On Hypersequents and Labelled Sequents
Translating Labelled Sequent Proofs to Hypersequent Proofs

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Extensions of Gentzen-style Sequent Calculi

Extensions to Gentzen-style sequent calculi obtained by changing to specific syntactic features [Paoli] in order to control proof search for non-classical logics, such as:

- Labelled Systems
- Multiple Sequents (e.g. higher-order sequents, hypersequents)
- Multi-sided Sequents
- Multi-arrow Sequents (e.g. sequents of relations)
- Multi-comma Systems (e.g. Display Logics)
- Deep Inference Systems (e.g. Calculus of Structures)

Many systems are hybrids of these, such as nested sequents or relational hypersequents.
Why Compare Formalisms?

- Interface vs implementation (automated proof assistants)
- Translating proofs of meta properties.
- Novel and interesting rules obtained from other formalisms.
- Formal criteria for comparing formalisms.
- Illuminate the meaning of particular syntactic features.
- Use abstraction to conceive of new extensions? (akin to juggling notation...)
- Develop a hierarchy of the strength of proof systems.
Why Compare Labelled Sequents and Hypersequents?

- Folklore about relationship, but no published formal comparison beyond specific calculi (mainly for \( S5 \)).
- There are labelled and hypersequent calculi for overlapping sets of logics. (Here we look at some Int\( ^* \) logics.)
- A comparison of the rules for some logics suggests a relationship...
Labelled Systems

- The language of formulae is extended with a language of annotations to control inference, e.g.

\[
\frac{\Gamma \Rightarrow \Delta, A^y}{\Gamma \Rightarrow \Delta, \Box A^x} \quad \mathcal{R}\Box
\]

where \( y \) is fresh for the conclusion.
- Additional kinds of formulae based on labels may be used for controlling inference, e.g. \( \mathcal{R}xy \).
- Easily obtained using the relational semantics of a logic.
Syntax of Labelled Sequents

- Formulae in a sequent are annotated with **labels**, e.g. $A^x$.

  \[ \Gamma_1^{x_1}, \ldots, \Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_n} \]

- Sequents may also contain **relational formulae** which indicate a relationship between labels, e.g. $R_{xy}$.

  \[ R_{x_1 i_1 y_1}, \ldots, R_{x_k i_k y_k}, \Gamma_1^{x_1}, \ldots, \Gamma_n^{x_n} \Rightarrow \Delta_1^{x_1}, \ldots, \Delta_n^{x_n} \]

- In some calculi, labels may be complex expressions, or may contain variables...

- ...relational formulae may be $n$-ary, occur on either side, or even be “first class” and combined with formulae, e.g. $R_{xy} \land (A \lor B)^x$. 

The Simple Relational Calculus G3I

▶ A labelled calculus with atomic labels and binary relations.
▶ A fragment of the calculus G3I from [Negri, 2005]:

\[ R_{xy}, \Sigma; P^x, \Gamma \Rightarrow \Delta, P^y \]

\[ R_{xy}, \Sigma; (A \supset B)^x, \Gamma \Rightarrow \Delta, A^y \]
\[ R_{xy}, \Sigma; (A \supset B)^x, B^y, \Gamma \Rightarrow \Delta \]
\[ R_{xy}, \Sigma; (A \supset B)^x, \Gamma \Rightarrow \Delta \]

\[ L \supset \]

\[ R_{xy}, \Sigma; A^y, \Gamma \Rightarrow \Delta, B^y \]
\[ \Sigma; \Gamma \Rightarrow \Delta, (A \supset B)^x \]
\[ R \supset \]

The rules for \( \land \), \( \lor \) and \( \bot \) are standard.

▶ The **pure relational rules** (or “ordering rules”):

\[ R_{xx}, \Sigma; \Gamma \Rightarrow \Delta \]
\[ \Sigma; \Gamma \Rightarrow \Delta \]
\[ \text{refl} \]

\[ R_{xz}, R_{xy}, R_{yz}, \Sigma; \Gamma \Rightarrow \Delta \]
\[ R_{xy}, R_{yz}, \Sigma; \Gamma \Rightarrow \Delta \]
\[ \text{trans} \]
[Pinto & Uustalu, 2009] give a similar calculus for \textbf{BiInt}, with (aside from the dual of $\supset$) contraction as a primitive rule and replacing the axiom with

$$\Sigma; A^x, \Gamma \Rightarrow \Delta, A^x$$

$$\frac{R_{xy}, \Sigma; A^x, A^y, \Gamma \Rightarrow \Delta}{R_{xy}, \Sigma; A^x, \Gamma \Rightarrow \Delta} \quad L_{\text{mono}} \quad \frac{R_{xy}, \Sigma; \Gamma \Rightarrow \Delta, A^x, A^y}{R_{xy}, \Sigma; \Gamma \Rightarrow \Delta, A^y} \quad R_{\text{mono}}$$

The mono rules are derivable in \textbf{G3I} using cut, e.g.:

$$\vdots$$

$$\frac{R_{xy}, \Sigma; A^x, \Gamma \Rightarrow \Delta, A^y}{R_{xy}, \Sigma; A^x, A^y, \Gamma \Rightarrow \Delta} \quad \frac{R_{xy}, \Sigma; A^x, A^y, \Gamma \Rightarrow \Delta}{R_{xy}, \Sigma; A^x, \Gamma \Rightarrow \Delta} \quad \text{cut}$$
Geometric Rules

- **A geometric rule** is a $\mathcal{G}3$-style rule of the form

$$
\frac{[\hat{z}/y] \Sigma_1, \Sigma_0, \Gamma \Rightarrow \Delta \ldots [\hat{z}/y] \Sigma_n, \Sigma_0, \Gamma \Rightarrow \Delta}{\Sigma_0, \Gamma \Rightarrow \Delta}
$$

where the variables $\hat{z}$ do not occur free in the conclusion, and each $\Sigma_i$ is a multiset of atoms.

- Geometric rules can be added to $\mathcal{G}3$-style calculi without affecting admissibility of cut, weakening or contraction. [Negri 2005] [Simpson 1994].

- **A geometric implication** [Palmgren 2002?] is a formula of the form $\forall \bar{x}.(A \supset B)$, without $\supset, \forall$ in subformulae of $A, B$. They are constructively equivalent to:

$$
\forall \bar{x}.((P_{10} \land \ldots \land P_{k0}) \supset \exists \bar{y}.((P_{11} \land \ldots \land P_{k1}) \lor \ldots \lor (P_{1n} \land \ldots \land P_{kn})))
$$

- Frame conditions of many logics in $\text{Int}^*$ are geometric implications.
Extending G3I for Geometric Intermediate Logics

- Adding rules that correspond to frame conditions of logics...

  - Adding the “directedness” rule yields a calculus for \textit{Jan}:
    \[
    \frac{Rx\hat{z}, Ry\hat{z}, Rxw, Ryw, \Sigma; \Gamma \Rightarrow \Delta}{Rxw, Ryw, \Sigma; \Gamma \Rightarrow \Delta} \quad \text{dir}
    \]

  - Adding the “linearity rule” yields a calculus for \textit{GD}:
    \[
    \frac{Rxy, \Sigma; \Gamma \Rightarrow \Delta \quad Ryx, \Sigma; \Gamma \Rightarrow \Delta}{\Sigma; \Gamma \Rightarrow \Delta} \quad \text{lin}
    \]

  - Adding the “symmetry” rule yields a calculus for \textit{Cl}:
    \[
    \frac{Rxy, Ryx, \Sigma; \Gamma \Rightarrow \Delta}{Rxy, \Sigma; \Gamma \Rightarrow \Delta} \quad \text{sym}
    \]

- Weakening, contraction and cut admissibility is preserved.
Hypersequents

- Attributed to [Avron] although similar calculi occur in earlier work by [Beth], [Sambin & Valentini], [Pottinger].
- A hypersequent is a non-empty list/multiset of sequents

\[ \Gamma_1 \Rightarrow \Delta_1 \mid \ldots \mid \Gamma_n \Rightarrow \Delta_n \]

called its components.
- A hypersequent \( \mathcal{H} \) is true in an interpretation \( \mathcal{I} \) iff one of its components, \( \Gamma_i \Rightarrow \Delta_i \in \mathcal{H} \) is true in that interpretation, i.e.

\[(\forall \Gamma_1 \supset \forall \Delta_1) \lor \ldots \lor (\forall \Gamma_n \supset \forall \Delta_n)\]
Syntax of Hypersequents

- **Internal rules** are (structural) rules which have one active component in each premiss, and one principal component in the conclusion. **External rules** are (structural) rules which are not internal rules.

- The **standard external rules** are

  \[
  \frac{\mathcal{H}}{\mathcal{H} \vdash \Gamma \Rightarrow \Delta} \quad \text{EW} \quad \frac{\mathcal{H} \vdash \Gamma \Rightarrow \Delta}{\mathcal{H} \vdash \Gamma \Rightarrow \Delta} \quad \text{EC} \quad \frac{\mathcal{H} \vdash \Gamma \Rightarrow \Delta}{\mathcal{H} \vdash \Gamma \Rightarrow \Delta} \quad \text{EP}
  \]

  where \( \mathcal{H}, \mathcal{H}' \) denote the **side components**.

- The **hyperextension** of a sequent calculus is its extension as a hypersequent calculus by adding hypercontexts to rules and the standard external rules.
A Hyperextension of a Calculus for Int

\[ \frac{\Gamma, P \Rightarrow P, \Delta}{\text{Ax}} \]

\[ \frac{\Gamma, \bot \Rightarrow \Delta}{L\bot} \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow \Delta, \bot}{R\bot} \]

\[ \frac{\mathcal{H}|\Gamma, A \Rightarrow \Delta \quad \mathcal{H}|\Gamma, B \Rightarrow \Delta}{\mathcal{H}|\Gamma, A \lor B \Rightarrow \Delta} \quad \text{L\lor} \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow A, \Delta \quad \mathcal{H}|\Gamma \Rightarrow B, \Delta}{\mathcal{H}|\Gamma \Rightarrow A \lor B, \Delta} \quad \text{R\lor}_1 \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow A \lor B, \Delta}{R\lor_2} \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow \Delta, A \quad \mathcal{H}|\Gamma, B \Rightarrow \Delta}{\mathcal{H}|\Gamma, A \supset B \Rightarrow \Delta} \quad \text{L\supset} \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow A \supset B, \Delta}{\text{R\supset}} \]

\[ \frac{\mathcal{H}|\Gamma \Rightarrow \Delta}{\mathcal{H}|\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{W} \]

\[ \frac{\mathcal{H}|\Gamma, \Gamma' \Rightarrow \Delta, \Delta', \Delta'}{\mathcal{H}|\Gamma, \Gamma' \Rightarrow \Delta, \Delta'} \quad \text{C} \]

plus the dual \( \land \) rules and standard external rules and (hyperextended) cut.
Extensions for Some Intermediate Logics

- Adding the LQ rule yields a calculus for \textbf{Jan}:

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow}{\mathcal{H}|\Gamma_1 \Rightarrow |\Gamma_2 \Rightarrow} \quad \text{LQ}
\]

- Adding the communication rule yields a calculus for \textbf{GD}:

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1 |\Gamma_2 \Rightarrow \Delta_2} \quad \text{Com}
\]

- Adding the split rule yields a calculus for \textbf{Cl}:

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1 |\Gamma_2 \Rightarrow \Delta_2} \quad \text{S}
\]
The Labelled and Hypersequent Rules Look Similar

Hypersequent Rule

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow}{\mathcal{H}|\Gamma_1 \Rightarrow |\Gamma_2 \Rightarrow}
\]

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1 \quad \mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_2}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1 |\Gamma_2 \Rightarrow \Delta_2}
\]

\[
\frac{\mathcal{H}|\Gamma_1, \Gamma_2 \Rightarrow \Delta_1, \Delta_2}{\mathcal{H}|\Gamma_1 \Rightarrow \Delta_1 |\Gamma_2 \Rightarrow \Delta_2}
\]

Relational Rule

\[
\frac{R_{x\hat{z}}, R_{y\hat{z}}, R_{wx}, R_{wy}, \Sigma; \Gamma \Rightarrow \Delta}{R_{wx}, R_{wy}, \Sigma; \Gamma \Rightarrow \Delta}
\]

\[
\frac{R_{xy}, \Sigma; \Gamma \Rightarrow \Delta \quad R_{yx}, \Sigma; \Gamma \Rightarrow \Delta}{\Sigma; \Gamma \Rightarrow \Delta}
\]

\[
\frac{R_{xy}, R_{yx}, \Sigma; \Gamma \Rightarrow \Delta}{R_{xy}, \Sigma; \Gamma \Rightarrow \Delta}
\]

Components roughly correspond to labels, and relational formula roughly correspond to subset relations.
Translation of Labelled Sequents to Hypersequents

- We want a translation of proofs in labelled systems like $G3I^*$ to (familiar) hypersequent systems.
- Each label corresponds to a component.
- Relations are translated using monotonicity: $R_{xy}$ is translated by including the antecedent (r. succedent) of the component for $x$ (r. $y$) as a subset of the antecedent (r. succedent) of the component for $y$ (r. $x$). e.g.,

$$R_{xy}, A^x, B^y \Rightarrow C^x, D^y \quad \mapsto \quad A \Rightarrow C, D \mid A, B \Rightarrow D$$

The process is called transitive unfolding.
- The translation makes an explicit relationship between labels into an implicit relationship between components.
Labelled Calculi are More Expressive than Hypersequents

- The two labelled sequents,

\[ \mathcal{R}_{xy}, \mathcal{R}_{xz}; \Gamma^x \Rightarrow \mathcal{R}_{xy}, \mathcal{R}_{yz}; \Gamma^x \Rightarrow \]

both translate to the same hypersequent,

\[ \Gamma \Rightarrow | \Gamma \Rightarrow | \Gamma \Rightarrow \]

- What do relations mean w.r.t. hypersequents? e.g. The following holds for \textbf{Int} models:

\[ \mathcal{R}_{xy}; (A \lor B)^x, (B \supset C)^y \Rightarrow A^x, C^y \]

but the corresponding hypersequent is not derivable for \textbf{Int}:

\[ A \lor B \Rightarrow A, C \mid A \lor B, B \supset C \Rightarrow C \]
Hypersequents and Monotonicity

- Ideally, we’d like hypersequent rules to act on multiple components in accordance with monotonicity, just as labelled rules do.
- But the following rule is not valid for Int:

\[
\frac{\mathcal{H}|A, \Gamma \Rightarrow \Delta, \Delta'|A, \Gamma, \Gamma'| \Rightarrow \Delta'}{\mathcal{H}|A, \Gamma \Rightarrow \Delta, \Delta'|\Gamma, \Gamma'| \Rightarrow \Delta'} \quad L \subseteq
\]

- A simple counterexample is

\[
\frac{A \Rightarrow A \land B | A, B \Rightarrow A \land B}{A \Rightarrow A \land B | B \Rightarrow A \land B} \quad L \subseteq
\]

which is valid for \( \text{GD} = \text{Int} + (A \supset B) \lor (B \supset A) \).
The Translation Requires Communication

**Theorem**

*Labelled proofs in G3I*\(^*\) (that do not contain ordering rules with principal relational formulae) can be translated into hypersequent proofs in a corresponding calculus augmented with the Com rule,

\[
\frac{\mathcal{H}|\Gamma \Rightarrow \Delta, \Delta'|\Gamma, \Gamma' \Rightarrow \Delta' \quad \mathcal{H}|\Gamma, \Gamma' \Rightarrow \Delta | \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H}|\Gamma \Rightarrow \Delta | \Gamma' \Rightarrow \Delta'} \quad \text{Com}
\]

- Labelled rules and proofs for some logics Int\(^*\) can be translated into hypersequent proofs for GD\(^*\).
- The restriction on ordering rules has to do with the admissibility of cut. A rule such as

\[
\frac{\mathcal{R}_{yx}, \mathcal{R}_{xy}, \mathcal{R}_{yz}; \Gamma \Rightarrow \Delta \quad \mathcal{R}_{zy}, \mathcal{R}_{xy}, \mathcal{R}_{yz}; \Gamma \Rightarrow \Delta}{\mathcal{R}_{xy}, \mathcal{R}_{yz}; \Gamma \Rightarrow \Delta} \quad \text{bd}_2
\]

translates to hypersequent rules with duplicated metavariables in the conclusion, and that may affect cut admissibility. (⋆)
Translation of Proofs

- Note that this work is about translating *proofs* of arbitrary labelled sequents (with relations) into hypersequents.
- The communication rule allows us to derive hypersequent variants of the labelled rules.
- We proceed by transitive unfolding the premisses of each labelled inference and then applying the hypersequent variant of the inference rule, to obtain a conclusion that is the transitive unfolding of the conclusion of the labelled inference.
- The refl, trans and mono rules are ignored as they are implicit in the translation. (⋆)
Monotonicity Rules

Lemma
The rules

\[
\frac{\mathcal{H} | A, \Gamma \Rightarrow \Delta, \Delta' | A, \Gamma, \Gamma' \Rightarrow \Delta'}{\mathcal{H} | A, \Gamma \Rightarrow \Delta, \Delta' | \Gamma, \Gamma' \Rightarrow \Delta'} \quad L\subset
\]

\[
\frac{\mathcal{H} | \Gamma \Rightarrow \Delta, \Delta' | \Gamma, \Gamma' \Rightarrow \Delta'}{\mathcal{H} | \Gamma \Rightarrow \Delta, \Delta' | \Gamma, \Gamma' \Rightarrow \Delta', A} \quad R\subset
\]

are derivable using Com.

Proof.

\[
\frac{\mathcal{H} | A, \Gamma \Rightarrow \Delta, \Delta' | A, \Gamma, \Gamma' \Rightarrow \Delta'}{\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' | A, \Gamma, \Gamma' \Rightarrow \Delta'} \quad W \quad (RS)
\]

\[
\frac{\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' | A, \Gamma, \Gamma' \Rightarrow \Delta'}{\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'^2} \quad C
\]

\[
\frac{\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta'}{\mathcal{H} | A, \Gamma, \Gamma' | \Gamma' \Rightarrow \Delta^2, \Delta'} \quad EW
\]

\[
\mathcal{H} | A, \Gamma \Rightarrow \Delta, \Delta' | \Gamma, \Gamma' \Rightarrow \Delta' \quad W
\]

\[
\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad EW
\]

\[
\mathcal{H} | A, \Gamma, \Gamma' \Rightarrow \Delta, \Delta' \quad C
\]

\[
\mathcal{H} | A, \Gamma \Rightarrow \Delta, \Delta' | \Gamma, \Gamma' \Rightarrow \Delta' \quad L\subset
\]

The proof of \( R\subset \) is similar. \(\square\)
Lemma

The rule

\[
\frac{H|A, \Gamma_1 \Rightarrow \Delta_1| \ldots |A, \Gamma_k \Rightarrow \Delta_k \quad H|B, \Gamma_1 \Rightarrow \Delta_1| \ldots |B, \Gamma_k \Rightarrow \Delta_k}{H|A \lor B, \Gamma_1 \Rightarrow \Delta_1| \ldots |A \lor B, \Gamma_k \Rightarrow \Delta_k} \quad L \lor \star
\]

where $\Gamma_i \subseteq \Gamma_{i+1}$ and $\Delta_{i+1} \subseteq \Delta_i$, is derivable using Com.

The dual rule $R \land \star$ is similarly derivable.

A $L \supset \star$ rule is also derivable, using the derived monotonicity rules.
An Example Translation

\[ \mathcal{R}_{yy}, \mathcal{R}_{xy}; B^x, B^y, (B \supset C)^y \Rightarrow B^y \quad \mathcal{R}_{yy}; C^y, (B \supset C)^y \Rightarrow C^y \]

\[ \mathcal{R}_{xx}, \mathcal{R}_{xy}; A^x \Rightarrow A^x, C^y \quad \mathcal{R}_{xx}, \mathcal{R}_{xy}; B^x, B^y, (B \supset C)^y \Rightarrow C^y \]

\[ \mathcal{R}_{xy}; A^x \Rightarrow A^x, C^y \quad \mathcal{R}_{xy}; B^x, B^y, (B \supset C)^y \Rightarrow C^y \]

\[ \mathcal{R}_{xy}; (A \lor B)^x, (B \supset C)^y \Rightarrow A^x, C^y \]

\[ B \Rightarrow A, C \mid B, B \supset C \Rightarrow B, C \quad B \Rightarrow A, C \mid C, B \supset C \Rightarrow C \]

\[ A \Rightarrow A, C \mid A, B \supset C \Rightarrow C \quad B \Rightarrow A, C \mid B, B \supset C \Rightarrow C \]

\[ A \Rightarrow A, C \mid A, B \supset C \Rightarrow C \quad B \Rightarrow A, C \mid B, B \supset C \Rightarrow C \]

\[ A \lor B \Rightarrow A, C \mid A \lor B, B \supset C \Rightarrow C \]
Related Work (1)

- Hypersequents and labelled calculi for $S_5$, [Avron, 1996], etc.
- Hypersequents and Display Logics for specific logics, [Wansing, 1998], and labelled calculi for $S_5$, [Restall, 2006].
- Hypersequents and labelled calculi for $A$ and $L$, [Metcalf et al, 2002].
- Starred sequents, hypersequents and indexed sequents for $S_5$ and $N_3$, [P. Girard, 2005].
- Relationship between labelled calculi and nested sequents for modal logics [Fitting, 2010].
Related Work (2)

- Obtaining labelled calculi from non-labelled (e.g. Hilbert and sequent) calculi, [Gabbay, 1996].
- Obtaining (hyper)sequent rules from Hilbert-style axioms [Ciabattoni et al, 2008].
- Syntactic conditions for cut admissibility [Ciabattoni et al, 2009].
- Labelled sequent calculi with geometric rules, for non-classical logics [Negri, 2005], spec. for intermediate logics [Dyckhoff & Negri, 2010 (MS)].
Open Questions and Future Work

- Do rules with non-linear conclusions (e.g. $bd_2$) admit cut in the presence of Com?
- Can hypersequent proofs of single components be transformed so that they do not have Com, for logics weaker than $GD$?
- Can transformation of labelled proofs into hypersequent proofs give a technique for parallelising programs?