A useful characterization of the feasible region of binary linear programs

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1 Introduction

Given a Binary Linear Programming (BLP) problem in the following general form:

$$\left.\begin{aligned} \min_x \ & cx \\ \text{s.t.} \ & Ax \le b \\ & x \in \{0,1\}^n, \end{aligned}\right\} [P] \tag{1}$$

(where x are the decision variables, c is a rational cost n-vector, A is a rational $m \times n$ matrix, and b is a rational m-vector), the convex hull is the convex combination of all feasible integral points; its importance lies in the fact that the relaxed solution of the continuous relaxation of (1) subject to the convex hull of all its feasible integral points is integer.

In view of providing an explicit representation of the convex hull by listing all the facets, it is interesting to describe the integral feasible region in terms of interior points, i.e. hypercube vertices which are feasible in (1) and such that all their adjacent hypercube vertices are also feasible in (1) and exterior points, for which there is at least one infeasible adjacent hypercube vertex. Whereas interior points belong to trivial facets of the convex hull (i.e. those facets which are also hypercube facets), exterior points define all the non-trivial facets. In this work we use a particular type of rounding along the hypercube edges (called *flattening*) to derive all exterior points of the feasible region of BLPs. We also show how to exploit this characterization to derive practically useful valid inequalities passing through hypercube vertices, and their relation to Balas' intersection cuts [1]. Other works in the literature which are closely related to this topic are geometric [3] and canonical [2] cuts; both of these also pass through hypercube vertices, and therefore also identify exterior points.

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The main idea in this work is that if an intersection point p between a hyperplane $A_i x = b_i$ (arising from the inequality $A_i x \leq b_i$ of the relaxed feasible polyhedron) with the edge segments of the unit hypercube is not integral, then it has a unique fractional component. The two integral neighbouring hypercube points x_1, x_2 are then separated by $A_i x = b_i$; assuming $A_i x_1 \leq b_i$ and $A_i x_2 > b_i$, and supposing that x_1 is feasible in (1), we "flatten" the constraint $A_i x \leq b_i$ in the direction of the feasible point x_1. Flattening inequalities are designed to intersect feasible integral points, hence they are likely to provide fast convergence for a cutting plane algorithm whenever the current relaxed solution is near a hypercube edge; however, because they are not guaranteed to be valid cuts, they need to be paired with general-purpose valid cuts separating the current relaxed solution, such as intersection cuts [1].

Let $C^n = (V, E)$ be the graph structure of the unit hypercube in n dimensions, and $\iota : V \to \{0,1\}^n$ be the (invertible) map sending each vertex of the hypercube graph into the corresponding unit hypercube vertex in Euclidean space. We denote the set of adjacent vertices of v as $\delta(v)$. Given distinct $x, y \in \mathbb{R}^n$ we let $[x, y]$ be the closed segment joining x, y ((x, y) is an open segment, $(x, y]$ and $[x, y)$ are semi-closed segments). For each $\{u, v\} \in E$ we let $[u, v] = [\iota(u), \iota(v)]$, and $\overline{[u, v]}$ be the line containing $[u, v]$. We denote by $H^n = \{\iota(v) \mid v \in V\}$ and by $\bar{H}^n = \bigcup_{\{u,v\} \in E} [u, v]$. Given a set $T \in \mathbb{R}^n$ of n linearly independent points, we let $(\pi(T), \pi_0(T)) \in \mathbb{R}^{n+1}$ be a vector $(\pi_1, \ldots, \pi_n, \pi_0)$ such that $\pi x = \pi_0$ is the hyperplane passing through all the points in T. Given $y \notin \text{aff}(T)$, let $(\pi(T, y), \pi_0(T, y)) \in \mathbb{R}^{n+1}$ be such that $\pi x = \pi_0$ for all $x \in T$ and $\pi y > \pi_0$. For all $j \in \{1, \ldots, n\}$ we denote by e_j the j-th unit coordinate direction vector $(0_1, \ldots, 1_j, \ldots, 0_n)$, and let $e = \sum_{j=1}^n e_j$ be the vector with all entries set to 1. Let $F = \{x \in \{0,1\}^n \mid Ax \leq b\}$ be the feasible region of problem P, which we assume to be non-empty, and $\bar{F} = \{x \in [0,1]^n \mid Ax \leq b\}$ its continuous relaxation. The continuous relaxation $\bar{P}$ of P is the problem $\min\{cx \mid x \in \bar{F}\}$. Let $F^\circ = \{x \in F \mid \delta(\iota^{-1}(x)) \in F\}$ be the *integral interior* of F, namely the set of hypercube points feasible in P such that their n adjacent points in C^n are also feasible in P. For all $i \in \{1, \ldots, m\}$ let A_i be the i-th row of A, so that $A_i x \leq b_i$ is the i-th problem constraint; let $R_i = \{x \in \mathbb{R}^n \mid A_i x = b_i\}$ and $\bar{R}_i = \{x \in \mathbb{R}^n \mid A_i x \leq b_i\}$. Given a solution x' of $\bar{P}$, let $I(x')$ be the set of active constraint indices.

2 The flattening operator

For $i \leq m$ and $\{u, v\} \in E$, we consider the set $N_i^{uv} = R_i \cap [u, v]$. The following facts hold:

(1) N_i^{uv} is either a single point, or empty, or the whole segment $[u, v]$.
(2) If $|N_i^{uv}| = 1$, R_i is a separating hyperplane for the singleton sets $\{\iota(u)\}$,

$\{\iota(v)\}$; furthermore, $A_i\iota(u) \le b_i \Leftrightarrow (\iota(v)-\iota(u))A_i > 0$, and $A_i\iota(u) > b_i \Leftrightarrow (\iota(v)-\iota(u))A_i < 0$.

(3) If $N_i^{uv} = [u,v]$, then $(\iota(v)-\iota(u))A_i = 0$ and both $\iota(u), \iota(v)$ are in $\bar{R}_i$.

For all $i \le m$ let

$$N_i = \bigcup_{\{u,v\}\in E} N_i^{uv}.$$

Lemma 1 *For all $i \le m$ and $p \in N_i$, there exists at most one component of p that is fractional.*

For $i \le m$ and $p \in N_i$ such that p is not integral, we denote by $f(p)$ the unique fractional component index of p. Define:

$$\lfloor p \rfloor = (p_1, \ldots, \lfloor p_{f(p)} \rfloor, \ldots, p_n).$$
$$\lceil p \rceil = (p_1, \ldots, \lceil p_{f(p)} \rceil, \ldots, p_n).$$

For integral p, we let $\lfloor p \rfloor = \lceil p \rceil = p$ and $f(p) = -1$. For $\{u,v\} \in E$ we define $u < v$ if there is $j \le n$ such that $(\iota(v)-\iota(u)) = e_j$, and $u > v$ if there is $j \le n$ such that $(\iota(v)-\iota(u)) = -e_j$. Assuming $u < v$ and $f(p) \ge 0$, it is straightforward to show that $\lfloor p \rfloor = \iota(u)$ and $\lceil p \rceil = \iota(v)$; furthermore, $(\lceil p \rceil - \lfloor p \rfloor) = e_{f(p)}$.

To each $p \in N_i$ $(i \le m)$ we associate the "closest" feasible integral point. For $i \le m$ and $p \in N_i$ we define the *flattening of p* as:

$$\Phi(p) = \begin{cases} \lfloor p \rfloor & \text{if } \lfloor p \rfloor \in \bar{R}_i, \lceil p \rceil \notin \bar{R}_i \\ \lceil p \rceil & \text{if } \lceil p \rceil \in \bar{R}_i, \lfloor p \rfloor \notin \bar{R}_i \\ \{\lfloor p \rfloor, \lceil p \rceil\} & \text{if } \lfloor p \rfloor, \lceil p \rceil \in \bar{R}_i \end{cases}$$

Let $N = \bigcup_{i \le m} N_i$ be the set of all intersection points of the hyperplanes defining the problem constraints with the unit hypercube edges. The *flattening of N* is $\Phi(N) = \{\Phi(p) \mid p \in N\} \cap \bar{F}$.

Theorem 2 *$\{F^\circ, \Phi(N)\}$ is a partition of F.*

The main limitation of Thm. 2 is that for any given $i \le n$ and $\{u,v\} \in E$, $|N_i^{uv}|$ is generally not polynomial in n, but depends on the number of edges in the unit hypercube, which is $\sum_{d=1}^{n} \binom{n}{d} d$.

We recall that a facet of P is a hyperplane $\pi x = \pi_0$ such that $\dim \operatorname{aff}(\{x \mid \pi x = \pi_0\} \cap \operatorname{conv}(F)) = n$. The following results characterizes the extent to which flattened points can be used to derive facets of $\operatorname{conv}(F)$.

Theorem 3 *Assume dim aff$(F) = n$. Let $W \subseteq \Phi(N)$ such that (a) $|W| = n$, (b) $\forall\ w \in F^\circ\ (\pi(W)w \neq \pi_0(W))$. Then $\pi(W)x = \pi_0(W)$ is a facet of conv(F).*

In practice, Thm. 3 cannot really be used to derive facets because testing condition (b) would yield exponential time complexity. We can, however, derive some cutting planes by flattening just one point at a time.

Proposition 4 *For $i \leq m$ and $p \in N_i$ such that p is not integral, let $q_1, \ldots, q_{n-1}$ be the vertices adjacent to p in the $(n-1)$-dimensional polyhedron $R_i \cap \bar{H}^n$, and $W = \{\Phi(p), q_1, \ldots, q_{n-1}\}$. Then $\pi(W, p)x \leq \pi_0(W, p)$ is a cutting plane for P.*

The cutting planes described in Prop. 4 are called *flattening inequalities*. Their most interesting feature is that they always pass through a polyhedron vertex. It is reasonable to expect that they should contribute to a faster convergence of cutting planes type algorithms by accelerating the identification of the optimal integral solution specially towards the end of the search (when the relaxed optima are expected to be nearer hypercube edges).

3 Conclusion

In this paper we discussed a characterization of the feasible region of Binary Linear Programming problems in terms of interior and exterior points, and showed that this characterization is useful to derive some cutting planes.

References

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