Acceleration Methods based on Interval Arithmetic in Deterministic Global Optimization

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Introduction

$\Longrightarrow$ Slove NLP

Problem

$$\begin{cases} \min\limits_{x \in X \subset \mathbb{R}^n} & f(x) \\ s.t. & \\ & g_i(x) \leq 0 \ , \ \forall i \in \{1, \ldots, n_g\} \\ & h_j(x) = 0 \ , \ \forall j \in \{1, \ldots, n_h\} \end{cases}$$

- Principe: Interval Branch&Bound Algorithm (IBBA)
- Problem: exponential complexity of memory and of time

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$\Longrightarrow$ acceleration techniques

Outline

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1. IBBA
   - Principe
   - Interval Arithmetic
2. Constraints Propagation Techniques
   - Principe
   - Example
3. Automatic Relaxation Techniques
   - Affine Arithmetic
   - Graphical Views
   - Reformulation Method
   - Example

IBBA applied to a problem with constraints

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- STOP $\Longrightarrow \max_{(Z,z)\in\mathcal{L}} Wid(Z) \leq \epsilon_L$
  $\Longrightarrow \widetilde{f(x)} - \min_{(Z,z)\in\mathcal{L}} z \leq \epsilon_f$

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$$\begin{aligned}\text{Example :}1/3 &\rightarrow [0.33333331, 0.33333335]\\ &\rightarrow [0.33333333333333331, 0.33333333333333338]\end{aligned}$$

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Every real number is represented by an interval of two floating point numbers which encloses it.
$\Rightarrow$ take care about numerical errors during calculations

$$f = 33.75y^6 + x^2(11x^2y^2 - y^6 - 121y^4 - 2) + 5.5y^8 + \frac{x}{2y}$$

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For $x = 77617$ and $y = 33096$, with FORTRAN double precision,

we obtain : $f = 1.1726039400531...$

But the real value is : $f = -\frac{54767}{66192} = -0.8273960599$

Interval Arithmetic

$$\begin{cases} [a,b]+[c,d]=[a+c,b+d] \\ [a,b]-[c,d]=[a-d,b-c] \\ [a,b]\times[c,d]=[\min\{a\times c,a\times d,b\times c,b\times d\}, \\ \qquad\qquad\qquad \max\{a\times c,a\times d,b\times c,b\times d\}] \\ [a,b]\div[c,d]=[a,b]\times[\frac{1}{d},\frac{1}{c}]\ si\ 0\notin[c,d] \end{cases}$$

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- Example :

$$\begin{aligned} [1;2]+[3;4] &= [4;6] \\ [4;6]-[3;4] &= [0;3] \\ [1;2]\times[3;4] &= [3;8] \\ [3;8]\div[3;4] &= [3/4;8/3] \\ [-1;1]\times([1;2]+[3;4]) &= [-6;4] \\ [-1;1]\times[1;2]+[-1;1]\times[3;4] &= [-6;6] \end{aligned}$$

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The natural extension into intervals of an expression of a function provides lower and upper bounds of this function over a box.

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Exact Bounds, $\forall x \in [1;2] \times [3;4]\ ,\ f(x) \in [6.75;13]$

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Propagation Techniques

$c(x) \in [a, b]$ is a contraint $\Longrightarrow$ implicit (or explicit) relations between the variables of the problem.

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Idea: use some deduction steps for reducing the box X.

Linear case: if $c(x) = \sum_{i=1}^{n} a_i x_i = [a, b]$ then:

$$X_k := \left(\frac{[a, b] - \sum_{i=1, i \neq k}^{n} a_i X_i}{a_k} \right) \cap X_k, \text{ si } a_k \neq 0. \tag{1}$$

where k is in $\{1, \cdots, n\}$ and X_i is the i^{th} component of X.

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Non-linear case:
Idea (E. Hansen): one linearizes using T_1 (or T_2). Then one solves a linear system with interval coefficients.
Other Idea: construction of the calculus tree and propagation.

Example of Propagation Technique based on the Calculus Tree

Let $c(x) = 2x_3x_2 + x_1$ and

$$c(x) = 3$$

where $x_i \in [1, 3]$ for all $i \in \{1, 2, 3\}$.

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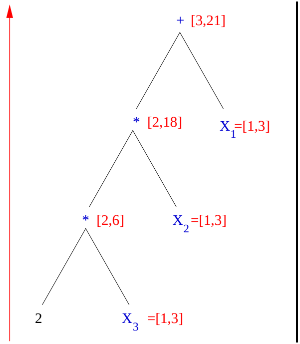

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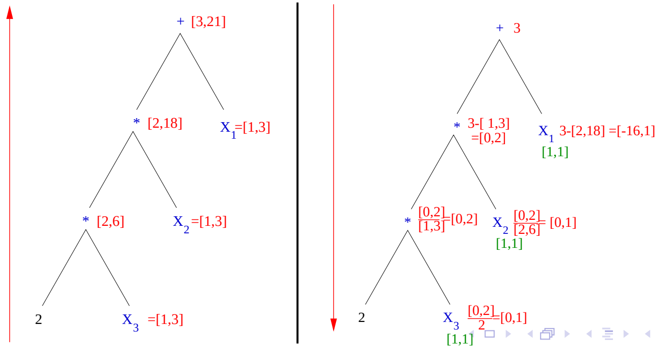

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Affine Arithmetic: MVA Andrade, J Comba, J Stolfi (1994)

Definition

All real numbers are represented by an affine form $\hat{x}$

$$\hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i$$

$$\textit{with } \forall i \in [1; n], x_i \in \mathbb{R} \textit{ and } \epsilon_i = [-1; 1]$$

- Example : with $n = 1$

$$\begin{array}{rcl} 1/3 & \rightarrow & 0.333 + 0.001 * [-1; 1] \\ log(2) & \rightarrow & 0.693 + 0.001 * [-1; 1] \end{array}$$

Affine Operator

$$\hat{x} \pm \hat{y} = (x_0 \pm y_0) + \sum_{i=1}^{n} (x_i \pm y_i)\epsilon_i$$

$$a \pm \hat{x} = (a \pm x_0) + \sum_{i=1}^{n} x_i \epsilon_i$$

$$a \times \hat{x} = a x_0 + \sum_{i=1}^{n} a x_i \epsilon_i$$

- Example : $A = [1;3]$ and $B = [-2;0]$

$$\begin{aligned} \hat{A} &\rightarrow 2 + \epsilon_1 \\ \hat{B} &\rightarrow -1 + \epsilon_2 \\ \hat{A} + \hat{B} &= 1 + \epsilon_1 + \epsilon_2 \end{aligned}$$

Non-Affine Operator

$$\begin{aligned} x \times y &= (x_0 + \sum_{i=1}^{n} x_i \epsilon_i) \times (y_0 + \sum_{i=1}^{n} y_i \epsilon_i) \\ &= x_0 y_0 + \sum_{i=1}^{n} (x_0 y_i + x_i y_0) \epsilon_i + \left(\sum_{i=1}^{n} \mid x_i \mid \times \sum_{i=1}^{n} \mid y_i \mid \right) \epsilon_{n+1} \end{aligned}$$

$$\hat{f}(\hat{x}) = \zeta + \alpha \hat{x} + \delta \epsilon_{n+1}$$

$$\textit{with } \alpha, \delta, \zeta \in \mathbb{R} \textit{ and } \hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i$$

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$\Longrightarrow$ All non-affine operations add a new variable

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In real, $f(x) \in [6.75;13]$

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Difficulties with AF

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- AF2 : $\hat{x} = x_0 + \sum_{i=1}^{n} x_i \epsilon_i + e\epsilon_{n+1} + e_+\epsilon_{n+2} + e_-\epsilon_{n+3}$

  with $\epsilon_{n+1} = [-1;1]$, $\epsilon_{n+2} = [0;1]$ and $\epsilon_{n+3} = [-1;0]$

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With AF1, $f(x) \in [5.5; 13]$

With AF2, $f(x) \in [6; 13]$

In real, $f(x) \in [6.75; 13]$

Visualization using a tree

$f(x_1, x_2) = x_1 * x_2 + \exp(x_1 + x_2)$, $x_1 = [-3, -1]$ and $x_2 = [3, 5]$

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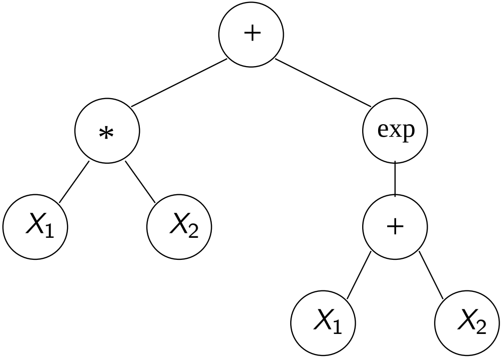

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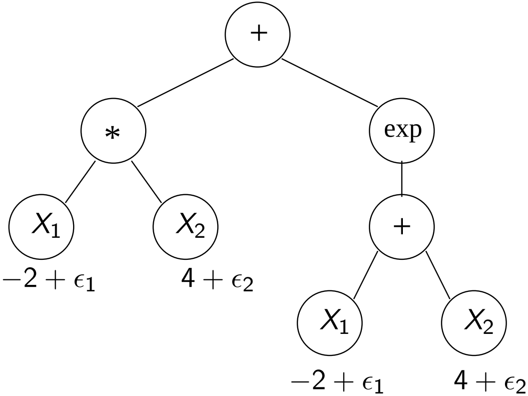

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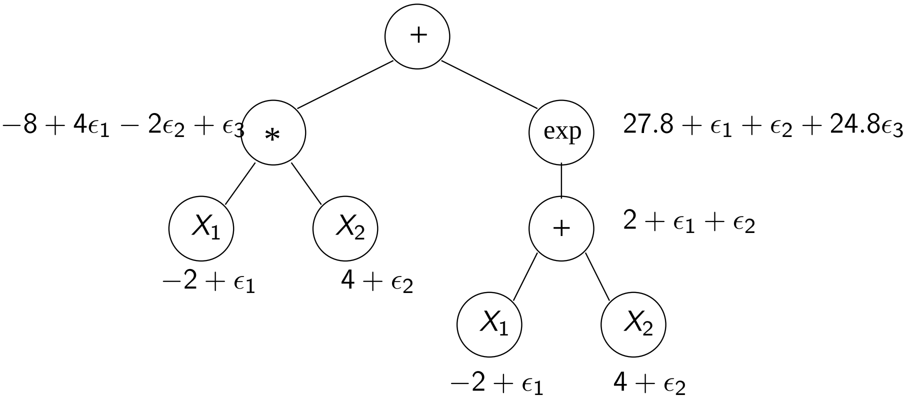
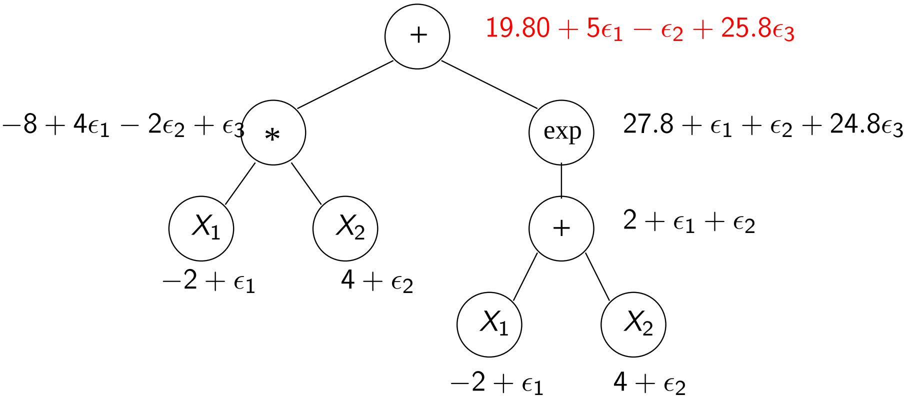
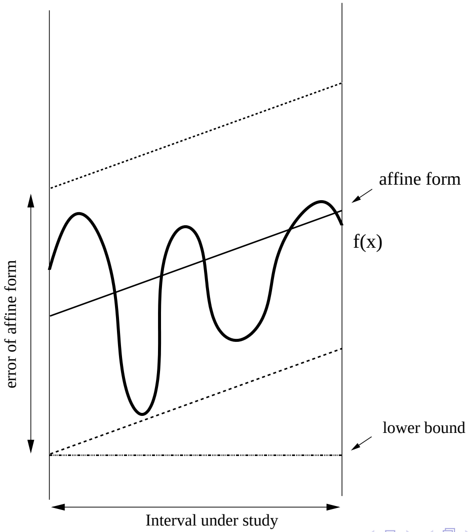


Visualization using a tree

$$f(x_1, x_2) = x_1 * x_2 + \exp(x_1 + x_2) \ , \ x_1 = [-3, -1] \text{ and } x_2 = [3, 5]$$



Graphical view



Notation

$$\forall x \in X \subset \mathbb{R}^n, f(x) - L_f(T(x)) \in E_f$$

where X is the domain under study,
T is the affine transformation of X to $[-1;1]^n$,
L_f is a linear function of$[-1;1]^n$ to $\mathbb{R}^n$,
$E_f = [\underline{E}_f; \overline{E}_f]$ is the interval corresponding to the error generated.

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With AF1, $L_f(\epsilon) = \sum_{i=1}^{n} x_i \epsilon_i$ and $E_f = x_0 + x_{n+1}\epsilon_{n+1}$

With AF2, $L_f(\epsilon) = \sum_{i=1}^{n} x_i \epsilon_i$ and $E_f = x_0 + e\epsilon_{n+1} + e_+\epsilon_{n+2} + e_-\epsilon_{n+3}$

Reformulation

$$\begin{cases} \min\limits_{x \in X^n} & f(x) \\ s.\ t. & \\ & g_i(x) \leq 0 \ , \ \forall i \in \{1, ..., n_g\} \\ & h_j(x) = 0 \ , \ \forall j \in \{1, ..., n_h\} \end{cases}$$

$$\begin{cases} \min\limits_{\epsilon \in [-1;1]^n} & L_f(\epsilon) \\ s.\ t. & \\ & L_{g\,i}(\epsilon) \leq -\underline{E}_{g_i} \ , \quad \forall i \in \{1, ..., n_g\} \\ & L_{hj}(\epsilon) \leq -\underline{E}_{h_j} \ , \quad \forall j \in \{1, ..., n_h\} \\ & -L_{hj}(\epsilon) \leq \overline{E}_{h_j} \ , \quad \forall j \in \{1, ..., n_h\} \end{cases}$$

$$\left\{ \begin{array}{ll} \min_{x \in [1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1 x_4 \\ & x_1 x_2 x_3 x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

$$x_3 + (x_1+x_2+x_3)x_1x_4 \Leftrightarrow \quad 18.98 + 3.44\epsilon_1 + 0.39\epsilon_2 + 0.64\epsilon_3 + 3.05\epsilon_4 + 1.13\epsilon_5$$

$$\left\{ \begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array} \right.$$

$$\begin{array}{rl} x_3 + (x_1 + x_2 + x_3)x_1x_4 \Leftrightarrow & 18.98 + 3.44\epsilon_1 + 0.39\epsilon_2 + 0.64\epsilon_3 + 3.05\epsilon_4 + 1.13\epsilon_5 \\ 25 - x_1x_2x_3x_4 \Leftrightarrow & -2.83 - 5.57\epsilon_1 - 1.46\epsilon_2 - 1.86\epsilon_3 - 5.57\epsilon_4 + 2.71\epsilon_5 \end{array}$$

$$\begin{cases} \displaystyle\min_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{cases}$$

$$\begin{aligned} x_3 + (x_1 + x_2 + x_3)x_1x_4 \Leftrightarrow & \quad 18.98 + 3.44\epsilon_1 + 0.39\epsilon_2 + 0.64\epsilon_3 + 3.05\epsilon_4 + 1.13\epsilon_5 \\ 25 - x_1x_2x_3x_4 \Leftrightarrow & \quad -2.83 - 5.57\epsilon_1 - 1.46\epsilon_2 - 1.86\epsilon_3 - 5.57\epsilon_4 + 2.71\epsilon_5 \\ x_1^2 + x_2^2 + x_3^2 + x_4^2 - 40 \Leftrightarrow & \quad -0.25 + 0.62\epsilon_1 + 2.38\epsilon_2 + 1.88\epsilon_3 + 0.63\epsilon_4 + 0.25\epsilon_5 \end{aligned}$$

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2=40 \end{array}\right.$$

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$$\left\{ \begin{array}{lll} \min\limits_{\epsilon\in[-1;1]^4} & 3.44\epsilon_1 + 0.39\epsilon_2 + 0.64\epsilon_3 + 3.05\epsilon_4 & \\ & -5.57\epsilon_1 - 1.46\epsilon_2 - 1.86\epsilon_3 - 5.57\epsilon_4 & \leq 5.54 \\ & 0.62\epsilon_1 + 2.38\epsilon_2 + 1.88\epsilon_3 + 0.63\epsilon_4 & \leq 0.5 \\ & -0.62\epsilon_1 - 2.38\epsilon_2 - 1.88\epsilon_3 - 0.63\epsilon_4 & \leq 0 \end{array} \right.$$

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

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Solution of linear program :

$$\begin{array}{ll} \epsilon_R & = (-1; -0.24; 1; -0.26) \\ L_f(\epsilon_R) & = -3.70 \end{array}$$

$$\left\{\begin{array}{ll}\min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40\end{array}\right.$$

$$\left\{\begin{array}{lll}\min\limits_{\epsilon\in[-1;1]^4} & 3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4 & \\ & -5.57\epsilon_1-1.46\epsilon_2-1.86\epsilon_3-5.57\epsilon_4 & \leq 5.54 \\ & 0.62\epsilon_1+2.38\epsilon_2+1.88\epsilon_3+0.63\epsilon_4 & \leq 0.5 \\ & -0.62\epsilon_1-2.38\epsilon_2-1.88\epsilon_3-0.63\epsilon_4 & \leq 0\end{array}\right.$$

Solution of linear program : $\begin{array}{ll}\epsilon_R & = (-1; -0.24; 1; -0.26) \\ L_f(\epsilon_R) & = -3.70\end{array}$

lower bound $= L_f(\epsilon_R) + E_f^L$

$$\left\{\begin{array}{ll} \displaystyle\min_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3 + (x_1 + x_2 + x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2 + x_2^2 + x_3^2 + x_4^2 = 40 \end{array}\right.$$

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Solution of linear program : $\epsilon_R = (-1; -0.24; 1; -0.26)$
$L_f(\epsilon_R) = -3.70$

Lower bound by IA = 12.5 ; by AF1 = 10.34375

$$\left\{\begin{array}{ll} \min\limits_{x\in[1;1.5]\times[4.5;5]\times[3.5;4]\times[1;1.5]} & x_3+(x_1+x_2+x_3)x_1x_4 \\ & x_1x_2x_3x_4 \geq 25 \\ & x_1^2+x_2^2+x_3^2+x_4^2 = 40 \end{array}\right.$$

$$\left\{\begin{array}{lll} \min\limits_{\epsilon\in[-1;1]^4} & 3.44\epsilon_1+0.39\epsilon_2+0.64\epsilon_3+3.05\epsilon_4 & \\ & -5.57\epsilon_1-1.46\epsilon_2-1.86\epsilon_3-5.57\epsilon_4 & \leq 5.54 \\ & 0.62\epsilon_1+2.38\epsilon_2+1.88\epsilon_3+0.63\epsilon_4 & \leq 0.5 \\ & -0.62\epsilon_1-2.38\epsilon_2-1.88\epsilon_3-0.63\epsilon_4 & \leq 0 \end{array}\right.$$

Solution of linear program : $\begin{array}{ll} \epsilon_R & = (-1;-0.24;1;-0.26) \\ L_f(\epsilon_R) & = -3.70 \end{array}$

Lower bound by IA $=$ 12.5 ; by AF1 $=$ 10.34375

New lower bound $=$ 14.15

Numerical Test

name	n	m	CP		AFFINE2+CP		AFFINE2	
			T(s)	Iter	T(s)	Iter	T(s)	Iter
TUY	4	2			4.95	**58 022**	18.40	134 219
2_1_1	5	1	0.02	26 309	0.00	**149**	0.00	149
2_1_2	6	2	0.03	118 209	1.08	113 545	0.90	**31 610**
2_1_3	13	9			1.25	114 439	0.43	**41 434**
2_1_4	6	5	0.00	9 881	0.10	**2 407**	0.15	2 598
2_1_5	10	11	1.83	1 449 757	37.45	**354 376**	45.25	392 234
2_1_6	10	5			0.03	**659**	0.02	660
3_1_2	5	6	0.00	11 330	0.03	**464**	0.12	595
3_1_3	6	6	0.02	34 148	0.25	**18 356**	0.18	3 794
3_1_4	3	3	0.02	35 902	0.02	**249**	0.03	291
7_3_1	4	7	0.05	30799	0.08	3592	0.12	**3444**
7_3_2	4	7	0.00	768	0.00	**155**	0.03	587
7_3_3	5	8	0.05	86282	0.03	**550**	0.12	1535
14_2_1	5	7	32.40	14 659 121	44.38	**944 184**	$>$	
14_2_2	4	5	0.02	11 708	0.10	**2 719**	0.18	6 378
14_2_3	6	9			110.03	2 177 710	37.25	**661 868**
14_2_4	5	7	11.92	2 284 630	1.10	**31 337**	2.58	54 418
14_2_5	4	5	0.03	20 656	0.07	**2 193**	0.25	5 129

Conclusion

Acceleration Methods based on expression trees:

- Constraints Propagation Techniques
- Automatic Relaxation Techniques