesigning the partitioning scheme in order to induce the overall sequence of relaxations generated to become exact in the limit. Further extensions for solving nonlinear factorable programs for which the objective and constraint functions involve sums of products of univariate functions have also been developed [49]. Here, suitable under/over-approximating nonconvex polynomial functions are derived for the defining univariate functions in the problem, and then an appropriate partitioning scheme is devised that drives the errors from these approximations and those for the RLT process applied to the resulting polynomial program simultaneously to zero in the limit, in order to obtain a global optimum for the given factorable program. For nonconvex programs that are defined in terms of black-box functions, a new concept of a pseudo-global RLT approach has been developed by Sherali and Ganesan [36], which has been successfully applied to the design of containerships.

A special application of the RLT to mixed-integer quadratic problems subject to linear equality constraints that yields exact reformulations having fewer quadratic terms and some additional supporting RLT constraints has been developed to produce tighter convex relaxations [10, 11, 12, 15, 14]. More precisely, we multiply a subset of equality constraints $Ax = b$ by an appropriate subset of problem variables $\{x_k \mid k \in K\}$, to obtain a *reduced RLT system* $\forall k \in K(Aw_k = bx_k)$, where $w_k \equiv (x_k x_1, \ldots, x_k x_n)$ for all $k \in K$. This is equivalent to the homogeneous linear system $\forall k \in K(Az_k = 0)$ where $z_k = (w_{k1} - x_k x_1, \ldots, w_{kn} - x_k x_n)$, which may be written in a more compact way as $A'z = 0$. If we partition A' into basic and nonbasic submatrices B, N, and accordingly partition z into z_B and z_N, we have $(B|N)z = 0$, whence $Nz_N = 0$ implies that $Bz_B = 0$. We therefore conclude that enforcing the reduced RLT system and the subset of quadratic relations $w_{ki} = x_k x_i$ for (k, i) corresponding to nonbasic columns of N is enough to infer $w_{ki} = x_k x_i$ for all (k, i). In other words, by letting the RLT process ensure that $z_N = 0$, we automatically obtain as an implication of the RLT linearized constraints that the quadratic relation $z_B = 0$ will hold true as well.

For the continuous case, there exist special instances where RLT can produce convex hull or convex envelope representations [17, 28]. Various classes of applications have been studied for which specialized RLT designs have been used to develop enhanced effective algorithms. This list, which is ever expanding, includes bilinear programming problems [28, 15], general indefinite quadratic programming problems [47, 13, 12], location-allocation problems employing different distance metrics [20, 41, 46, 29], water distribution network design problems [43, 44], the solution of Hartree-Fock equations in quantum chemistry [14], the linear complementarity problem [37], and hard and fuzzy clustering problems [31, 32]. References [23, 24, 25, 11, 19, 33] provide expository discussions and a survey of RLT theory and applications.

References

[1] Adams, W. P. and Guignard, M. and Hahn, P. M. and Hightower, W. L.: 'A Level-2 Reformulation-Linearization Technique Bound for the Quadratic Assignment Problem', *European Journal of Operational Research*, to appear.

[2] Adams, W. P. and Johnson, T. A. 'Improved linear programming-based lower bounds for the

quadratic assignment problem', *DIMACS Series in Discrete Mathematics and Theoretical Computer Science, Quadratic Assignment and Related Problems*, P. M. Pardalos and H. Wolkowicz (eds.), **16**, 43-75 (1994).

[3] Adams, W. P. and Lassiter, J. B. and Sherali, H. D.: 'Persistency in 0-1 optimization', *Mathematics of Operations Research*, **23**(2), 359-389 (1998).

[4] Adams, W. P. and Sherali, H. D.: 'A tight linearization and an algorithm for zero-one quadratic programming problems', *Management Science*, **32**(10), 1274-1290 (1986).

[5] Adams, W. P. and Sherali, H. D.: 'Linearization strategies for a class of zero-one mixed integer programming problems', *Operations Research*, **38**(2), 217-226 (1990).

[6] Adams, W. P. and Sherali, H. D.: 'Mixed-integer bilinear programming problems', *Mathematical Programming*, **59**(3), 279-306 (1993).

[7] Adams, W. P. and Sherali, H. D.: 'A hierarchy of relaxations leading to the convex hull representation for general discrete optimization problems', *Annals of Operations Research*, **140**(1), 21-47 (2005).

[8] Balas, E. and Ceria, S. and Cornuejols, G.: 'A lift-and-project cutting plane algorithm for mixed 0-1 programs', *Mathematical Programming*, **58**(3), 295-324 (1993).

[9] Boros, E. and Crama, Y. and Hammer, P. L.: 'Upper bounds for quadratic 0-1 maximization problems', *Operations Research Letters*, **9**, 73-79 (1990).

[10] Liberti, L.: 'Reduction constraints for the global optimization of NLPs', *International Transactions in Operations Research*, **11**(1), 34–41 (2004).

[11] Liberti, L.: 'Reformulation and convex relaxation techniques for global optimization', *4OR*, **2**, 255–258 (2004).

[12] Liberti, L.: 'Linearity embedded in nonconvex programs', *Journal of Global Optimization*, **33**(2), 157–196 (2005).

[13] Liberti, L.: 'Compact linearization of binary quadratic problems', *4OR*, DOI 10.1007/s10288-006-0015-3 (published online 2007).

[14] Liberti, L. and Lavor, C. and Maculan, N. and Chaer Nascimento, M. A.: 'Reformulation in mathematical programming: an application to quantum chemistry', *Discrete Applied Mathematics*, submitted.

[15] Liberti, L. and Pantelides, C. C.: 'An exact reformulation algorithm for large nonconvex NLPs involving bilinear terms', *Journal of Global Optimization*, **36**, 161–189 (2006).

[16] Lovasz, L. and Schrijver, A.: 'Cones of matrices and set functions, and 0-1 optimization', *SIAM Journal on Optimization*, **1**, 166-190 (1990).

[17] Sherali, H. D.: 'Convex envelopes of multilinear functions over a unit hypercube and over special discrete sets', *Acta Mathematica Vietnamica*, **22**(1), 245-270 (1997).

[18] Sherali, H. D.: 'Global optimization of nonconvex polynomial programming problems having rational exponents', *Journal of Global Optimization*, **12**(3), 267-283 (1998).

[19] Sherali, H. D.: 'Tight Relaxations for nonconvex optimization problems using the reformulation-linearization/convexification technique (RLT)', *Handbook of Global Optimization, Volume 2: Heuristic Approaches*, P. M. Pardalos and H. E. Romeijn (eds.), Kluwer Academic Publishers, Dordrecht/London/Boston, 1-63 (2002).

[20] Sherali, H. D. and Adams, W. P.: 'A decomposition algorithm for a discrete location-allocation problem', *Operations Research*, **32**(4), 878-900 (1984)

[21] Sherali, H. D. and Adams, W. P.: 'A hierarchy of relaxations between the continuous and convex hull representations for zero-one programming problems', *SIAM Journal on Discrete Mathematics*, **3**(3), 411-430 (1990).

[22] Sherali, H. D. and Adams, W. P.: 'A hierarchy of relaxations and convex hull characterizations for mixed-integer zero-one programming problems', *Discrete Applied Mathematics*, **52**, 83-106 (1994).

[23] Sherali, H. D. and Adams, W. P.: A reformulation- linearization technique (RLT) for solving discrete and continuous nonconvex programming problems', *XII-A Mathematics Today*, special issue on Recent Advances in Mathematical Programming, ed. O. Gupta, 61-78 (1994).

[24] Sherali, H. D. and Adams, W. P.: 'Computational advances using the reformulation-linearization technique (RLT) to solve discrete and continuous nonconvex problems', OPTIMA, **49**, 1-6 (1996).

[25] Sherali, H. D. and Adams, W. P.: A reformulation-linearization technique for solving discrete and continuous nonconvex problems, *Kluwer Academic Publishers*, Dordrecht (1999).

[26] Sherali, H. D. and Adams, W. P.: 'Extension of a reformulation-linearization technique (RLT) to semi-infinite and convex programs under mixed 0-1 and general discrete restrictions', Manuscript, Dept. of Industrial Systems Engineering, Virginia Polytechnic Institute, Blacksburg, VA 24061 (2006).

[27] Sherali, H. D. and Adams, W. P. and Driscoll, P.: 'Exploiting special structures in constructing a hierarchy of relaxations for 0-1 mixed integer problems', *Operations Research*, **46**(3), 396-405 (1998).

[28] Sherali, H. D. and Alameddine, A.: 'A new reformulation- linearization technique for solving bilinear programming problems', *Journal of Global Optimization*, **2**, 379-410 (1992).

[29] Sherali, H. D. and Al-Loughani, I. and Subramanian, S.: 'Global optimization procedures for the capacitated Euclidean and l_p distance multifacility location-allocation problems', *Operations Research*, **50**(3), 433-448 (2002).

[30] Sherali, H. D. and Brown, E. L.: 'A quadratic partial assignment and packing model and algorithm for the airline gate assignment problem', *DIMACS Series in Discrete Mathematics and Theoretical Computer Science', 'Quadratic assignment and related problems'* P. M. Pardalos and H. Wolkowicz (eds.), **16**, 343-364 (1994).

[31] Sherali, H. D. and Desai, J.: 'A global optimization RLT-based approach for solving the hard clustering problem', *Journal of Global Optimization*, **32**, 281-306 (2005).

[32] Sherali, H. D. and Desai, J.: 'A global optimization RLT-based approach for solving the fuzzy clustering problem', *Journal of Global Optimization*, **33**, 597-615 (2005).

[33] Sherali, H. D. and Desai, J.: 'On solving polynomial, factorable, and black-box optimization problems using the RLT methodology', *Essays and Surveys on Global Optimization*, C. Audet, P. Hansen, and G. Savard (eds)., Springer, New York, 131-163 (2005).

[34] Sherali, H. D. and Driscoll, P. J.: 'On tightening the relaxation of Miller-Tucker-Zemlin formulations for asymmetric traveling salesman problems', *Operations Research*, **50**(4), 656-669 (2002).

[35] Sherali, H. D. and Fraticelli, B. M. P.: 'Enhancing RLT relaxations via a new class of semidefinite cuts', *Journal of Global Optimization*, **22**(1-4), 233-261 (2002).

[36] Sherali, H. D. and Ganesan, V.: 'A pseudo-global optimization approach with application to the design of containerships', *Journal of Global Optimization*, **26**(4), 335-360 (2003).

[37] Sherali, H. D. and Krishnamurty, R. and Al-Khayyal, F. A.: 'Enumeration approach for linear complementarity problems based on a reformulation-linearization technique', *Journal of Optimization Theory and Applications* **99**(2), 481-507 (1998).

[38] Sherali, H. D. and Lee, Y.: 'Sequential and simultaneous liftings of minimal cover inequalities for GUB constrained knapsack polytopes', *SIAM Journal on Discrete Mathematics*, **8**(1), 133-153 (1995).

[39] Sherali, H. D. and Lee, Y.: 'Tighter representations for set partitioning problems via a reformulation-linearization approach', *Discrete Applied Mathematics*, **68**, 153-167 (1996).

[40] Sherali, H. D. and Lee, Y. and Adams, W. P.: 'A Simultaneous lifting strategy for identifying new classes of facets for the Boolean quadric polytope', *Operations Research Letters*, **17**(1), 19-26 (1995).

[41] Sherali, H. D. and Ramachandran, S. and Kim, S.: 'A localization and reformulation discrete programming approach for the rectilinear distance location-allocation problem', *Discrete Applied Mathematics*, **49**(1-3), 357-378 (1994).

[42] Sherali, H. D. and Sarin, S. C. and Tsai, P. F.: 'A class of lifted path and flow-based formulations for the asymmetric traveling salesman problem with and without precedence constraints', *Discrete Optimization*, **3**, 20-32 (2006).

[43] Sherali, H. D. and Smith, E. P.: 'A global optimization approach to a water distribution network problem', *Journal of Global Optimization*, **11**, 107-132 (1997).

[44] Sherali, H. D. and Subramanian, S. and Loganathan, G. V: 'Effective relaxations and partitioning schemes for solving water distribution network design problems to global optimality', *Journal of Global Optimization*, **19**, 1-26 (2001).

[45] Sherali, H. D. and Tuncbilek, C. H.: 'A global optimization algorithm for polynomial programming problems using a reformulation-linearization technique', *Journal of Global Optimization*, **2**, 101-112 (1992).

[46] Sherali, H. D. and Tuncbilek, C. H.: 'A squared-Euclidean distance location-allocation problem', *Naval Research Logistics*, **39**, 447-469 (1992).

[47] Sherali, H. D. and Tuncbilek, C. H.: 'A reformulation- convexification approach for solving nonconvex quadratic programming problems', *Journal of Global Optimization*, **7**, 1-31 (1995).

[48] Sherali, H. D. and Tuncbilek, C. H.: 'New reformulation- linearization technique based relaxations for univariate and multivariate polynomial programming problems', *Operations Research Letters*, **21**(1), 1-10 (1997).

[49] Sherali, H. D. and Wang, H.: 'Global optimization of nonconvex factorable programming problems', *Mathematical Programming*, **89**(3), 459-478 (2001).

[50] Stubbs, R. A. and Mehrotra, S. : 'A branch-and-cut method for 0-1 mixed convex programming', *Mathematical Programming*, **86**, 515-532 (1999).