From non-linear Perron-Frobenius theory to static analysis

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Survey of joint works with: Éric Goubault, Sylvie Putot, Sarah Zennou, Akur Taly, and Assale Adje.

this talk: **correspondence** (almost an isomorphism) between

Zero sum games

and

fixed point problems in **static analysis** by abstract interpretation

Tools: **nonlinear Perron-Frobenius theory**, convex analysis.

Nonlinear Perron-Frobenius theory...

Krein-Rutman, Krasnoselskii, . . . see Nussbaum 88 (AMS memoir)

a cone C (convex, closed, $C \cap (-C) = \{0\}$)

most important example $C = \mathbb{R}^n_+$

Define: $x \leq y \iff y - x \in C$. If $C = \mathbb{R}^n_+$, this is the standard order.

$f : C \to C$ is **monotone** if $x \leq y \implies f(x) \leq f(y)$.

Find x: $f(x) = x$? Asymptotics $f^k(y), k \to \infty$?

The classical Perron-Frobenius theorem

Perron (1907) and Frobenius (1912) studied the special case: $C = \mathbb{R}^n_+$ and $f(x) = Ax$ linear.

f monotone $\implies A_{ij} \geq 0$.

Eigenproblem: $Au = \lambda u, \;\; u \in \mathbb{R}^n_+ \setminus 0$.

1) Existence holds in general, the uniqueness of u is guaranteed if $G := \{(i,j) \mid \; A_{ij} > 0\}$ is strongly connected (A irreducible), then $u \in (\mathbb{R}^*_+)^n$;

2) If $\rho(A) = 1$ (then $\lambda = 1$) and A is irreducible, for all x, $A^k x$ converges as $k \to \infty$ to a periodic orbit the length of which divides the gcd of the lengths of circuits of $G(A)$.

Non-linear case

Monotonicity is not enough

One may require f to be not only monotone but *homogeneous*

(H): $f(\alpha x) = \alpha f(x),\ \alpha > 0$

Alternatively, one may look at $C = \mathbb{R}^n_+$ with **logarithmic glasses**:

$F := \log \circ f \circ \exp : \mathbb{R}^n \to \mathbb{R}^n$,

(M): $x \leq y \implies F(x) \leq F(y)$

If f is (H), then F is additively homogeneous

(AH): $F(\alpha + x) = \alpha + F(x)$

(M)+(AH) $\implies$ (N): $\|f(x) - f(y)\|_\infty \leq \|x - y\|_\infty$ (nonexpansiveness in the sup norm)

Motivations of nonlinear Perron-Frobenius theory

- Population dynamics . . .

- Mathematical economics . . .

- Diffusion on fractals . . .

- Discrete event systems . . .

- **Optimal control and game theory**

Dynamic operators of zero-sum repeated games with state space $\{1, \ldots, n\}$ are of the form:

$$f : \mathbb{R}^n \to \mathbb{R}^n \qquad f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + \alpha P_i^{ab} x)$$

$A(i)$ action space of player I, $B(i, a)$ action space of player II

$P_i^{ab} := (P_{ij}^{ab})$, proba. of moving $i \to j$,

$\sum_j P_{ij}^{ab} \leq 1$, case < 1 models termination/death.

r_i^{ab}: payment of Player I to player II

$0 < \alpha \leq 1$ discount factor, $-\log \alpha$ discount rate > 0, in the *infinite horizon game*

$$r_{i_0}^{a_0 b_0} + \alpha r_{i_1}^{a_1 b_1} + \alpha^2 r_{i_2}^{a_2 b_2} + \cdots$$

The value of the infinite horizon game is the fixed point v of f: $v = f(v)$

Rubinov and Singer have shown that any order preserving sup-norm nonexpansive map can be represented by such a game - even with deterministic transition probabilities P_i^{ab}.

Neyman, Sorin, Rosenberg, SG and Gunawardena, . . . have developed an "operator approach" of games using these properties.

Example: two player, deterministic.

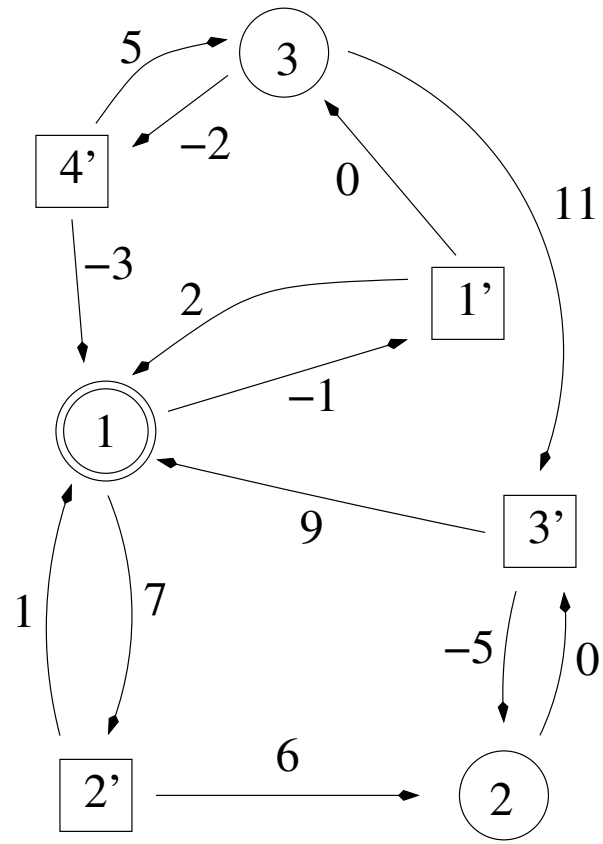
$G = (V, E)$ directed bipartite graph, r_{ij} weight of arc $(i, j) \in E$.

Two players, "Max", and "Min", move a pawn.

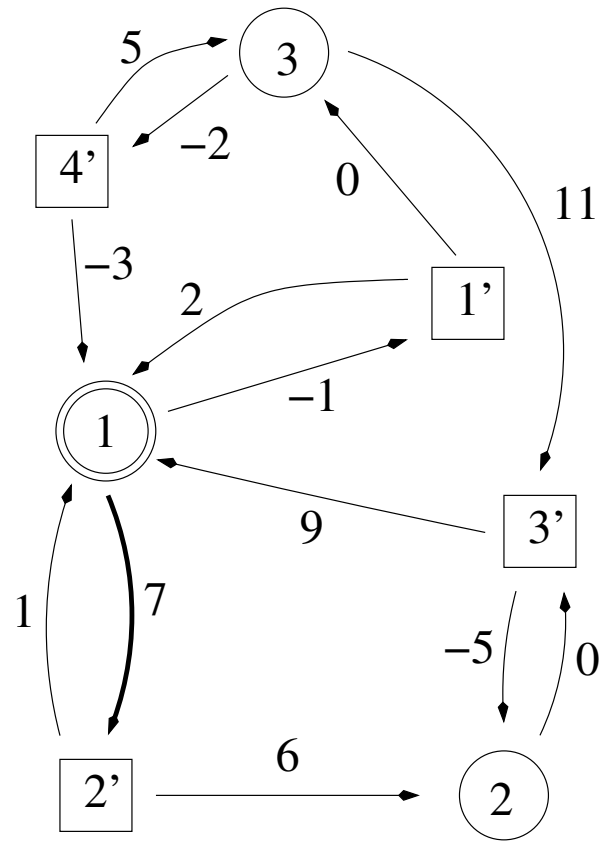
The pawn is initially at a given node $i_0 \in V$. The player who plays first, chooses an arc (i_0, i_1) in E, moves the pawn from i_0 to i_1, and Min pays $r_{i_0 i_1}$ to him. Then, the other player chooses an arc (i_1, i_2) in E, moves the pawn from i_1 to i_2, and pays $r_{i_1 i_2}$ to Max, etc.

The reward of Max (or the loss of Min) after k turns is

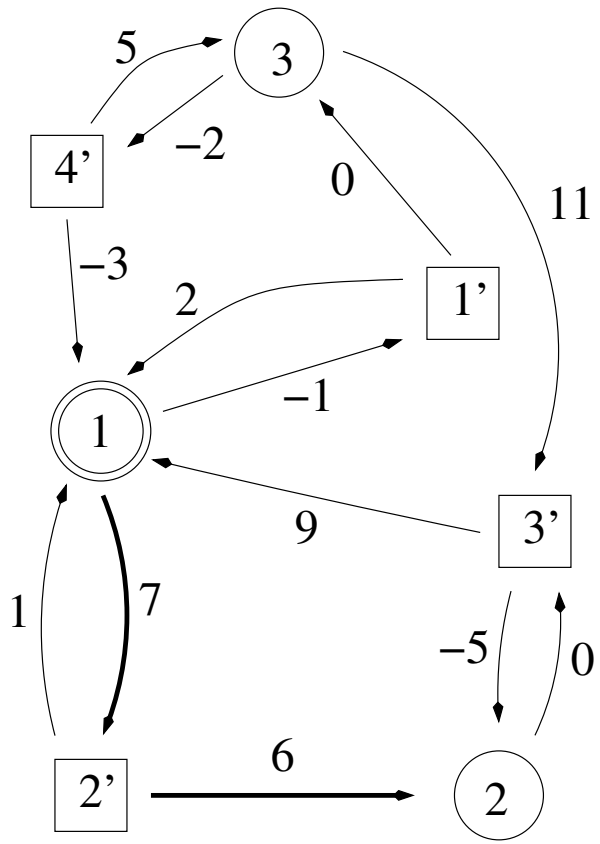
$$r_{i_0 i_1} + \cdots + r_{i_{2k-1} i_{2k}} \ .$$



The circles (resp. squares) represent the nodes at which Max (resp. Min) can play. The initial node, "1", is indicated by a double circle.



If Max initialy moves to $2'$. . .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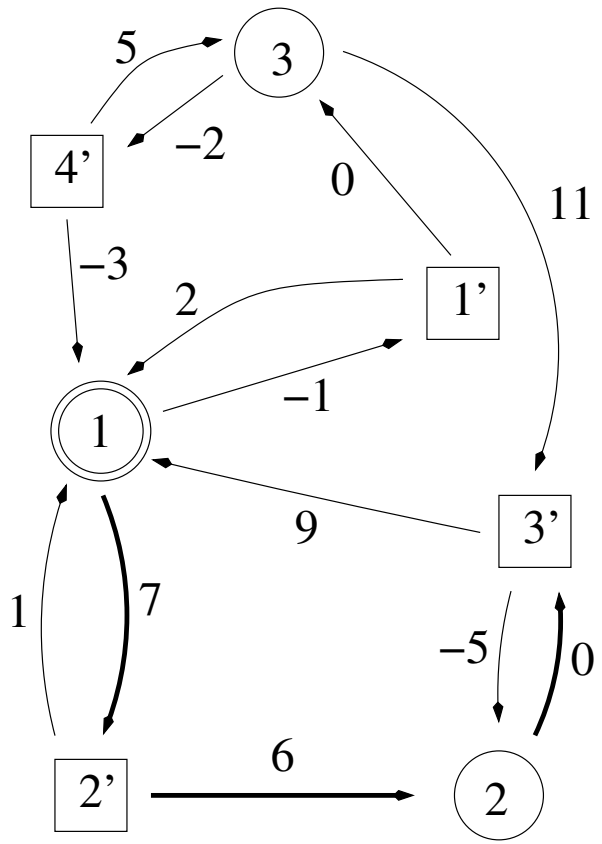
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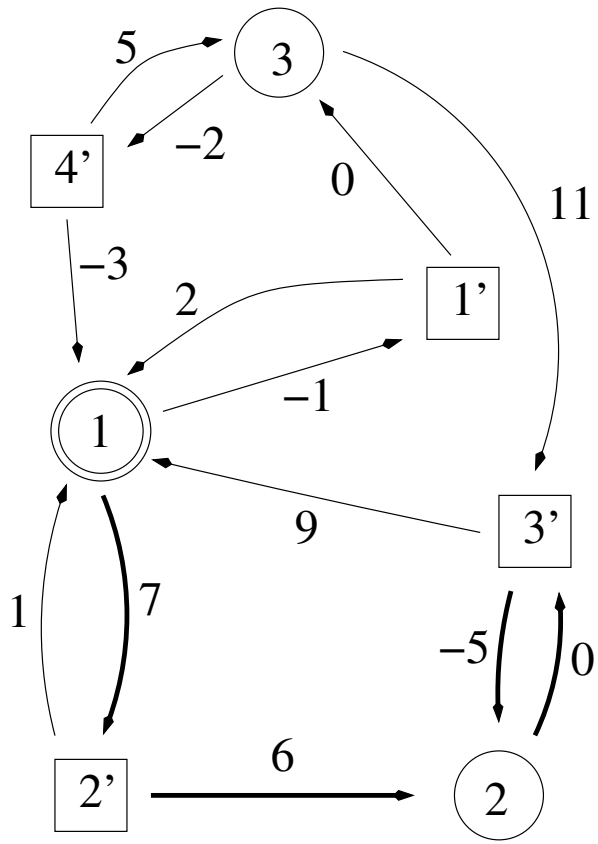
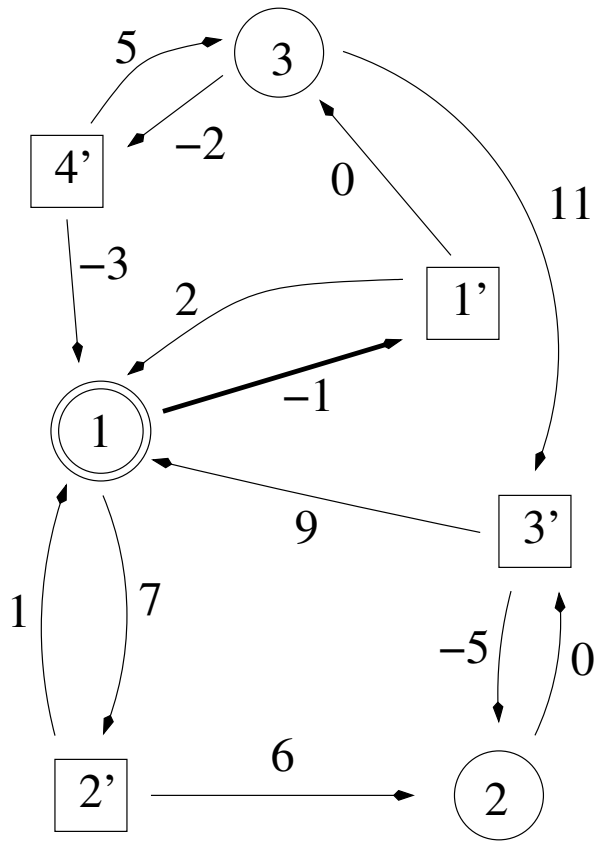
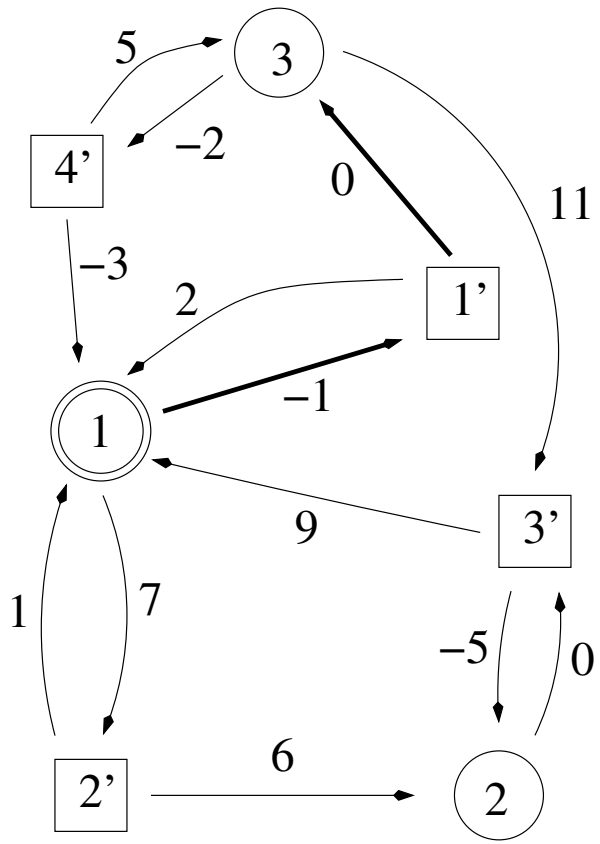


he eventually looses 5 per turn.



But if Max initialy moves to $1'$. . .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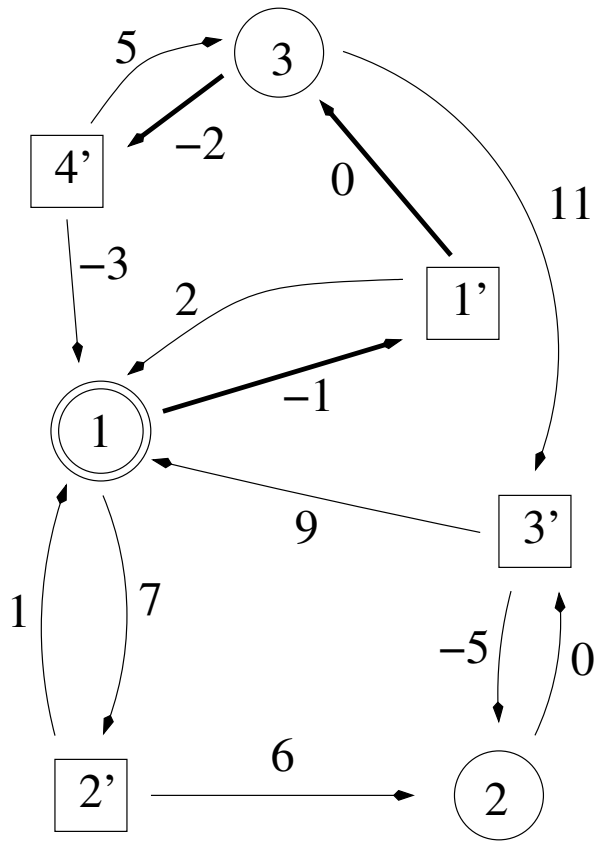
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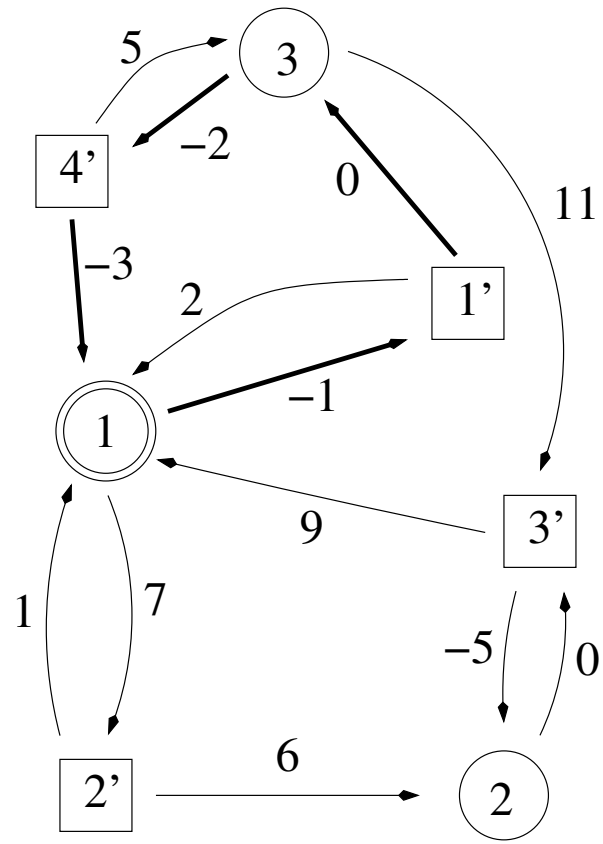


he only looses eventually $(1 + 0 + 2 + 3)/2 = 3$ per turn.

So the *value* of the game for Max, defined in terms of his mean payoff per turn, is -3.

Ehrenfeucht and Mycielski (Internat. J. Game Theory, 79): the value does exist for such games.

$$V = \{1, \ldots, p\} \cup \{1', \ldots, n'\}$$

$1, \ldots, p$: nodes in which it is Max's turn to play
$1', \ldots, n'$: nodes in which it is Min's turn to play.

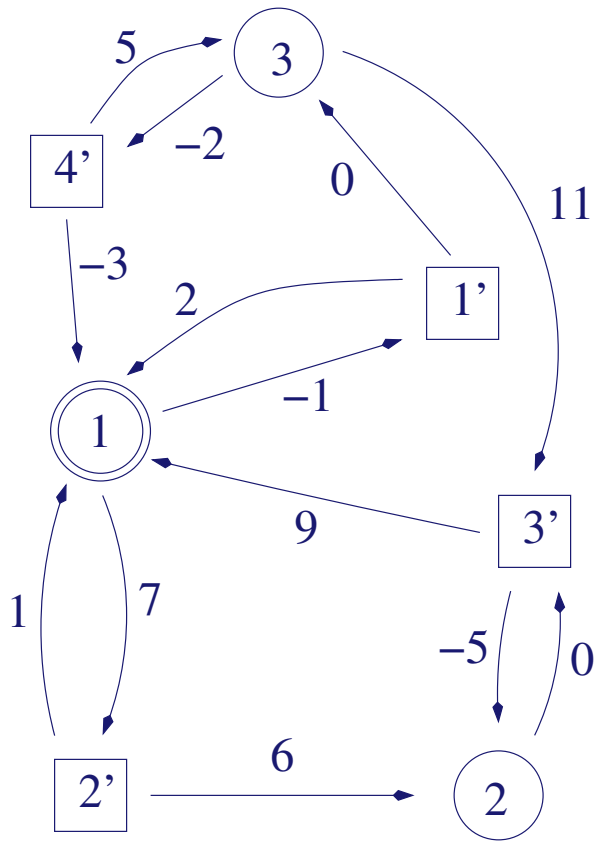
$v_j^N :=$ value of Max in horizon N, initial state j' (Min plays). The *value function* $v^N := (v_j^N)_{1 \le j \le n} \in \mathbb{R}^n$ satisfies

$$v^N = f(v^{N-1}), \qquad v^0 = 0, \qquad \text{where } f = g \circ g'$$

$$g : \mathbb{R}^p \to \mathbb{R}^n, \quad g_j(y) = \min_{(j',i) \in E} r_{j'i} + y_i$$

$$g' : \mathbb{R}^n \to \mathbb{R}^p, \quad g'_i(x) = \max_{(i,j') \in E} r_{ij'} + x_j$$

f is a min-max function: Olsder (JDEDS, 91) Gunawardena (JDEDS, 94).



$$g(y) = \begin{pmatrix} (2+y_1) \wedge y_3 \\ (1+y_1) \wedge (6+y_2) \\ (9+y_1) \wedge (-5+y_2) \\ (-3+y_1) \wedge (5+y_3) \end{pmatrix},$$

$$g'(x) = \begin{pmatrix} (-1+x_1) \vee (7+x_2) \\ x_3 \\ (-2+x_4) \vee (11+x_3) \end{pmatrix}$$

Let η_j denote the mean payoff with initial state j' for Min.

It is characterized by:

$\exists v \in \mathbb{R}^n, f(v + t\eta) = v + (t+1)\eta, \forall t \geq 0$

[Kohlberg (Maths OR, 80)]

$f(v) = v$ has a solution $v \in \mathbb{R}^n$ iff the mean payoff game has zero value for all initial states, i.e. $\eta = 0$

Static analysis of programs by abstract interpretation

Cousot: finding invariants of a program reduces to compute the smallest fixed point of a monotone self-map of a complete lattice L

To each breakpoint i of the program, is associated a set $x^i \in L$ which is an overapproximation of the set of reachable values of the variables, at this breakpoint.

x^i may be a cartesian product of intervals (one interval for each variable of the program)

The best x is the smallest solution of a fixed point problem $x = f(x)$ with f order preserving $L^n \to L^n$ ($n \leq \sharp$ breakpoints).

```
void main() {
  int x=0;            // 1
  while (x<100) {     // 2
    x=x+1;            // 3
  }                   // 4
}
```

$$\begin{array}{rcr} x_1 & = & [0,0] \\ x_2 & = &]-\infty, 99] \cap (x_1 \cup x_3) \\ x_3 & = & x_2 + [1,1] \\ x_4 & = & [100, +\infty[\cap (x_1 \cup x_3) \end{array}$$

Let $x_2^+ := \max x_2$. After some elimination, we arrive at

$$x_2^+ = \min(99, \max(0, x_2^+ + 1)) \ .$$

The smallest x_2^+ is 99, it is the value of a zero-sum game with a stopping option.

When does the fixed point problem of abstract intepretation reduce to a game problem ?

Does it work for

More general programs ?

More general domains ?

Some useful domains

1. **Zones** (Miné). Sets of the form

$Z = \{x \in \mathbb{R}^n \mid \ x_i - x_j \leq M_{ij}\}$

a zone is coded by the matrix $M \in (\mathbb{R} \cup \{+\infty\})^{n \times n}$.

by setting $x_0 := 0$ and projecting, we see that Zones $\supset$ Intervals.

2. **Polyhedra** (Cousot, Halbwachs 78. . .)

but the number of extreme points or faces may grow exponentially

$\rightarrow$ not scalable

3. Templates S. Sankaranarayanan and H. Sipma and Z. Manna (VMCAI'05)

almost as expressive as polyhedra but scalable.

I'll give a convex analytic view of templates.

The **support function** σ_X of $X \subset \mathbb{R}^n$ is defined by

$\sigma_X(p) = \sup_{x \in X} p \cdot x$

Legendre-Fenchel duality tells that $\sigma_X = \sigma_Y$ iff X and Y have the same closed convex hull.

$\sigma_X(\alpha p) = \alpha \sigma_X(p)$ for $\alpha > 0$, so it is enough to know $\sigma_X(p)$ for all p in the unit sphere.

Idea: discretize the unit sphere and represent X by σ_X restricted to the discretization points.

So fix $\mathcal{P} \subset \mathbb{R}^n$ a finite set of directions.

$L(\mathcal{P})$ lattice of sets of the form

$Z = \{x \mid p \cdot x \leq \gamma(p),\ \forall p \in \mathcal{P}\}, \quad \gamma : \mathcal{P} \to \mathbb{R} \cup \{+\infty\}.$

Z is coded by $\gamma := \sigma_Z \mid_{\mathcal{P}}$.

Z is a polyhedron every facet of which is orthogonal to some $p \in \mathcal{P}$.

Specialization: $\mathcal{P} = \{\pm e_i,\ i = 1, \ldots, n\}$ gives intervals, $\mathcal{P} = \{\pm(e_i - e_j),\ 1 \leq i < j \leq n\}$ gives Miné's templates.

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$\gamma(e_1) = +\infty$$
$$\gamma(-e_1) = -1$$
$$\gamma(e_2) = 10$$
$$\gamma(-e_2) = -\infty$$
$$\gamma(e_1 - e_2) = 0$$
$$\gamma(e_1 + 2e_2) = 21$$
$$\gamma(-e_1 - 2e_2) = -21 \ .$$

$\mathcal{P} = \{\pm e_1, \pm e_2, e_1 - e_2, \pm(e_1 + 2e_2)\}$, γ: breakpoint 1.

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

$$i \leq +\infty$$

$$i \geq 1$$

$$j \leq 10$$

$$j \geq -\infty$$

$$i \leq j$$

$$i + 2j \leq 21$$

$$i + 2j \geq 21$$

$(i, j) \in [(1, 10), (7, 7)]$ (exact result).

To reach this conclusion, we have to solve the fixed point problem:

$$\gamma(p) = ((1,10)\cdot p) \vee (\bar{\gamma}(p) + (2,-1)\cdot p), \qquad \forall p \in \mathcal{P} \setminus \{e_1 - e_2\}$$
$$\gamma(e_1 - e_2) = 0 \wedge (-9 \vee (\bar{\gamma}(e_1 - e_2) - 3))$$
$$\bar{\gamma} = \text{convex hull}(\gamma)$$

```
void main() {
 i = 1; j = 10;
 while (i <= j){ //1
  i = i + 2;
  j = j - 1; }
}
```

Correspondence theorem (SG, Goubault, Taly, Zennou, ESOP'07) When the arithmetics of the program is *affine* (no product or division of variables), abstract interpretation over a lattice of templates reduces to finding the smallest fixed point of a map $f : (\mathbb{R} \cup \{+\infty\})^n \to (\mathbb{R} \cup \{+\infty\})^n$ of the form

$$f_i(x) = \inf_{a \in A(i)} \sup_{b \in B(i,a)} (r_i^{ab} + M_i^{ab} x)$$

with $M_i^{ab} := (M_{ij}^{ab})$, $M_{ij}^{ab} \geq 0$, but possibly $\sum_j M_{ij}^{ab} > 1$

$\to$ game in infinite horizon with a "negative discount rate".

Sketch of proof.

$y = Ax + b$; If $x \in Z^1 := \{z \mid p \cdot z \leq \gamma^1(z),\ \forall p \in \mathcal{P}\}$, find the best $Z^2 := \{z \mid p \cdot z \leq \gamma^2(z),\ \forall p \in \mathcal{P}\}$ such that $y \in Z^2$.

$\gamma^2(p) = \sup_{x \in Z^1} p \cdot (Ax + b) = \sup p \cdot (Ax + b);\ p \cdot x \leq \gamma^1(p),\ \forall p \in \mathcal{P}$

by the strong duality theorem

$= \inf p \cdot b + \sum_{q \in \mathcal{P}} \lambda(q)\gamma^1(q);\ \ \lambda(q) \geq 0,\ A^T p = \sum_{q \in \mathcal{P}} \lambda(q) q$

The inf is attained at an extreme point of the feasible set, so this is in fact a min over a finite set.

$\sigma_{X \cap Y} =$ convex hull$(\inf(\sigma_X, \sigma_Y))$.

Convex hull reduces to a finite min by a similar argument.

Modelling the dataflow yields maxima, because $\sigma_{X \cup Y} = \sup(\sigma_X, \sigma_Y)$

How to solve the fixed point problem ?

Classically: Kleene (fixed point iteration) is slow or may even not converge, so widening and narrowing have been used, leading to an overapproximation of the solution.

An alternative: **Policy iteration.**

method developed by Howard (60) in stochastic control, extended by Hofman and Karp (66) to some special (nondegenerate) stochastic games. Extension to Newton method $\Longrightarrow$ fast. complexity still open.

extended by Costan, SG, Goubault, Martel, Putot, CAV'05) to fixed point problems in static analysis (difficulty: what are the strategies?)

experiments: PI often yields more accurate fixed points (because it avoids widening), small number of iterations.

A **strategy** is a map π which to a state i associates an action $\pi(i) \in A(i)$.

Consider the one player dynamic programming operator:

$$f_i^\pi(x) := \sup_{b \in B(i,\pi(i))} (r_i^{\pi(i)b} + M_i^{\pi(i)b} x)$$

$$f = \inf_\pi f^\pi$$

and the set $\{ f^\pi \mid \pi \text{ strategy } \}$ has a *selection*:

$$\forall v \in \mathbb{R}^n, \ \exists \pi \qquad f(v) = f^\pi(v) \ .$$

Since f^π is convex and piecewise affine, finding the smallest finite fixed point of f^π (if any) can be done by linear programming:

$$\min \sum_i v_i; \quad f^\pi(v) \leq v \ .$$

Can we compute the smallest fixed point of f from the smallest fixed points of the f^π?

We denote by f^- the smallest fixed point of a monotone self-map f of a complete lattice $\mathcal{L}$, whose existence is guaranteed by Tarski's fixed point theorem.

Theorem (Costan, SG, Goubault, Martel, Putot CAV'05). *Let $\mathcal{G}$ denote a family of monotone self-maps of a complete lattice $\mathcal{L}$ with a lower selection, and let $f = \inf \mathcal{G}$. Then $f^- = \inf_{g \in \mathcal{G}} g^-$.*

The input of the following algorithm consists of a finite set $\mathcal{G}$ of monotone self-maps of a lattice $\mathcal{L}$ with a lower selection. When the algorithm terminates, its output is a fixed point of $f = \inf \mathcal{G}$.

1. *Initialization*. Set $k = 1$ and select any map $g_1 \in \mathcal{G}$.

2. *Value determination*. Compute a fixed point x^k of g_k.

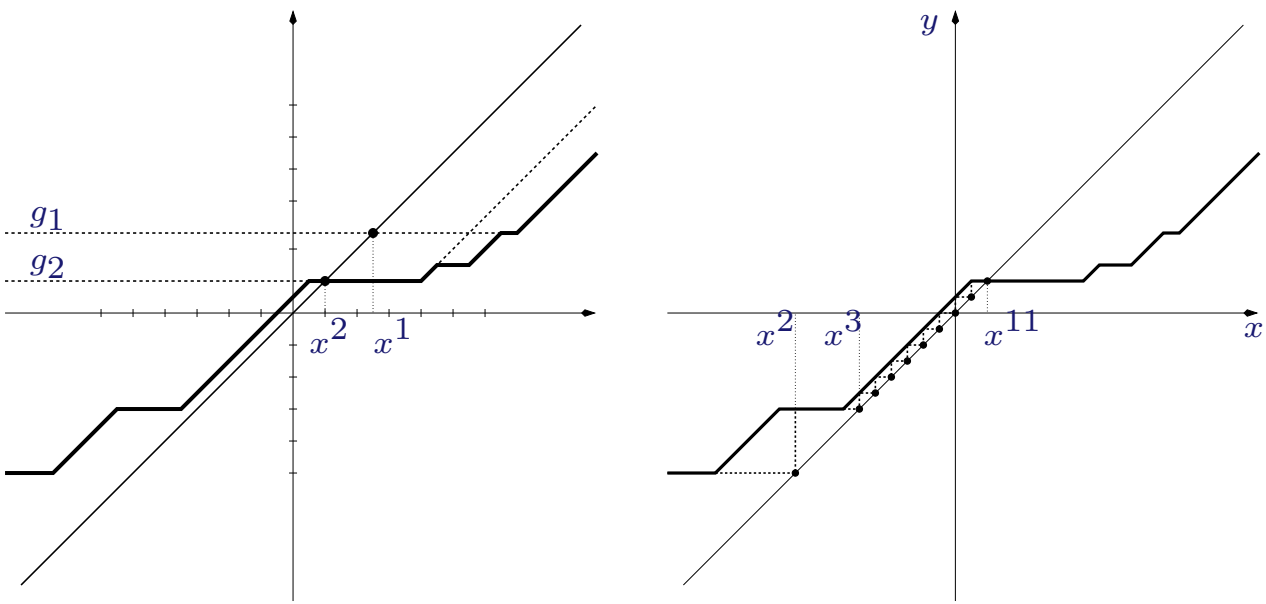
3. Compute $f(x^k)$.

4. If $f(x^k) = x^k$, return x^k.

5. *Policy improvement*. Take g_{k+1} such that $f(x^k) = g_{k+1}(x^k)$. Increment k and goto Step 2.

The algorithm does terminate when at each step, the smallest fixed-point of g_k, $x^k = g_k^-$ is selected.

Example. Take $\mathcal{L} = \overline{\mathbb{R}}$, and consider the self-map of $\mathcal{L}$, $f(x) = \inf_{1 \le i \le m} \max(a_i + x, b_i)$, where $a_i, b_i \in \mathbb{R}$. The set $\mathcal{G}$ consisting of the m maps $x \mapsto \max(a_i + x, b_i)$ admits a lower selection.



Experimentally fast, but the worst case complexity is not known. Condon showed: mean payoff games is in NP $\cap$ co-NP, same with positive discount. Much current work: (Zwick, Paterson, TCS 96), (Jurdziński, Paterson, Zwick, SODA'06), (Bjorklund, Sandberg, Vorobyov, preprint 04),

PI often more accurate than Klenne+widening/narrowwing:

```
0     i = 150;
1     j = 175;
2     while (j >= 100){
3       i++;
4       if (j<= i){
5           i = i - 1;
6           j = j - 2;
7       }
8   }
9
```

$$
\begin{array}{lcl}
M_0 & = & \mathit{context\_initialization} \\
M_2 & = & (\mathit{Assignment}\ (i \leftarrow 150,\ \ j \leftarrow 175)(M_0))^* \\
M_3 & = & ((M_2 \sqcup M_8) \sqcap (j \geq 100))^* \\
M_4 & = & (\mathit{Assignment}\ (i \leftarrow i+1)(M_3))^* \\
M_5 & = & (M_4 \sqcap (j \leq i))^* \\
 & & \\
M_7 & = & (\mathit{Assignment}\ (i \leftarrow i-1,\ \ j \leftarrow j-2)(M_5))^* \\
M_8 & = & ((M_4 \sqcap (j > i))^* \sqcup M_7 \\
M_9 & = & ((M_2 \sqcup M_8) \sqcap (j < 100))^*
\end{array}
$$

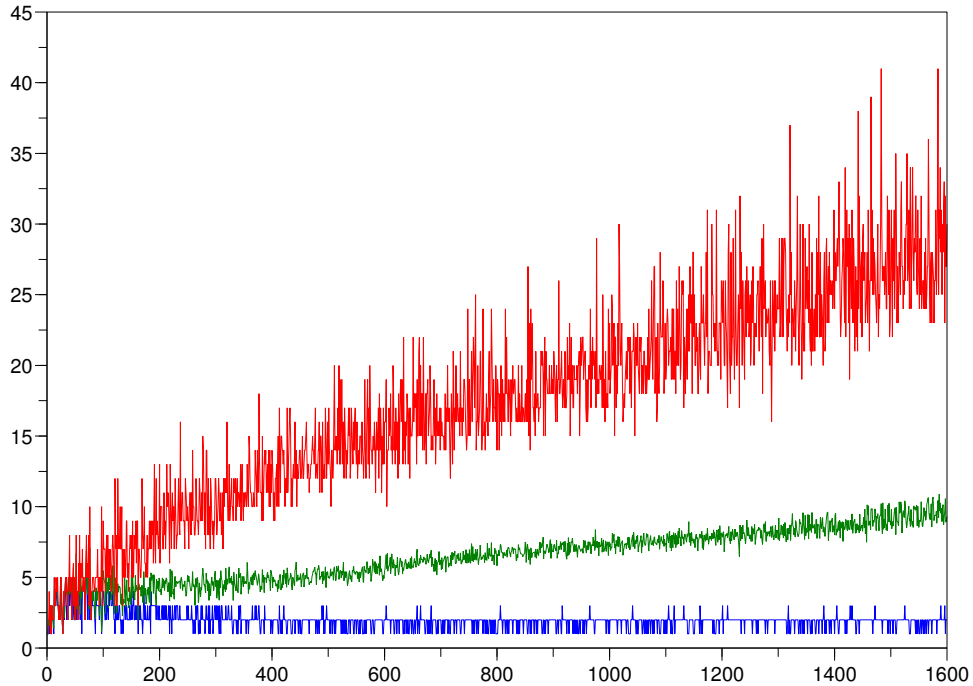
IP $\left\{ \begin{array}{c} 150 \leq i \leq 174 \\ 98 \leq j \leq 99 \\ -76 \leq j - i \leq -51 \end{array} \right.$

Mine's Octogon $\left\{ \begin{array}{c} 150 \leq i \\ 98 \leq j \leq 99 \\ j - i \leq -51 \\ 248 \leq j + i \end{array} \right.$

Experiments (deterministic games)

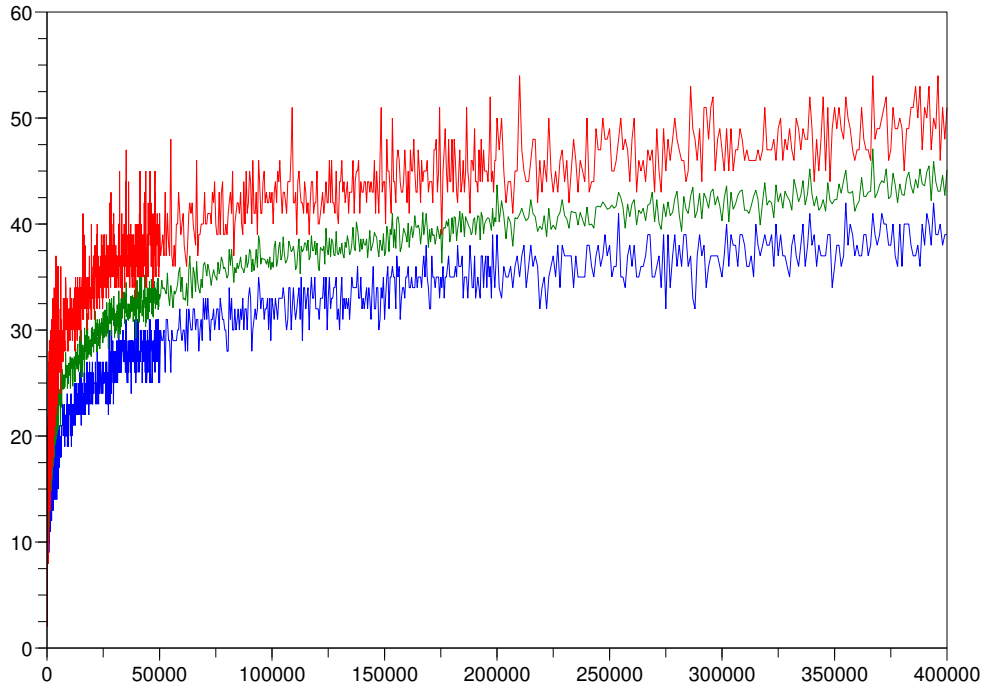
SG, Dhingra (Valuetools'06).

Complete bipartite graphs, in which $n = p$. Random weights (uniform).



$N_{\min}$= number of strategies of Min before the algorithm terminates. 100 graphs for each n, max, average, and min of $N_{\min}$ shown.

Sparse bipartite graphs. n nodes of each kind, every node has exactly 2 successors drawn at random; random weights.



Number of iterations of minimizer $N_{\min}$

Difficulty

PI may return a nonminimal fixed point.

We know there is a policy yielding the minimal fixed point.

How to find it?

Theorem (Costan, SG, Goubault, Martel, Putot CAV'05). *If f is nonexpansive (N) in the sup norm, if $f(v) = v$ with $v \in \mathbb{R}^n$, and if $\exists! \pi,\ f^\pi(v) = f(v)$, then, v is the smallest real fixed point of f.*

Proof uses a special case of a theorem of Nussbaum: if f is M and N from $\mathbb{R}^n$ to $\mathbb{R}^n$, there is a M and N retract $r = r^2$ such that $r(\mathbb{R}^n)$ is the fixed point set of f. In particular, the fixed point set is connected by arcs.

This may not be true if the discount rate is negative.

Define the semiderivative of f at point v by

$f(v+x) = f(v) + f'_v(x) + o(\|x\|)$ where f'_v is homogeneous of degree one, continuous, but possibly not linear.

E.g., $f(x) = \min(\max(1+x_1, x_2), x_3)$, $f'_{(0,0,0)}(x) = \min(x_1, x_3)$

Theorem (Assale Adje 07). *Assume that f is N (always in the sup norm) and semidifferentiable, and let v be such that $f(v) = v$.*

Then, v is the smallest real fixed point of f if: ($f'_v(x) = x$ and $x \leq 0$ implies $x = 0$).

We have to solve an auxiliary (simpler) fixed point problem,

Without nonexpansiveness, we have only a local minimality.

This condition can be checked using results on spectral radius.

Let $h : C \to C$ be M, continuous, and homogeneous of degree 1 (H).

Bonsall spectral radius

$\tilde{r}(h) := \lim_k \|h^k\|^{1/k}$ where

$\|h\| = \sup_x \|h(x)\|/\|h\|$

Cone eigenvalue spectral radius

$r(h) := \sup\{\lambda \mid \exists v \in C \setminus 0,\ h(v) = \lambda v\}$

Collatz-Wielandt value

$\bar{r}(h) := \inf\{\lambda \mid \exists v \in \text{int}\, C\ h(v) \leq \lambda v\}$

$r(h) \leq \tilde{r}(h) \leq \bar{r}(h)$

Nussbaum and Mallet-Parret . . . showed that these coincide in reasonable circumstances ($C = \mathbb{R}^n_+$ OK).

Let $C := \mathbb{R}^n_-$. To check that f'_v does not have a fixed point in $C \setminus 0$, it suffices that $\tilde{r}(f'_v) < 1$. Collatz-Wielandt value allows precisely to check that (termination).

Eigenvectors give descent direction allowing one to improve the strategy.

Eigenvector can be computed by the power algorithm, or by policy iteration algorithms for general population growth models (convergence proof only in a degenerate case).

Open question. How to find the smallest fixed point if f is not nonexpansive?

Should make big jumps.

See also Gawlitza and Seidl (ESOP'07) in another context.

Generalizations. Could repace linear maps by nonlinear maps in templates:

$Z = \{x \mid p(x) \leq \alpha(p),\ \forall p \in \mathcal{P}\}$.

E.g. p quadratic, can use Shor SDP relaxation.

But the original (linear) templates yield the only miraculous case in which no relaxation is needed!

That's all...