ARS Workshop

Markov Random Fields minimization and minimal cuts in image restoration

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Context

Exact total variation minimization

Total variation and regularization

TV models

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Conclusion

Conclusion

Perspectives
Main context

Image degradation

\[ v = Hu + \eta \]

- \( v \) → Observed image
- \( u \) → Original image
- \( \eta \) → Noise
- \( H \) → Linear degradation

Goal

Obtain the best estimation \( \bar{u} \) from \( u \) when \( H = \text{identity} \).
Energy minimization

First approach

Restoration corresponds to find the minimum of

\[ E(u, v) = \sum_{p \in \Omega} F_p(u_p, v_p) \]

with \( \Omega \subset \mathbb{R}^2 \).

- Inverse problem (Hadamard) \( \Rightarrow \) noise amplification (when \( H \neq \text{Id} \)).
- Need to regularize the solution.

\[ E(u, v) = \sum_{p \in \Omega} F_p(u_p, v_p) + \beta \cdot \sum_{p, q \in \Omega \atop \{p, q\} \in N} G_{p, q}(u_p, u_q) \quad \forall \beta \in \mathbb{R}^+ \]

\( \{p, q\} \in N \) indicates the neighborhood relation.

**Notations**
- General principle
- Maximum flow / minimal cut
- Energy representation
- Results
- More results
- Further results for 3D images

**Conclusion**
- Conclusion
- Perspectives
## Energy minimization

### Standard minimization methods

- **Continuous**
  - Gradient descent.
  - Graduated Non Convexity (GCN).

- **Discrete**
  - Dynamic programming (only in 1D).
  - Simulated annealing.
  - Iterated Conditional Modes.

### Problems

- No or poor convergence guarantees.
- Solution not ever optimal.
Exact total variation minimization
Regularization (Tikhonov)

- From: Introduce by A. N. Tikhonov in 1963
- Goal: Consider restoration as find the minimum of

\[ E(u) = \| u - v \|_{L^2}^2 + \beta \cdot \| \nabla u \|_{L^2}^2 \]

where \( \| u \|_{L^p} = \left( \int_{\Omega} |u(x)|^p \, dx \right)^{\frac{1}{p}} \)

Problem

"Cubes" image \( \sigma_b = 30 \)

Tikhonov restoration

Solution

- Regularize differently.
- Decrease the weight of big gradients.
**BV Space**

**Definition**

$BV \Rightarrow \text{Space of functions with bounded variations.}$

$$BV(\Omega) = \{ u \in L^1(\Omega) \mid \int_{\Omega} |\nabla u| < +\infty \}$$

Exact definition uses duality, because $|\nabla u|$ can be a measure.

with the semi-norm

$$|u|_{BV} = \int_{\Omega} |\nabla u| = TV(u) \quad \Rightarrow \quad \text{Total Variation}$$

**Advantages**

- Discontinuities are authorized along curves.
- Good space for geometric images.
- Existence and unicity of the solution.
Total Variation

Definition (co-area – continuous)

Let \( u \in BV(\Omega) \). Total variation of \( u \) is

\[
TV(u) = \int_{\Omega} |\nabla u| = \int_{\mathbb{R}} \int_{d\{u \leq \lambda\}} ds \ d\lambda,
\]

where \( \{u \leq \lambda\} \) is equivalent to \( \{u(x) \in \Omega \mid u(x) \leq \lambda\} \).

Definition (co-area – discrete)

Let \( u \) be a discrete function. Total variation of \( u \) is

\[
TV(u) = \sum_{\lambda=0}^{L-2} \sum_{\{p,q\} \in \mathcal{N}} w_{p,q} |u_p^\lambda - u_q^\lambda| \quad \text{where} \quad u_p^\lambda = \mathbf{1}_{\{u_p \geq \lambda\}}
\]

Remarks

- (-) Details suppression (textures).
- (+) Allows sharp contours.
**TV models**

**Definition**

Let $v \in L^1(\Omega)$ the observed image. The TV model consist of finding

$$\arg\min_{u \in BV(\Omega)} TV(u) + \beta \|u - v\|_{L^\alpha} \quad \alpha \in \{1, 2\}$$

**TV + $L_2$ Model / ROF (Rudin Osher Fatemi 92)**

- (+) Strictly convex $\Rightarrow$ unicity.
- (-) Lost of contrast (iterative regularization).
- Gaussian noise.

**TV + $L_1$ Model (Nikolova 2004)**

- (-) Convex $\Rightarrow$ not unicity.
- (+) No contrast lost.
- Impulsive noise.
Level set approach

**Principle**

1. Decompose the image in order to solve a succession of quadratic binary optimization problems $\bar{u}^\lambda$ (MRF)
2. Solve each problem $\bar{u}^\lambda$ where the solution is a level set
3. Reconstruct $\bar{u}$ from $\bar{u}^\lambda$ (trivial)

**Level set decomposition $\lambda$**

- Upper-set $\rightarrow U^\lambda(u) = \{p \in \Omega \mid u_p \geq \lambda\}$
- Lower-set $\rightarrow L^\lambda(u) = \{p \in \Omega \mid u_p \leq \lambda\}$

**Reconstruction**

$$u_p = \sup\{\lambda \in \mathcal{L} \mid p \in U^\lambda(u)\} \quad \forall p \in \Omega$$
Level set approach

Reformulation - $TV + L_1$

$$\arg\min_{u^\lambda \in \{0,1\}^N} E_1^\lambda(u^\lambda) = TV(u^\lambda) + \beta \sum_{p \in \Omega} [(1 - y_p)u_p^\lambda + y_p(1 - u_p^\lambda)]$$

with

$$y_p = 1\{v_p \geq \lambda\}$$

Reformulation - $TV + L_2$

$$\arg\min_{u^\lambda \in \{0,1\}^N} E_2^\lambda(u^\lambda) = TV(u^\lambda) + 2\beta \sum_{p \in \Omega} \left((\lambda - 0.5)u_p^\lambda + v_p(1 - u_p^\lambda)\right)$$
Level set approach

Reformulation

- Here, MRF are positive-negative quadratic pseudo-boolean functions, ie all the linear terms are positive and all the quadratic terms are negative (equivalent to submodular functions).
- Solve MRF is thus equivalent to find a maximal independent set in a bipartite graph, ie find a maximal flow – minimal cut in an associated graph.

Theorem

Minimizing $E$ is equivalent to minimizing all the $E^\lambda$ for each level.

Total energy $E(u) = \sum_{\lambda=0}^{L-2} E^\lambda(u^\lambda)$ can be minimized because $\{u^\lambda\}_{\lambda=0}^{L-2}$ is monotonous, ie:

$$\bar{u}^\lambda \leq \bar{u}^\mu \quad \forall \lambda < \mu.$$ 

The optimal solution is given by

$$\forall p \in \Omega, \bar{u}_p = \text{max}\{\lambda, \bar{u}^\lambda = 1\}.$$
Minimal cut (graph cut) as energy minimization
**Notations**

\( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) is a directed weighted graph with two terminals \( s, t \) where

- \( \mathcal{V} = \{1, \ldots, k\} \cup \{s\} \cup \{t\}, \ n = |\mathcal{V}| \)
- \( \mathcal{E} = \{(i, j) \mid 1 \leq i, j \leq n, i \neq j\}, \ m = |\mathcal{E}| \)
- **Capacity** \( \Rightarrow c : \mathcal{E} \rightarrow \mathbb{R}^+ \cup +\infty \)
- **Flow** \( \Rightarrow f : \mathcal{E} \rightarrow \mathbb{R} \)

**Vocabulary**

<table>
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<th>Node</th>
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<tr>
<td>s</td>
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<td>N-links</td>
<td>arcs ((i, j))</td>
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<tr>
<td>T-links</td>
<td>arcs ((s, i)) and ((i, t))</td>
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### Definitions

**Definition (flow)**

Let \( G \) be a graph. \( f(i, j) \) must verify

1) **Capacity constraints**

\[
f(i, j) \leq c(i, j)
\]

\( \forall i, j \in \mathcal{V} \) et \( \forall (i, j) \in \mathcal{E} \)

2) **Flow symmetry**

\[
f(i, j) = -f(j, i)
\]

\( \forall i, j \in \mathcal{V} \) et \( \forall (i, j) \in \mathcal{E} \)

3) **Kirchhoff law**

\[
\sum_{j \in \mathcal{V} - \{s, t\}}_{(i, j) \in \mathcal{E}} f(i, j) = 0
\]

\( \forall i \in \mathcal{V} - \{s, t\} \)

**Definition (cut)**

Cut is a partition \( C = (S, T) \) of \( \mathcal{V} \) such

\[
s \in S, \ t \in T \quad \text{et} \quad S \cap T = \emptyset, \ S \cup T = \mathcal{V}
\]

**Definition (Cut capacity)**

The capacity of a cut \( C \) is

\[
|C| = \sum_{i \in S, j \in T \atop (i, j) \in \mathcal{E}} c(i, j)
\]
General principle

**Theorem (Energy minimization (Greig Porteous Seheult 89))**

Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed weighted graph and $E$ be an energy function. $E$ can be minimized using a minimal cut in $\mathcal{G}$ for the image binary case.

**Principle**

1. Construct a graph $\mathcal{G}$.
2. Compute a minimal cut $\mathcal{C} = (\mathcal{S}, \mathcal{T})$ in $\mathcal{G} \Rightarrow$ minimize $E$.
3. Assign a value to each $u_p$ such that

$$
\begin{align*}
    u_p &= 0 \quad \text{if } p \in \mathcal{S} \\
    u_p &= 1 \quad \text{if } p \in \mathcal{T}
\end{align*}
$$
### Maximum flow algorithms

- **Augmenting paths**
  - **Principle**: Find iteratively a non saturated path from $s$ to $t$ in $\mathcal{G}$.
  - **Algorithms**:
    - Ford-Fulkerson $\rightarrow O(m \cdot f)$, where $f = \text{maximum flow}$
    - Edmonds-Karp $\rightarrow O(nm^2)$
    - Dinic $\rightarrow O(n^2m)$
    - Boykov-Kolmogorov $\rightarrow O(n^2m|\mathcal{C}|)$

- **Push-relabel**
  - **Principe**: Propagate an excess of flow repeatedly from $s$ to $t$ in $\mathcal{G}$.
  - **Algorithms**:
    - General push flow relabel $\rightarrow O(n^2m)$
    - Push flow relabel with dynamic trees $\rightarrow O(nm\log(n))$
Energy representation

Questions
- Which energies can be minimized via minimal cuts?
- How construct the graph to minimize $E$?

Definition (representation (Kolmogorov Zabih 02))

Let $E$ be an energy function with $n$ binary variables

$$E(x_1, \ldots, x_n) = \sum_i E_i(x_i) + \sum_{i<j} E_{i,j}(x_i, x_j) \quad \text{with} \quad x_i \in \{0, 1\}.$$  

Every function with one variable can be represented by a graph.
Every function with two variables can be represented by a graph iff

$$E_{i,j}(0, 0) + E_{i,j}(1, 1) \leq E_{i,j}(0, 1) + E_{i,j}(1, 0) \quad \text{(submodular)}$$

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Energy representation

\[ E^{i,j} = \begin{bmatrix} E^{i,j}(0,0) & E^{i,j}(0,1) \\ E^{i,j}(1,0) & E^{i,j}(1,1) \end{bmatrix} = \begin{bmatrix} A \\ C \end{bmatrix} \begin{bmatrix} B \\ D \end{bmatrix} \]

Energy \( E_i \) with
\[ E_i(0) < E_i(1) \]

Energy \( E_i \) with
\[ E_i(0) > E_i(1) \]

Energy \( E_{ij} \) with \( C > A \) and
\( C > D \)

Energy representation

Results
More results
Further results for 3D images

Conclusion
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Minimization algorithms

Sequential algorithm

- **Proposed by**: Darbon, Chambolle, Zalesky.
- **Principle**: Do $L$ independant optimizations.
- **Complexity**: $O(L \times F)$ with $O(F)$ the complexity to find the maximal flow – minimal cut.
- **Execution time**: $< 1$ min.

Dyadic algorithm

- **Proposed by**: Darbon, Chambolle, Hochbaum.
- **Principle**: Use the overlap between the level sets.
- **Complexity**: $O(\log_2(L))$.
Results

Tests characteristics
- Computer: AMD Athlon 64 X2 Dual Core 6000+, 2Go of RAM.
- Implementation under MegaWave2.
- Kolmogorov et al. library to compute the maximum flow.
- Images $256^2$ and $512^2$.
- Averages over 10 launchings.

Images
- Image “Circles”
- Image “Man”
- Image “Elaine”

Context
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## Results - $TV + L_1$

### Neighborhood: 4-connexity

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### Neighborhood: 8-connexity

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Results - $TV + L_2$

Neighborhood: 4-connexity

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More results

Tests characteristics
- Impulsive noise: $d_b = 20\%$ and $d_b = 40\%$.
- Gaussian noise: $\sigma_b = 15$ and $\sigma_b = 30$.
- Images $512^2$.
- Connexity 8.

Images
- Image “Cubes”
- Image “Man”

Context
Exact total variation minimization
- Total variation and regularization
- TV models
- Minimization

Minimal cut (graph cut) as energy minimization
- Notations
- General principle
- Maximum flow / minimal cut
- Energy representation
- Results

More results
- Further results for 3D images

Conclusion
- Conclusion
- Perspectives
Results - $TV + L_1 - d_b = 20\%$

Context

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Perspectives
Results - $TV + L_1 - d_b = 40\%$

Noise - $SNR = 0.47$

Result - $\beta = 0.65$

Level lines

Noise - $SNR = -0.88$

Result - $\beta = 2.8$
Results - $TV + L_2 - \sigma_b = 15$

Noise - SNR = 14.70

Result - $\beta = 0.04$

Level lines

Noise - SNR = 12.1

Result - $\beta = 0.1$
Results - $TV + L_2 - \sigma_b = 30$

Noise - $SNR = 8.82$

Result - $\beta = 0.03$

Level lines

Noise - $SNR = 6.25$

Result - $\beta = 0.06$
3D Images - Results - $TV + L^1$

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**Table:** Computation times (seconds) for $TV + L^1$ with 6 connexity. 3D images: $40^3$ and $80^3$. 

**Context**

**Exact total variation minimization**
- Total variation and regularization
- $TV$ models
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**Minimal cut (graph cut) as energy minimization**
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**Conclusion**
- Conclusion
- Perspectives
3D Images - Results - $TV + L^1$

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**Table:** Computation times (seconds) for $TV + L^1$ with 26 connexity. 3D images: $40^3$ and $80^3$. 
## 3D Images - Results - $TV + L^2$

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**Table:** Computation times (seconds) for $TV + L^2$ with 6 connectivity. 3D images: $40^3$ and $80^3$. 
3D Images - Results - $TV + L^2$

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<td></td>
<td>Dyadic</td>
<td>3.05</td>
<td>3.43</td>
<td>3.76</td>
<td>5.24</td>
<td>3.03</td>
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<tr>
<td>CELLULES-80</td>
<td>Sequential</td>
<td>262.14</td>
<td>277.04</td>
<td>373.07</td>
<td>294.75</td>
<td>239.85</td>
</tr>
<tr>
<td></td>
<td>Dyadic</td>
<td>50.98</td>
<td>61.38</td>
<td>126.01</td>
<td>67.98</td>
<td>48.13</td>
</tr>
</tbody>
</table>

Table: Computation times (seconds) for $TV + L^2$ with 26 connexity. 3D images: $40^3$ and $80^3$. 
Conclusion
Conclusion

**TV minimization**

- (+) Exact solutions.
- (+) Quick results.
- (-) Restricted energy classes.
- (-) Over-smoothing along the discontinuities.
Perspectives

Parametric flow

Objectif: re-use the flow value.

Conditions:
- Arcs \((s, i)\) → non-increasing capacities.
- Arcs \((i, t)\) → non-decreasing capacities.
- Arcs \((i, j)\) → constant capacities.

Results: Less improvements than for the dyadic technique (Darbon Chambolle 08).

Applications: interactive segmentation, video segmentation.
**Perspectives**

**Extension to multiway cut**

Multi-labelling

**Extension to other operators**

**Goal**: generalize restoration to other operators $H$ (convolution, sampling).

**Applications**: confocal microscopy, IRM.

**Extension to other energy minimization models**

Potts model