Curry-Howard Correspondence for Classical Logic

CENTRE NATIONAL DE LA RECHERCHE SCIENTIFIQUE

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Practicalities

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Slides . . .

Schedule: Tuesdays, from 16:15 to 19:15

7th December, 14th December, 3rd January, 10th January (guest lecture by Beniamino Accattoli)

Room : 2035, Sophie Germain building

Lecture I
Classical logic as a typing system

Contents

I. Introduction

Curry-Howard correspondence for Classical Logic

These lectures are part of the course

Logique linéaire et paradigmes logiques du calcul

(mostly)

Although, *polarity* and *focusing* -from linear logic- have played a major part in the understanding of C-H correspondence for Classical Logic.

(see e.g. Olivier Laurent's PhD work Laurent [2003])

Curry-Howard correspondence

One of two sides of the coin

"**computational interpretation** of a logic"

output of a computation = cut-free proof

Side 1 computation as proof search

(starting from a formula to prove)

Logic programming (see e.g. Dale Miller's course)

Side 2 computation as *composition* of proofs / cut-elimination

(starting from a proof with cuts)

Curry-Howard (see e.g. this course)

II. What works and what does not

Where it all works smoothly

Intuitionistic/minimal logic

Logic	*Programming language* λ-calculus	*Categories*
Propositions	Types	Objects
Proofs	Typed programs λ-terms	Morphisms
Cut/Composition	Program composition β-redex	Morphism composition
Where the stuff happens (Cut-elimination)	Execution β-reduction	Equality of morphisms (commuting diagrams)

Original correspondence

For minimal logic:

Curry	Combinators (S,K,I)	$\leftrightarrow$	Hilbert-style system
Howard Howard [1980]	Typed λ-terms	$\leftrightarrow$	Natural Deduction

I : $A \rightarrow A$

K : $A \rightarrow B \rightarrow A$ (provides erasure)

S : $(A \rightarrow (B \rightarrow C)) \rightarrow (A \rightarrow B) \rightarrow (A \rightarrow C)$ (provides duplication)

$$\mathbf{I}\ M \longrightarrow M$$

$$\mathbf{K}\ M\ N \longrightarrow M$$

$$\mathbf{S}\ M\ N\ P \longrightarrow M\ P\ (N\ P)$$

Original correspondence

For minimal logic:

Curry	Combinators (S,K,I)	$\leftrightarrow$	Hilbert-style system
Howard Howard [1980]	Typed λ-terms	$\leftrightarrow$	Natural Deduction

$$\frac{}{\Gamma, x{:}A \vdash x : A}$$

$$\frac{\Gamma, x{:}A \vdash M : B}{\Gamma \vdash \lambda x.M : A \to B} \qquad \frac{\Gamma \vdash M : A \to B \quad \Gamma \vdash N : A}{\Gamma \vdash M\ N : B}$$

$$(\lambda x.M)\ N \quad \longrightarrow_\beta \quad \{^N\!/_x\}M$$

$$\lambda x.M\ x \quad \longrightarrow_\eta \quad M \qquad \text{if } x \notin \mathsf{FV}(M)$$

$=_{\beta\eta}$ sound and complete for Cartesian Closed Categories (CCC)

Generalising the approach

Decorate proofs with syntactic terms: $\Gamma \vdash A$ becomes $\Gamma \vdash M : A$

Proof transformations

Reductions (execution of) M: $M \longrightarrow_{\mathcal{S}} N$

given by rewrite system $\mathcal{S}$.

Desired properties of the reduction system

- Progress, i.e. any term containing undesirable structures can be reduced.
- Subject reduction property, i.e. preservation of typing:

  If $\Gamma \vdash M : A$ and $M \longrightarrow_{\mathcal{S}} N$ then $\Gamma \vdash N : A$

1. Confluence, programs are deterministic
2. Normalisation (strong), i.e. execution of programs terminates.

Example

From minimal logic to *intuitionistic logic* = *minimal logic* + *"Ex falso Quodlibet"*

add rule:

$$\frac{\Gamma \vdash M : \bot}{\Gamma \vdash \text{abort}(M) : A}$$

Computational behaviour:

$$\text{abort}(M)\ N \longrightarrow\ \text{abort}(M)$$

In category theory: add to a CCC an *initial* object $\bot$

(i.e. for every object A there is a unique morphism $\bot \longrightarrow A$)

Now, remember that $\neg A$ is $A \rightarrow \bot$

How to get classical logic

Either add axiom schemes

$$(\neg\neg A) \Rightarrow A \qquad \text{(Elimination of double negation)}$$

$$((A \Rightarrow B) \Rightarrow A) \Rightarrow A \qquad \text{(Peirce's law)}$$

$$A \vee \neg A \qquad \text{(Law of excluded middle)}$$

In presence of "Ex falso Quodlibet" ($\bot \Rightarrow A$): *All equivalent*

Without "Ex falso Quodlibet": *only EDN⇒PL⇒LEM*

...Or with inference rules, for instance:

$$\frac{\Gamma, \neg A \vdash \bot}{\Gamma \vdash A} \text{ for EDN}$$

or by the structure of formalism cf. classical sequent calculus and right-contraction

The bad news

Take a CCC with initial object $\bot$

Require that for all object A, A is naturally isomorphic to $(A \Rightarrow \bot) \Rightarrow \bot$

The category collapses to a boolean algebra:

at most 1 morphism between any 2 objects

= cannot distinguish 2 proofs of the same theorem

Quite useless for a theory of proofs, or for the proofs-as-program paradigm

Has classical logic a computational content?

Girard, Lafont, Taylor *Proofs and types*: No Girard et al. [1989]

III. A bit of history

The notion of continuation

Program execution flow:

$\downarrow$	code P that has been executed, producing data v
v	its output
$\downarrow$	code E that remains to be executed, consuming data v

Continuation

= programming environment/context within which some code is executed

The notion of continuation

. . . is also useful for compiling recursive calls

```
myfunction(a1,...,an){
 some code;
 x = myfunction(a1',...,an');
 some code possibly using x;
}
```

When executing recursive call, whole environment must be saved to resume computation (code that remains to be executed + state of memory).

Not needed if `some code possibly using x` is empty (tail recursion).

Trick = pass it to the recursive call as a “continuation” `c'` :

```
myfunction(a1,...,an,c){
 some code;
 return myfunction(a1',...,an',c');
}
```

The notion of continuation in λ-calculus

Program execution flow:

$\downarrow$ code P that has been executed, producing data v

v its output

$\downarrow$ code E that remains to be executed, consuming data v

can be seen in

- P is a λ-term that is reduced
- E is the context, in the syntactic sense (a term with a hole $E[\ ]$)

$E[\ ]$ can also be seen as a function $\lambda x.E[x]$

The notion of control

In pure λ-calculus, P has no knowledge of $E[\,]$ while being evaluated.

Control =

letting a program know and manipulate its environment/continuation

getting "unpure features", modelling `goto` instructions

- Reynolds Reynolds [1972], Strachey-Wadsworth Strachey and Wadsworth [2000] (re-edition of 74)

  on continuations and *call-with-current-continuation (call-cc): cc*

  Added to programming langage `Scheme`

- Felleisen's PhD work Felleisen [1987] on Syntactic Theory of Control: the $\mathcal{C}$ *operator*

Connection with Logic (89-90)

The general idea:

$$
\begin{array}{lcl}
E[\text{abort}(M)] & \longrightarrow & M \\
E[\text{cc } M] & \longrightarrow & E[M\ (\lambda x.E[x])] \\
E[\mathcal{C}\ M] & \longrightarrow & M\ (\lambda x.E[x])
\end{array}
$$

In presence of abort( ), cc and $\mathcal{C}$ are interdefinable:

$$
\begin{array}{lcll}
\mathcal{C}\ M & := & \text{cc}\ (\lambda k.\text{abort}(M\ k)) & k \notin \text{FV}(M) \\
\text{cc}\ M & := & \mathcal{C}\ (\lambda k.k\ (M\ k)) & k \notin \text{FV}(M)
\end{array}
$$

Griffin Griffin [1990]:

cc can be typed by $((A \to B) \to A) \to A$

$\mathcal{C}$ can be typed by $(\neg\neg A) \to A$

Central question about control

$$E[\textsf{abort}(M)] \longrightarrow M$$
$$E[\textsf{cc}\ M] \longrightarrow E[M\ (\lambda x.E[x])]$$
$$E[\mathcal{C}\ M] \longrightarrow M\ (\lambda x.E[x])$$

Above rules are not “standard” rewrite rules. . .

What exactly does $E[\,]$ stand for / range over?

More fundamentally:

What kind of continuation can be captured by a control operator?

Is the capture delimited? undelimited? etc

One proposed formalisation: Parigot's $\lambda\mu$-calculus Parigot [1992]

$$\begin{array}{lrcl} \text{Terms} & M, N, P \ldots & ::= & x \mid \lambda x.M \mid M\ N \mid \mu\alpha.c \\ \text{Commands} & c & ::= & [\alpha]M \end{array}$$

$$\frac{}{\Gamma, x : A \vdash x : A; \Delta}$$

$$\frac{\Gamma, x : A \vdash M : B; \Delta}{\Gamma \vdash \lambda x.M : A \to B; \Delta} \qquad \frac{\Gamma \vdash M : A \to B; \Delta \quad \Gamma \vdash N : A; \Delta}{\Gamma \vdash M\ N : B; \Delta}$$

$$\frac{c : (\Gamma \vdash\ ; \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A; \Delta} \qquad \frac{\Gamma \vdash M : A; \alpha : A, \Delta}{[\alpha]M : (\Gamma \vdash\ ; \alpha : A, \Delta)}$$

Parigot's $\lambda\mu$-calculus Parigot [1992]

Extra reduction rules:

$$
\begin{array}{lcl}
(\mu\alpha.c)\ N & \longrightarrow & \mu\beta.\{{}^{[\beta]M\ N}/_{[\alpha]M}\}\,c \\
[\beta]\mu\alpha.c & \longrightarrow & \{{}^{\beta}/_{\alpha}\}\,c
\end{array}
$$

Integrates Peirce's law: cc $:= \ \lambda x.\mu\alpha.[\alpha](x\ \lambda y.\mu\beta.[\alpha]y)$ $\quad : ((A\to B)\to A)\to A$

Consider that contexts are of the form $E[\,] = [\gamma]([\,]\ N_1 \ldots N_n)$

If given a top-level continuation variable top $:\bot$ (Ariola-Herbelin Ariola and Herbelin [2003]),

- integrates "Ex falso quod libet" $\quad \lambda x.\mu\alpha.[\text{top}]x \quad : \bot\to A$
- integrates DNE $\quad \mathcal{C} := \ \lambda x.\mu\alpha.[\text{top}](x\ \lambda y.\mu\beta.[\alpha]y) \quad : (\neg\neg A)\to A$

So far, so good

Symmetry

There's something symmetric about classical logic:

- Boolean algebras
- De Morgan duality:

$$\neg(A \wedge B) = \neg A \vee \neg B$$

$$\neg(A \vee B) = \neg A \wedge \neg B$$

- Classical Sequent calculus LK

  left-contraction symmetric to right-contraction (≠ intuitionistic logic)

So far, not explicit in our proof theory + term calculi

Filinski Filinski [1989]: first formalisation of a duality between

- functions as values
- functions as continuations

Symmetric λ-calculus, with explicit conversions from one view to the other

No explicit connection with logic. Is there one?

Barbanera - Berardi and after

Yes, there's one.

See BB's symmetric λ-calculus Barbanera and Berardi [1996]: Natural Deduction with continuations

It is more like a one-sided sequent calculus

Symmetry of the calculus corresponds to symmetry/duality of LK

Other calculi for (bi-sided) versions of LK, with cut-elimination as computation:

- Urban's calculus Urban [2000],
- Curien-Herbelin's $\overline{\lambda}\mu\widetilde{\mu}$ Curien and Herbelin [2000] for $\Rightarrow$ (easier in bi-sided sequent calculus),

  later extended by Wadler Wadler [2003] for $\wedge$ and $\vee$ (connecting to De Morgan)

Two independent works

Curien-Herbelin's aim:

Express duality of computation syntactically (with a Filinski-like calculus)

Semantics, no proof of SN.

Urban's aim:

Have a typing system as close as possible to LK, have a reduction system as close as possible to basic cut-elimination procedures

SN, but no semantics.

Then: a broad literature on comparing such calculi.

Curien-Herbelin-Wadler - syntax

terms	t	$::= x \mid \mu\beta.c \mid \lambda x.t \mid \langle t_1, t_2 \rangle \mid \mathsf{inj}_i(t)$
continuations	e	$::= \alpha \mid \mu x.c \mid t :: e \mid \langle e_1, e_2 \rangle \mid \mathsf{inj}_i(e)$
commands	c	$::= \langle t \bullet e \rangle$

Intuition:

$x, y, \ldots$:	inputs (variables standing for terms)
$\alpha, \beta, \ldots$:	outputs (variables standing for continuations)
terms =	some inputs (free term variables)
	+ one main output
	+ alternative outputs (free continuation variables)
continuations =	one main input
	+ additional inputs (free term variables)
	+ some outputs (free continuation variables)
commands =	a term facing a continuation (this interaction creates computation)

Curien-Herbelin-Wadler - typing

$$\frac{}{\Gamma, x : A \vdash x : A \;;\; \Delta} \qquad \frac{}{\Gamma \;;\; \alpha : A \vdash \alpha : A, \Delta}$$

$$\frac{\Gamma, x : A \vdash t : B \;;\; \Delta}{\Gamma \vdash \lambda x.t : A \to B \;;\; \Delta} \qquad \frac{\Gamma \vdash t : A \;;\; \Delta \quad \Gamma \;;\; e : B \vdash \Delta}{\Gamma \;;\; t :: e : A \to B \vdash \Delta}$$

$$\frac{\Gamma \vdash t_1 : A_1 \;;\; \Delta \quad \Gamma \vdash t_2 : A_2 \;;\; \Delta}{\Gamma \vdash \langle t_1, t_2 \rangle : A_1 \wedge A_2 \;;\; \Delta} \qquad \frac{\Gamma \;;\; e : A_i \vdash \Delta}{\Gamma \;;\; \mathsf{inj}_i(e) : A_1 \wedge A_2 \vdash \Delta}$$

$$\frac{\Gamma \vdash t : A_i \;;\; \Delta}{\Gamma \vdash \mathsf{inj}_i(t) : A_1 \vee A_2 \;;\; \Delta} \qquad \frac{\Gamma \;;\; e_1 : A_1 \vdash \Delta \quad \Gamma \;;\; e_2 : A_2 \vdash \Delta}{\Gamma \;;\; \langle e_1, e_2 \rangle : A_1 \vee A_2 \vdash \Delta}$$

$$\frac{c : (\Gamma \vdash \alpha : A, \Delta)}{\Gamma \vdash \mu\alpha.c : A \;;\; \Delta} \qquad \frac{c : (\Gamma, x : A \vdash \Delta)}{\Gamma \;;\; \mu x.c : A \vdash \Delta}$$

$$\frac{\Gamma \vdash t : A \;;\; \Delta \quad \Gamma \;;\; e : A \vdash \Delta}{\langle t \bullet e \rangle : (\Gamma \vdash \Delta)}$$

Example: Law of Excluded Middle

A story: The devil, the fool, and the $1000000. (borrowed from Phil Wadler)

- I have an offer for you! My promise is:

Either I offer you $1000000

or, if you give me $1000000

then I will grant you any wish

I choose to offer you the latter.

- Here's $1000000! I want immortality.

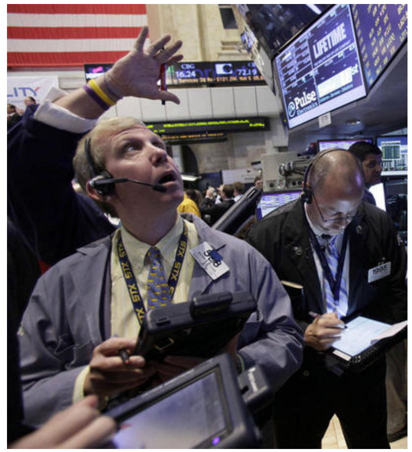
- Well done and thank you!

Now, I've changed my mind.

I've now decided to fulfill my promise

by offering you $1000000.

Here is your money back!



Questions?

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