MPRI C.2.3 - Concurrency

Probabilistic models and applications
Lecture 3

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Outline of the lectures

- Dec 13
- Dec 20
- Jan 10
- Jan 17
- Jan 24

Outline of the lectures

- The need for randomization
- Probabilistic automata
- Probabilistic bisimulation
- Probabilistic calculi
- Testing equivalence
- Introduction to probabilistic model checking and PRISM
- Metrics for probabilistic processes
- Verification of anonymity protocols: Dining Cryptographers, Crowds

Questions from the last lecture

Question 1:

- $P +_p Q \sqsubseteq_{\mathbf{may}} \tau.P + \tau.Q$
- $\tau.P + \tau.Q \sqsubseteq_{\mathbf{must}} P +_p Q$

Questions from the last lecture

Question 2: which of the following hold?

- $A\varphi \Leftarrow \mathcal{P}_{\geq\lambda}\varphi$?
- $A\varphi \Rightarrow \mathcal{P}_{\geq\lambda}\varphi$?
- $E\varphi \Leftarrow \mathcal{P}_{\geq\lambda}\varphi$?
- $E\varphi \Rightarrow \mathcal{P}_{\geq\lambda}\varphi$?

Questions from the last lecture

Question 3:

- $\Diamond\varphi \equiv \mathbf{true}\, U\varphi$
- $\Box\varphi \equiv \neg\Diamond\neg\varphi$
- $Pr_s^+\neg\psi = 1 - Pr_s^-\psi$
- $Pr_s^-\neg\psi = 1 - Pr_s^+\psi$

where the semantics of path formulas are extended with:
$s, s_1, \ldots \models \neg\psi$ iff $s, s_1, \ldots \not\models \psi$

Probabilistic bisimulation

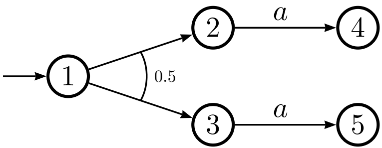
A relation $\mathcal{R} \subseteq S \times S$ is a *strong probabilistic bisimulation* iff for all $s_1, s_2 \in \mathcal{R}$ and for all $a \in A$

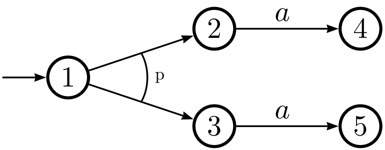
- if $s_1 \xrightarrow{a} \mu_1$ then $\exists \mu_2$ such that $s_2 \xrightarrow{a} \mu_2$ and $\mu_1 \, \mathcal{R} \, \mu_2$,
- if $s_2 \xrightarrow{a} \mu_2$ then $\exists \mu_1$ such that $s_1 \xrightarrow{a} \mu_1$ and $\mu_1 \, \mathcal{R} \, \mu_2$.

We write $s_1 \sim s_2$ if there is a strong bisimulation that relates them.

Probabilistic bisimulation

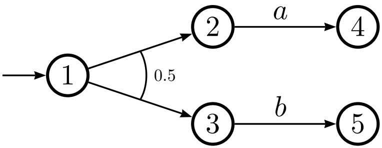
Transitions with different probabilities are allowed, as long as we go to equivalent states.

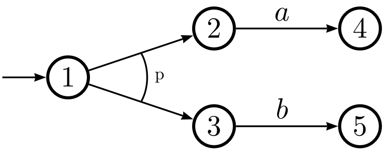




Probabilistic bisimulation

What about transitions to non-equivalent states?





We can argue that for p close to 0.5, the processes are "close".

Pseudometrics

$m : S \times S \to [0, \infty)$ s.t.

- $m(s, s) = 0$
- $m(s, t) = m(t, s)$
- $m(s_1, s_3) \leq m(s_1, s_2) + m(s_2, s_3)$

Goal: find a pseudometric m such that $m(s, t) = 0 \Leftrightarrow s \sim t$

Such a pseudometric is a metric on $S/\sim$

Metrics on probability distributions

- m: metric on S
- Goal: create metric $\hat{m}$ on $Disc(S)$
- $f : S \to \mathbb{R}$ is 1-Lipschitz wrt m iff

$$|f(s) - f(s')| \leq m(s, s') \qquad \forall s, s' \in S$$

- $f(\mu) = \sum_s \mu(s) f(s)$
- Kantorovich metric:

$$\hat{m}(\mu, \mu') = \sup\{|f(\mu) - f(\mu')| \ : f \text{ is 1-Lip wrt } m\}$$

Metrics on probability distributions

Kantorovich-Rubinstein theorem:

- Write $M(\mu, \mu')$ for the joint distributions $\alpha \in Disc(S \times S)$ with marginals μ, μ', i.e.

$$\alpha(s, S) = \mu(s) \qquad \alpha(S, t) = \mu'(t)$$

- Then:

$$\hat{m}(\mu, \mu') = \inf\{\sum_{s,t} \alpha(s,t) m(s,t) \mid \alpha \in M(\mu, \mu')\}$$

Metrics on probability distributions

$\hat{m}(\mu, \mu')$ can be computed as the solution to the following Linear program:

- Variables: $\alpha_{s,t}, s, t \in S$
- minimize $\sum_{s,t} \alpha_{s,t} m(s,t)$
- subject to:

$$\begin{aligned} \textstyle\sum_t \alpha_{s,t} &= \mu(s) && \forall s \in S \\ \textstyle\sum_s \alpha_{s,t} &= \mu'(t) && \forall t \in S \\ \alpha_{s,t} &\geq 0 && \forall s, t \in S \end{aligned}$$

Complete Lattices

- Partially ordered set $(L, \leq)$
  (reflexivity, antisymmetry, transitivity)
- All subsets of $A \subseteq L$ have a supremum $\bigvee A$ and an infimum $\bigwedge A$
- Examples:
  - 2^S with $\subseteq$
  - $[0, 1]$ with $\leq$
  - Equivalence relations ordered by refinement

  Question: what are the $\bigvee, \bigwedge$ in each case?

Complete Lattices

- $\mathcal{M}$: the set of all 1-bounded pseudometrics on S
- Ordered by: $m \leq m'$ iff $m(s,t) \geq m'(s,t)$ for all $s, t \in S$
- $(\mathcal{M}, \leq)$ is a complete lattice
- What are $\top, \bot, \bigvee, \bigwedge$?

Complete Lattices

Knaster-Tarski theorem:

- $(L, \leq)$ is a complete Lattice
- f is monotone: $a \leq b$ implies $f(a) \leq f(b)$
- Then f has a maximum and a minimum fixpoint
  (in fact the fixpoints form a complete Lattice under $\leq$)

The metric extension of bisimulation

General idea:

- Start from $m = \top$, i.e. everything is equivalent, which means distance 0 (similarly to the algorithm for computing bisimulation)
- The goal is that whenever $m(s,t) = a$ and $s \xrightarrow{a} \mu$, t should match it with a transition $t \xrightarrow{b} \mu'$ such that $\hat{m}(\mu, \mu') \leq a$
- $F : \mathcal{M} \to \mathcal{M}$ updates m so that the above property holds
- Our metric is the maximum fixpoint of F

Hausdorff distance

- Extend m from S to 2^S
- $m(A, B) = \max\{\sup_{s \in A} \inf_{t \in B} m(s, t), \sup_{t \in B} \inf_{s \in A} m(s, t)\}$

The metric extension of bisimulation

- Define $F : \mathcal{M} \to \mathcal{M}$ as $F(m)(s,t) < \epsilon$ iff
  - $\forall s \xrightarrow{a} \mu \; \exists t \xrightarrow{a} \mu' : \hat{m}(\mu, \mu') < \epsilon$
  - $\forall t \xrightarrow{a} \mu \; \exists s \xrightarrow{a} \mu' : \hat{m}(\mu, \mu') < \epsilon$
- Define $s \xrightarrow{a} \; = \; \{\mu \mid s \xrightarrow{a} \mu\}$
- Then

$$F(m)(s,t) = \max_a \hat{m}(s \xrightarrow{a}, t \xrightarrow{a})$$

The metric extension of bisimulation

- F is monotone, i.e. $m \leq m' \Rightarrow F(m) \leq F(m')$
- Hence, it has a maximum fixpoint
- We take m as the maximum fixpoint of F
- It can be computed by iterating F starting from $\top$

The metric extension of bisimulation

Lemma

R: equivalence relation on S, m: metric on S s.t. $m(s,t) = 0 \Leftrightarrow sRt$. Then

$$\hat{m}(\mu,\mu') \Leftrightarrow \mu R \mu'$$

Theorem

$m \sim t$ iff $m(s,t) = 0$