J Autom Reasoning (2008) 40:133–177
DOI 10.1007/s10817-007-9091-0

A Logical Characterization of Forward and Backward Chaining in the Inverse Method

Kaustuv Chaudhuri · Frank Pfenning · Greg Price

Received: 9 February 2007 / Accepted: 11 June 2007 / Published online: 24 January 2008

Abstract The inverse method is a generalization of resolution that can be applied to non-classical logics. We have recently shown how Andreoli's focusing strategy can be adapted for the inverse method in linear logic. In this paper we introduce the notion of focusing bias for atoms and show that it gives rise to forward and backward chaining, generalizing both hyperresolution (forward) and SLD resolution (backward) on the Horn fragment. A key feature of our characterization is the structural, rather than purely operational, explanation for forward and backward chaining. A search procedure like the inverse method is thus able to perform both operations as appropriate, even simultaneously. We also present experimental results and an evaluation of the practical benefits of biased atoms for a number of examples from different problem domains.

Keywords Inverse method · Focusing · SLD resolution · Hyperresolution · Intuitionistic linear logic

This work has been partially supported by the Office of Naval Research (ONR) under grant MURI N00014-04-1-0724 and by the National Science Foundation (NSF) under grant CCR-0306313. The first author was partially supported by a post-doctoral fellowship from INRIA-Futurs/École Polytechnique.

K. Chaudhuri
Laboratoire d'Informatique (LIX) École Polytechnique,
Palaiseau, France
e-mail: kaustuv@lix.polytechnique.fr

F. Pfenning (✉) · G. Price
Department of Computer Science, Carnegie Mellon University,
Pittsburg PA, USA
e-mail: fp@cs.cmu.edu

G. Price
e-mail: gprice@andrew.cmu.edu

1 Introduction

Designing and implementing an efficient theorem prover for a non-classical logic requires deep knowledge about the structure and properties of proofs in this logic. Fortunately, proof theory provides a useful guide, since it has isolated a number of important concepts that are shared between many logics of interest. The most fundamental is Gentzen's cut-elimination property [13] which allows us to consider only subformulas of a goal during proof search. Cut elimination gives rise to the inverse method [12] for theorem proving which applies to many non-classical logics. A more recent development is Andreoli's focusing property [1, 2] which allows us to translate formulas into derived rules of inference and then consider only the resulting big-step derived rules without losing completeness. Even though Andreoli's system was designed for classical linear logic, similar focusing systems for many other logics have been discovered [14, 16].

In prior work we have constructed a focusing system for *intuitionistic* linear logic which is consonant with Andreoli's classical version [8], and shown that restricting the inverse method to work only with big-step rules derived from focusing dramatically improves its efficiency [7]. The key feature of focusing is that each logical connective carries an intrinsic attribute called polarity that determines its behavior under focusing. In the case of linear logic, polarities are uniquely determined for each connective. However, as Andreoli noted, polarities may be chosen freely for atomic propositions as long as duality is consistently maintained. In this paper we prove that, despite the asymmetric nature of intuitionistic logic, a similar observation can be made here. Furthermore, we show that proof search on Horn propositions with the inverse method behaves either like hyperresolution or SLD resolution, depending on the chosen polarity for atoms. If different atoms are ascribed different polarities we can obtain combinations of these strategies that remain complete. The focused inverse method therefore directly generalizes these two classical proof search strategies. We also demonstrate through an implementation and experimental results that this choice can be important in practical proof search situations and that the standard polarity assumed for atoms in intuitionistic [15] or classical [22] logic programming is often the less efficient one.

The concept of viewing focused derivations as a means of constructing derived inference rules is not new. Andreoli himself has made similar observations for backward reasoning: see [2], for instance. Girard's *Ludics* [14] uses focusing as a foundational concept and takes it as an explanation for logic; in Ludics, "bipoles" or derived inference rules are the only rules that are syntactically allowed. Focusing for intuitionistic (including linear) logics was first investigated by Howe [16]; however, Howe did not allow atoms of different polarities.

The interaction of focusing and cut-elimination has been studied by Danos, Joinet and Schellinx [10, 11], though not in these precise terms. Although none of their translations explicitly use focusing, their calculi, particularly the constraints in the LK_p^η system bear unmistakable similarities to focusing. A more recent work by Jagadesan et al. [17] is the system λRCC, a logic programming language without focusing, but with atoms of different polarities. In λRCC the observation that switching polarity gives rise to forward- or backward-chaining is visible, though this observation is limited to the Horn-fragment of intuitionistic logic. Finally, a more recent work by Liang and Miller [19] uses biased focusing to give uniform interpretations of a number of linear and non-linear calculi such as the well-known LJQ and LJT sequent calculi.

Since focusing appears to be an almost universal phenomenon among non-classical logics, we believe these observations have wide applicability in constructing theorem provers. The fact that we obtain well-known standard strategies on the Horn fragment, where classical, intuitionistic, and even linear logic coincide, provides further evidence. We are particularly interested in intuitionistic linear logic and its extension by a monad, since it provides the foundation for the logical framework CLF [4] which we can use to specify stateful and concurrent systems. Theorem proving in CLF thereby provides a means for analyzing properties of such systems.

The remainder of the paper is organized as follows. In Section 2 we present the backward focusing calculus that incorporates a choice of polarity for atoms and describe the derived rules that are generated with atoms of different polarity. We then sketch the focused inverse method in Section 3, noting the key differences between sequents and rules in the forward direction from their analogues in the backward direction. In Section 4 we concentrate on the Horn fragment, where we show that the derived rules generalize hyperresolution (for negative atoms) and SLD resolution (for positive atoms). Finally, Section 5 summarizes our experimental results on an implementation of the inverse method presented in Section 3.

2 Biased Focusing

We consider intuitionistic linear logic including the following connectives: linear implication ($\multimap$), multiplicative conjunction ($\otimes, \mathbf{1}$), additive conjunction ($\&, \top$), additive disjunction ($\oplus, \mathbf{0}$), the exponential ($!$), and the first-order quantifiers ($\forall, \exists$). Quantification is over a simple term language consisting of variables and uninterpreted function symbols applied to a number of term arguments. Propositions are written using capital letters ($A, B, \ldots$), and atomic propositions with lowercase letters ($a, b, n, m, p, q, \ldots$).

We start with a standard dyadic sequent calculus for this logic consisting of sequents of the form $\Gamma; \Delta \Longrightarrow C$, where Γ and Δ are contexts of hypotheses and C is the single intuitionistic conclusion. The hypotheses in Δ are *linear*, i.e., each hypothesis must be consumed exactly once in the proof; those in Γ are *unrestricted*, i.e., each may be consumed as many times as necessary. The rules for this calculus are in Fig. 1. This calculus is known to have the usual nice properties: admissibility of cut, the identity principle, and admissible weakening and contraction for unrestricted hypotheses [5, 6].

In classical linear logic the synchronous or asynchronous nature of a given connective is identical to its polarity; the negative connectives ($\&, \top, \invamp, \bot, \forall$) are asynchronous, and the positive connectives ($\otimes, \mathbf{1}, \oplus, \mathbf{0}, \exists$) are synchronous. In intuitionistic logic, where the left- and right-hand side of a sequent are asymmetric and no involutive negation exists, we derive the properties of the connectives via the rules and phases of search: an asynchronous connective is one for which decomposition is complete in the *active phase*; a synchronous connective is one for which decomposition is complete in the *focused phase*. This definition happens to coincide with polarities for classical linear logic, although we know of no intrinsic

Judgmental rules

$$\frac{}{\Gamma \,;\, a \Longrightarrow a}\ \text{init} \qquad \frac{\Gamma, A \,;\, \Delta, A \Longrightarrow C}{\Gamma, A \,;\, \Delta \Longrightarrow C}\ \text{copy}$$

Multiplicative rules

$$\frac{\Gamma \,;\, \Delta \Longrightarrow A \quad \Gamma \,;\, \Delta' \Longrightarrow B}{\Gamma \,;\, \Delta, \Delta' \Longrightarrow A \otimes B}\ \otimes R \qquad \frac{\Gamma \,;\, \Delta, A, B \Longrightarrow C}{\Gamma \,;\, \Delta, A \otimes B \Longrightarrow C}\ \otimes L$$

$$\frac{}{\Gamma \,;\, \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Gamma \,;\, \Delta \Longrightarrow C}{\Gamma \,;\, \Delta, \mathbf{1} \Longrightarrow C}\ \mathbf{1}L$$

$$\frac{\Gamma \,;\, \Delta, A \Longrightarrow B}{\Gamma \,;\, \Delta \Longrightarrow A \multimap B}\ \multimap R \qquad \frac{\Gamma \,;\, \Delta \Longrightarrow A \quad \Gamma \,;\, \Delta', B \Longrightarrow C}{\Gamma \,;\, \Delta, \Delta', A \multimap B \Longrightarrow C}\ \multimap L$$

Additive rules

$$\frac{\Gamma \,;\, \Delta \Longrightarrow A \quad \Gamma \,;\, \Delta \Longrightarrow B}{\Gamma \,;\, \Delta \Longrightarrow A \,\&\, B}\ \& R \qquad \frac{\Gamma \,;\, \Delta, A_i \Longrightarrow C}{\Gamma \,;\, \Delta, A_1 \,\&\, A_2 \Longrightarrow C}\ \& L_i$$

$$\frac{\Gamma \,;\, \Delta \Longrightarrow A_i}{\Gamma \,;\, \Delta \Longrightarrow A_1 \oplus A_2}\ \oplus R_1 \qquad \frac{\Gamma \,;\, \Delta, A \Longrightarrow C \quad \Gamma \,;\, \Delta, B \Longrightarrow C}{\Gamma \,;\, \Delta, A \oplus B \Longrightarrow C}\ \oplus L$$

$$\frac{}{\Gamma \,;\, \Delta \Longrightarrow \top}\ \top R \qquad \frac{}{\Gamma \,;\, \Delta, \mathbf{0} \Longrightarrow C}\ \mathbf{0}L$$

Exponentials

$$\frac{\Gamma \,;\, \cdot \Longrightarrow A}{\Gamma \,;\, \cdot \Longrightarrow {!A}}\ !\,R \qquad \frac{\Gamma, A \,;\, \Delta \Longrightarrow C}{\Gamma \,;\, \Delta, !A \Longrightarrow C}\ !\,L$$

Quantifiers

$$\frac{\Gamma \,;\, \Delta \Longrightarrow [u/x]A}{\Gamma \,;\, \Delta \Longrightarrow \forall x.A}\ \forall R^u \qquad \frac{\Gamma \,;\, \Delta, [t/x]A \Longrightarrow C}{\Gamma \,;\, \Delta, \forall x.A \Longrightarrow C}\ \forall L$$

$$\frac{\Gamma \,;\, \Delta \Longrightarrow [t/x]A}{\Gamma \,;\, \Delta \Longrightarrow \exists x.A}\ \exists R \qquad \frac{\Gamma \,;\, \Delta, [u/x]A \Longrightarrow C}{\Gamma \,;\, \Delta, \exists x.A \Longrightarrow C}\ \exists L^u$$

Fig. 1 The backward sequent calculus for first-order linear logic

reason why this should be so. To maintain unity with the literature we use the terms *positive* and *negative* and call the positive or negative nature of a proposition its *polarity*. Note that because our backward linear sequent calculus is two-sided, positive (negative) propositions will be synchronous on the right (left) of the sequent arrow, and asynchronous on the left (right).

For an atomic proposition, we have a choice of polarities to assign to it; this choice we call a *bias*. A positive-biased atom behaves like a positive proposition in the sense its principal rule, an initial rule, must treat it as synchronous on the right, i.e., as a right focus. Dually, a negative-biased atom requires a left focus for its initial rule. Note that every atomic proposition is *either* positive-biased *or* negative-biased for the entire derivation. However, any arbitrary assignment of polarities to the atoms will guarantee completeness. This observation was already well made by Andreoli [1] for classical linear logic, but that it works just as well in the intuitionistic case is established in this work.[1]

To aid in clarity, we denote propositions of positive and negative polarities with the suggestive meta-variables P, Q, ... and N, M, ... respectively (lower-case used for the atoms):

$$\begin{array}{ll} \text{(positive)} & P, Q, \ldots ::= p \mid A \otimes B \mid \mathbf{1} \mid A \oplus B \mid \mathbf{0} \mid !A \mid \exists x. A \\ \text{(negative)} & N, M \ldots ::= n \mid A \,\&\, B \mid \top \mid A \multimap B \mid \forall x. A \end{array}$$

We will also write P^- for a positive proposition or a negative-biased atom, and N^+ for a negative proposition or a positive-biased atom.

The contexts in the sequents of the focusing calculus will be of three different kinds. We shall have the unrestricted contexts (Γ) as before. The linear context Δ will be restricted to contain only positive-biased atoms or negative propositions, i.e., of the form N^+. A third *active* context, written Ω, will be added for active sequents; this context will be ordered, indicated by a centered dot ($\cdot$) instead of a comma. Each hypothesis in the active context will also have to be consumed exactly once in the proof. The right-hand side of active sequents will be split into two kinds: a *passive* kind, written $\cdot; Q^-$, containing a positive propositions or negative-biased atom Q^-; and an *active* kind, written $A; \cdot$, where active rules may be applicable to A. If the precise form of the right-hand side does not matter, we shall write it as γ. The specific sequents in the focusing calculus are as follows:

$$\begin{array}{ll} \Gamma; \Delta \gg A & \textit{right-focal} \text{ sequent with } A \text{ under focus} \\ \Gamma; \Delta; A \ll Q^- & \textit{left-focal} \text{ sequent with } A \text{ under focus} \\ \Gamma; \Delta; \Omega \Longrightarrow \underbrace{\left\{ \begin{array}{l} \cdot;\ Q^- \\ C;\ \cdot \end{array} \right.}_{\gamma} & \textit{active} \text{ sequents} \end{array}$$

The full set of rules is in Fig. 2.

Active phase Leaving aside the exponential operator and unrestricted assumptions in Γ for the moment, right active propositions are decomposed until they become atomic or positive, i.e., a sequent of the form $\Gamma; \Delta; \Omega \Longrightarrow Q^-; \cdot$. The right hand side is then changed into the form $\cdot; Q^-$. Similarly, the propositions in Ω are decomposed except when the proposition is atomic or negative, in which case it is transferred to

[1]For recent developments along these lines since the conference version of this paper was published in [9], see [19, 30].

$\boxed{\Gamma ; \Delta \gg A \quad \text{right-focal}}$

$$\frac{}{\Gamma ; p \gg p}\ \text{init}^{+} \qquad \frac{}{\Gamma ; \cdot \gg \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Gamma ; \Delta_1 \gg A \quad \Gamma ; \Delta_2 \gg B}{\Gamma ; \Delta_1, \Delta_2 \gg A \otimes B}\ \otimes R$$

$$\frac{\Gamma ; \Delta \gg A_i}{\Gamma ; \Delta \gg A_1 \oplus A_2}\ \oplus R_i \qquad \frac{\Gamma ; \Delta \gg [t/x]A}{\Gamma ; \Delta \gg \exists x.A}\ \exists R \qquad \frac{\Gamma ; \cdot ; \cdot \Longrightarrow A ; \cdot}{\Gamma ; \cdot \gg\, !A}\ !R$$

$\boxed{\Gamma ; \Delta ; A \ll Q^{-} \quad \text{left-focal}}$

$$\frac{}{\Gamma ; \cdot ; n \ll n}\ \text{init}^{-} \qquad \frac{\Gamma ; \Delta ; A_i \ll Q^{-}}{\Gamma ; \Delta ; A_1 \,\&\, A_2 \ll Q^{-}}\ \& L_i$$

$$\frac{\Gamma ; \Delta_1 ; B \ll Q^{-} \quad \Gamma ; \Delta_2 \gg A}{\Gamma ; \Delta_1, \Delta_2 ; A \multimap B \ll Q^{-}}\ \multimap L \qquad \frac{\Gamma ; \Delta ; [t/x]A \ll Q^{-}}{\Gamma ; \Delta ; \forall x.A \ll Q^{-}}\ \forall L$$

$\boxed{\text{focus}}$

$$\frac{\Gamma ; \Delta \gg Q}{\Gamma ; \Delta ; \cdot \Longrightarrow \cdot ; Q}\ \text{focus}^{+} \qquad \frac{\Gamma ; \Delta ; N \ll Q^{-}}{\Gamma ; \Delta, N ; \cdot \Longrightarrow \cdot ; Q^{-}}\ \text{focus}^{-} \qquad \frac{\Gamma, A ; \Delta ; A \ll Q^{-}}{\Gamma, A ; \Delta \Longrightarrow \cdot ; Q^{-}}\ \text{focus}^{!}$$

$\boxed{\Gamma ; \Delta ; \Omega \Longrightarrow A ; \cdot \quad \text{right-active}}$

$$\frac{\Gamma ; \Delta ; \Omega \Longrightarrow A ; \cdot \quad \Gamma ; \Delta ; \Omega \Longrightarrow B ; \cdot}{\Gamma ; \Delta ; \Omega \Longrightarrow A \,\&\, B ; \cdot}\ \& R \qquad \frac{}{\Gamma ; \Delta ; \Omega \Longrightarrow \top ; \cdot}\ \top R$$

$$\frac{\Gamma ; \Delta ; \Omega \cdot A \Longrightarrow B ; \cdot}{\Gamma ; \Delta ; \Omega \Longrightarrow A \multimap B ; \cdot}\ \multimap R \qquad \frac{\Gamma ; \Delta ; \Omega \Longrightarrow [u/x]A ; \cdot}{\Gamma ; \Delta ; \Omega \Longrightarrow \forall x.\, A ; \cdot}\ \forall R^{u} \qquad \frac{\Gamma ; \Delta ; \Omega \Longrightarrow \cdot ; Q^{-}}{\Gamma ; \Delta ; \Omega \Longrightarrow Q^{-} ; \cdot}\ \text{act}\ R$$

$\boxed{\Gamma ; \Delta ; \Omega \cdot A \cdot \Omega' \Longrightarrow \gamma \quad \text{left-active}}$

$$\frac{\Gamma ; \Delta ; \Omega \cdot A \cdot B \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot A \otimes B \cdot \Omega' \Longrightarrow \gamma}\ \otimes L \qquad \frac{\Gamma ; \Delta ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot \mathbf{1} \cdot \Omega' \Longrightarrow \gamma}\ \mathbf{1}L$$

$$\frac{\Gamma ; \Delta ; \Omega \cdot A \cdot \Omega' \Longrightarrow Q \quad \Gamma ; \Delta ; \Omega \cdot B \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot A \oplus B \cdot \Omega' \Longrightarrow \gamma}\ \oplus L \qquad \frac{}{\Gamma ; \Delta ; \Omega \cdot \mathbf{0} \cdot \Omega' \Longrightarrow \gamma}\ \mathbf{0}L$$

$$\frac{\Gamma ; \Delta ; \Omega \cdot [u/x]A \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot \exists x.A \cdot \Omega' \Longrightarrow \gamma}\ \exists L^{u} \qquad \frac{\Gamma, A ; \Delta ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot !A \cdot \Omega' \Longrightarrow \gamma}\ !L \qquad \frac{\Gamma ; \Delta, N^{+} ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot N^{+} \cdot \Omega' \Longrightarrow \gamma}\ \text{act}\ L$$

$\boxed{\text{blur}}$

$$\frac{\Gamma ; \Delta ; P \Longrightarrow \cdot ; Q^{-}}{\Gamma ; \Delta ; P \ll Q^{-}}\ \text{blur}^{+} \qquad \frac{\Gamma ; \Delta ; \cdot \Longrightarrow N ; \cdot}{\Gamma ; \Delta \gg N}\ \text{blur}^{-}$$

Fig. 2 Backward linear focusing calculus

Δ. The two key judgmental rules that transfer atoms and synchronous propositions out of the active zones of the sequents are as follows:

$$\frac{\Gamma ; \Delta ; \Omega \Longrightarrow \cdot ; Q^{-}}{\Gamma ; \Delta ; \Omega \Longrightarrow Q^{-} ; \cdot}\ \text{act}\ R \qquad \frac{\Gamma ; \Delta ; N^{+} ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta ; \Omega \cdot N + \cdot \Omega' \Longrightarrow \gamma}\ \text{act}\ L$$

In the remaining active rules, a principal connective in an active proposition is decomposed using the corresponding rule in the backward sequent calculus. That is, modulo the distinction between Δ and Ω and the forms of the right-hand side, these rules are isomorphic to those of the non-focusing calculus. The following are two characteristic examples.

$$\frac{\Gamma;\Delta;\Omega \Longrightarrow A;\cdot \quad \Gamma;\Delta;\Omega \Longrightarrow B;\cdot}{\Gamma;\Delta;\Omega \Longrightarrow A\& B}\,\&R \qquad \frac{\Gamma;\Delta;\Omega\cdot A\cdot B\cdot \Omega' \Longrightarrow \gamma}{\Gamma;\Delta;\Omega\cdot A\otimes B\cdot \Omega'\gamma}\otimes L$$

Focal phase Eventually the active sequent is reduced to the form $\Gamma;\Delta;\cdot \Longrightarrow \cdot;Q^-$, which we call a *neutral sequent*. A focal phase is launched from such a neutral sequent by selecting a suitable proposition and giving it the corresponding left- or right-focus. This gives us the two focus rules

$$\frac{\Gamma;\Delta \gg Q}{\Gamma;\Delta;\cdot \Longrightarrow \cdot}\,\text{focus}^+ \qquad \frac{\Gamma;\Delta;N \ll Q^-}{\Gamma;\Delta,N;\cdot \Longrightarrow \cdot;Q^-}\,\text{focus}^-$$

Note the use of the syntactic classes in these rules: we never grant right focus to a negative atom, or left focus to a positive atom.

Once a proposition obtains focus, it is decomposed under focus until it becomes asynchronous or ends in an initial sequent. There are two forms of the initial sequent, corresponding to the two focusing biases.

$$\frac{}{\Gamma;\cdot;n \ll n}\,\text{init}^- \qquad \frac{}{\Gamma;p \gg p}\,\text{init}^+$$

A negative-biased atom thus has the interpretation from top-down (goal-directed) logic programming. Here, initial sequents have a *left* focus, and the right hand side is treated like an atomic goal to be matched with the head of a clause. On the other hand, a positive-biased atom has the interpretation from bottom-up logic programming. Here, the right-hand side is a passive goal and the linear hypotheses, which exactly represent a database, must evolve until they can match the right. This observation will be revisited in more detail in Section 4.

If the focal proposition has the opposite polarity, then we blur the focus and return to one of the active sequent forms.

$$\frac{\Gamma;\Delta;\cdot \Longrightarrow N;\cdot}{\Gamma;\Delta \gg N}\,\text{blur}^- \qquad \frac{\Gamma;\Delta;P \Longrightarrow \cdot;Q^-}{\Gamma;\Delta;P \ll Q^-}\,\text{blur}^+$$

This returns us to an active phase. We shall use the term *blur* to refer to the phenomenon of losing focus and transitioning to an active sequent.

We also have to account for the propositions in the unrestricted context Γ, which may be both synchronous and asynchronous. When we are in a neutral sequent, we may copy a proposition out of the unrestricted context and immediately focus on it.

$$\frac{\Gamma,A;\Delta;A \ll Q^-}{\Gamma,A;\Delta;\cdot \Longrightarrow \cdot;Q^-}\,\text{focus}^!$$

If this proposition is actually positive, then we immediately remove focus on it (using blur$^+$) and transition to an active phase.

Synchronous connectives are decomposed using non-invertible rules for that proposition, and focus is maintained where possible on the operands of the connective. For example, consider the $\&L_i$ rules:

$$\frac{\Gamma;\,\Delta;\,A \ll Q^-}{\Gamma;\,\Delta;\,A\&B \ll Q^-}\&L_1 \qquad \frac{\Gamma;\,\Delta;\,B \ll Q^-}{\Gamma;\,\Delta;\,A\&B \ll Q^-}\&L_2$$

Here we select (non-deterministically) an operand of the negative connective &, and then maintain focus on that selected operand. The next applicable rule *must* be applied to the selected operand, in this case A.

Focus can be propagated to multiple branches of the proof. For example:

$$\frac{\Gamma;\,\Delta_1 \gg A \quad \Gamma;\,\Delta_2 \gg B}{\Gamma;\,\Delta_1,\Delta_2 \gg A \otimes B} \otimes R$$

Here, both operands of $\otimes$ retain focus in their separate branches of the proof. In each branch, the rules are constrained to be applicable only to the respective operand.

There is only one subtlety in these focal rules regarding the exponential !. Although it is positive, the $!R$ rule cannot maintain focus on the operand using the following (incorrect!) rule:

$$\frac{\Gamma;\,\cdot \gg A}{\Gamma;\,\cdot \gg !A}!R$$

This calculus with this $!R$ rule is incomplete as there is no focused proof of the proposition $!(a \oplus b) \multimap !(b \oplus a)$, for example. To see why, consider the resulting neutral sequent $a \oplus b;\cdot;\cdot \Longrightarrow \cdot;\,!(b \oplus a)$. Now we have two choices. If we focus on $a \oplus b$ on the left, then we eventually obtain the neutral sequents $a \oplus b;a;\cdot \Longrightarrow \cdot;\,!(b \oplus a)$ and $a \oplus b;b;\cdot \Longrightarrow \cdot;\,!(b \oplus a)$.

$$\dfrac{\dfrac{\dfrac{\dfrac{a\oplus b\,;a\,;\cdot \Longrightarrow \cdot\,;!(b\oplus a)}{a\oplus b\,;\cdot\,;a \Longrightarrow \cdot\,;!(b\oplus a)}\text{act } L \quad \dfrac{a\oplus b\,;b\,;\cdot \Longrightarrow \cdot\,;!(b\oplus a)}{a\oplus b\,;\cdot\,;b \Longrightarrow \cdot\,;!(b\oplus a)}\text{act } L}{a\oplus b\,;\cdot\,;a\oplus b \Longrightarrow \cdot\,;!(b\oplus a)}\oplus L}{a\oplus b\,;\cdot\,;a\oplus b \ll !(b\oplus a)}\text{act } L}{a\oplus b\,;\cdot\,;\cdot \Longrightarrow \cdot\,;!(b\oplus a)}\text{focus}^!$$

In either case, focusing on the left yields nothing, and the $!R$ rule cannot be applied after a right focus because the linear context is not empty. The only remaining possibility is to start with a right focus instead of the left, i.e., with $a \oplus b;\cdot \gg !(b \oplus a)$. If we decompose this with $!R$, we get $a \oplus b;\cdot \gg b \oplus a$. Because $b \oplus a$ has focus, we are forced to use a $\oplus R$ rule to choose either b or a to prove; however, neither b nor a is provable from $a \oplus b$.

The fix is to blur the right focus on $b \oplus a$ in the $!R$ rule, i.e., to use the following version of the rule:

$$\frac{\Gamma;\,\cdot;\,\cdot \Longrightarrow A;\,\cdot}{\Gamma;\,\cdot \gg !A}!R$$

We can then focus on the left and get two provable sequents in the premisses of $\oplus L$. One explanation for this focus-removing nature of ! in a judgmental framework [5] is that there is a hidden transition from "$(!A)$ true" to the categorical judgment

"A valid" which in turn reduces to "A true". We may think of them as two rules, one decomposing the proposition and one changing the judgment:

$$\frac{\Gamma \Longrightarrow A \text{ valid}}{\Gamma; \cdot \Longrightarrow (!A) \text{ true}} \qquad \frac{\Gamma; \cdot \Longrightarrow A \text{ true}}{\Gamma \Longrightarrow A \text{ valid}}$$

The first of these two rules is the internalisation of the categorical judgment and is synchronous; the second is the definition of the categorical judgment and is asynchronous. The exponential therefore has aspects of both synchronicity and asynchronicity: the overall composition is synchronous, but there is a phase change when applying the rule. Girard has made a similar observation that exponentials are composed of one micro-connective to change polarity, and another to model a given behavior [14, page 114]; this observation extends to other modal operators, such as why-not (?) of JILL [5] or the lax modality of CLF [29].

Soundness of this calculus with respect to the non-focusing calculus in Fig. 1 is a rather obvious property – forget the distinction between Δ and Ω, elide the phase transition rules, and the original backward calculus appears.

Theorem 1 (Soundness)

1. *If* $\Gamma; \gg A$ *then* $\Gamma; \Delta \Longrightarrow A$.
2. *If* $\Gamma; \Delta; A \ll Q^-$ *then* $\Gamma; \Delta, A \Longrightarrow Q^-$.
3. *If* $\Gamma; \Delta; \Omega \Longrightarrow C; \cdot$ *then* $\Gamma; \Delta, \Omega \Longrightarrow C$.
4. *If* $\Gamma; \Delta; \Omega \Longrightarrow \cdot; Q^-$ *then* $\Gamma; \Delta, \Omega \Longrightarrow Q^-$.

Proof By structural induction on the given focused derivation. Note that all the logical rules neatly fall into one of the above cases. To illustrate, consider the rule $\otimes R$, i.e, the derivation that ends with the following rule:

$$\frac{\Gamma; \Delta_1 \gg A \quad \Gamma; \Delta_2 \gg B}{\Gamma; \Delta_1, \Delta_2 \gg A \otimes B}$$

$\Gamma; \Delta_1 \Longrightarrow A$ and $\Gamma; \Delta_2 \Longrightarrow B$ i.h.

$\Gamma; \Delta_1, \Delta_2 \Longrightarrow A \otimes B$ $\otimes R$.

For phase transition rules (i.e., blur$^+$, blur$^-$, act L, act R, focus$^-$, and focus$^+$), the premiss and the conclusion of the rule both denote the same sequent in the non-focusing calculus. □

To prove completeness, we take a more circuitous path, using admissibility of cut in the focusing calculus to show the rules of the non-focusing calculus are admissible. First let us look admissibility of cut. A principal cut is one where the cut proposition is immediately decomposed in the two given derivations. All principal cuts will be between a focal sequent and an active sequent, because polarities are dualised on the two sides of the sequent arrow. For example, for the principal cut for $\otimes$, we have to consider the following pair of derivations.

$$\frac{\begin{array}{cc}\mathcal{D}_1 & \mathcal{D}_2 \\ \Gamma; \Delta_1 \gg A_1 & \Gamma; \Delta_2 \gg A_2\end{array}}{\Gamma; \Delta_1, \Delta_2 \gg A_1 \otimes A_2} \qquad \frac{\begin{array}{c}\mathcal{E} \\ \Gamma; \Delta'; \Omega \cdot A_1 \cdot A_2 \cdot \Omega' \Longrightarrow \gamma\end{array}}{\Gamma; \Delta'; \Omega \cdot A_1 \otimes A_2 \cdot \Omega' \Longrightarrow \gamma}$$

The cut is distributed to the component derivations $\mathcal{D}_1, \mathcal{D}_2$ and $\mathcal{E}$, which also maintain this form of cut for subformulas; for example, a cut between $\mathcal{D}_1$ and $\mathcal{E}$ uses the smaller cut proposition A_1. The result of these cuts will be active because the proposition under focus is cut.

We also have to include a few more general cuts for the commutative cases in the cut theorem. Primarily, we require cuts between two active sequents, the result of which will be another active sequent. In the proof we also need to consider special cases where the cut proposition is in a focal sequent but not itself under focus. For the induction in the cut theorem to work, these specific cases will have to redo the focusing steps for the proposition under focus. We call these kinds of *preservative cuts* as they preserve the focus of one of the component derivations.

The proof of cut-elimination requires one key lemma: that permuting the ordered context does not affect provability. This lemma thus allows cutting propositions from anywhere inside the ordered context, and also to re-order the context when needed.

Lemma 2 (Permutation) *If $\Gamma; \Delta; \Omega \Longrightarrow \gamma$, then $\Gamma; \Delta; \Omega' \Longrightarrow \gamma$ for any permutation Ω' of Ω.*

Proof By structural induction on the derivation of $\Gamma; \Delta; \Omega \Longrightarrow \gamma$ We give a representative case for $\otimes L$, where $\Omega = \Omega_1 \cdot A \otimes B \cdot \Omega_2$ and the last rule in the derivation was:

$$\frac{\Gamma; \Delta; \Omega_1 \cdot A \cdot B \cdot \Omega_2 \Longrightarrow \gamma}{\Gamma; \Delta; \Omega_1 \cdot A \otimes B \cdot \Omega_2 \Longrightarrow \gamma} \otimes L$$

Let a permutation Ω' of $\Omega_1 \cdot A \otimes B \cdot \Omega_2$ be given. It must have the form $\Omega'_1 \cdot A \otimes B \cdot \Omega'_2$ where $\Omega'_1 \cdot \Omega'_2$ is a permutation of $\Omega_1 \cdot \Omega_2$. Therefore $\Omega'_1 \cdot A \cdot B \cdot \Omega'_2$ is a permutation of $\Omega_1 \cdot A \cdot B \cdot \Omega_2$. Therefore, by the induction hypothesis, hypothesis, $\Gamma; \Delta; \Omega'_1 \cdot A \cdot B \cdot \Omega'_2 \Longrightarrow \gamma$. Then use $\otimes L$. □

We also note (omitting its proof) a trivial corollary of this lemma; it will be useful during some cases in the proof of cut admissibility.

Corollary 3 (Inversion) All the active rules in Fig. 2 are invertible.

One consequence of Lemma 2 is that the order of the propositions in the active contexts does not matter. Therefore, we can always find a proof where the decompositions in the active phase fix a canonical order of decomposition. The $\otimes L$ rule, for example, could be restricted in an implementation to:

$$\frac{\Gamma; \Delta; \Omega \cdot A \cdot B \Longrightarrow_o \cdot; Q^-}{\Gamma; \Delta; \Omega . A \otimes B \Longrightarrow_o \cdot; Q^-} \otimes L$$

The ordered calculus operates on the right side of the sequent unless the right hand side is a positive proposition. Only then is the proposition on the right hand side moved into the passive zone and can propositions in Ω be decomposed. Any other fixed ordering would also work. Note that because the order of rules in the active context can be fixed based on the order of the active context, and the active context may be permuted arbitrarily by Lemma 2, it follows that the order of the active rules may also be permuted arbitrarily. For the purposes of the cut-admissibility theorem, such permutative variations are identified.

Definition 4 (Similar derivations) We define two derivations $\mathcal{D}_1$ and $\mathcal{D}_2$ of the same sequent to be *similar*, written $\mathcal{D}_1 \approx \mathcal{D}_2$, if they differ only in the order in which active rules are applied in the active phases.

Essentially, two derivations are similar if the only differences are in the inessential non-deterministic choices in the active phase. For the cut theorem, similar derivations are considered to be equal for the purposes of the lexicographic order. Note that no matter what order the active rules are done, the derivation will have the same neutral sequents, Furthermore, no copying of subformulas happens in the active rules, so the height of any active phase is bounded. Therefore, equating similar derivations for the purposes of the induction keeps the ordering well-founded.[2]

Theorem 5 (Cut) *If*

1. $\Gamma; \Delta \gg A$ *and*:
   (a) $\Gamma; \Delta'; \Omega \cdot A \cdot \Omega' \Longrightarrow \gamma$ *then* $\Gamma; \Delta, \Delta'; \Omega \cdot \Omega' \Longrightarrow \gamma$.
   (b) $\Gamma; \Delta', A; \Omega \Longrightarrow \gamma$ *then* $\Gamma; \Delta, \Delta'; \Omega \Longrightarrow \gamma$.
2. $\Gamma; \Delta; \Omega \Longrightarrow A; \cdot$ or $\Gamma; \Delta; \Omega \Longrightarrow \cdot; A$ *and*:
   (a) $\Gamma; \Delta'; A \ll Q^-$ *then* $\Gamma; \Delta, \Delta'; \Omega \Longrightarrow \cdot; Q^-$.
   (b) $\Gamma; \Delta'; \Omega' \cdot A \cdot \Omega'' \Longrightarrow \gamma$ *then* $\Gamma; \Delta, \Delta'; \Omega' \cdot \Omega \cdot \Omega'' \Longrightarrow \gamma$.
   (c) $\Gamma; \Delta', A; \Omega' \Longrightarrow \gamma$ *then* $\Gamma; \Delta, \Delta'; \Omega \cdot \Omega' \Longrightarrow \gamma$.
3. $\Gamma; \cdot \gg A$ *and* $\Gamma, A; \Delta; \Omega \Longrightarrow \gamma$ *then* $\Gamma; \Delta; \Omega \Longrightarrow \gamma$.
4. $\Gamma; \cdot; \cdot \Longrightarrow A; \cdot$ or $\Gamma; \cdot; \cdot \Longrightarrow \cdot; A$ *and* $\Gamma, A; \Delta; \Omega \Longrightarrow \gamma$ *then* $\Gamma; \Delta; \Omega \Longrightarrow \gamma$.

Proof By a nested induction as detailed below, after generalising the statement to include a number of additional *preservative* cuts. We name the three derivations in each case $\mathcal{D}$, $\mathcal{E}$ and $\mathcal{F}$, respectively, where $\mathcal{D}$ and $\mathcal{E}$ are given and $\mathcal{F}$ is constructed. We shall assume that the inductive hypothesis can be used whenever:

(a) The cut proposition is strictly smaller; or
(b) The cut proposition remains the same, but the inductive hypothesis is used for higher numbered cuts to justify a lower numbered cut (that is, a type 3 for a type 2b cut, etc.); or
(c) A preservative cut (see Appendix 6) is used to justify any of the above cuts; or
(d) The cut proposition and $\mathcal{E}$ remain the same, and $\mathcal{D}$ is similar to a strictly smaller first derivation; or
(e) The cut proposition and $\mathcal{D}$ remain the same, and $\mathcal{E}$ is similar to a strictly smaller second derivation.

The details of the proof are in Appendix. □

[2]In fact, if one were to construct an alternative formulation of this calculus where the details of the active rules were fully elided, then similar derivations would be syntactically equal.

We shall use the cut theorem to show that all rules of the non-focusing calculus are admissible in the focusing calculus by interpreting the non-focusing sequents as active sequents. To achieve this, we first need the equivalent of the identity principle for the focusing calculus: $\Gamma; \cdot; A \Longrightarrow A; \cdot$. In the focusing calculus this is not a straightforward induction because of the occurrence restrictions on focal sequents. To illustrate, Δ in $\Gamma; \Delta \gg A$ cannot contain any positive propositions, so the proof of $\Gamma; \cdot; A \otimes B \Longrightarrow A \otimes B; .$ is not simply a proof of $\Gamma; A \otimes B \gg A \otimes B$. We generalise the induction by furnishing a proof in terms of an *expansion* of these asynchronous propositions.

Definition 6 (Expansion)

1. The *left-expansion* of a proposition A, written $\mathrm{lexp}(A)$, is a set of two-zoned contexts defined inductively by the following equations.

$$\mathrm{lexp}(N^+) = \{(\cdot; N^+)\}$$
$$\mathrm{lexp}(A \otimes B) = \{(\Gamma_A, \Gamma_B; \Delta_A, \Delta_B) : (\Gamma_A; \Delta_A) \in \mathrm{lexp}(A) \text{ and } (\Gamma_B; \Delta_B) \in \mathrm{lexp}(B)\}$$
$$\mathrm{lexp}(\mathbf{1}) = \{(\cdot; \cdot)\}$$
$$\mathrm{lexp}(A \oplus B) = \mathrm{lexp}(A) \cup \mathrm{lexp}(B)$$
$$\mathrm{lexp}(\mathbf{0}) = \emptyset$$
$$\mathrm{lexp}(!A) = \{A; \cdot\}$$
$$\mathrm{lexp}(\exists x.A) = \mathrm{lexp}([u/x]A) \quad \text{for a fresh } u$$

2. The *right-expansion* of a proposition A, written $rexp(A)$, is a set of elements of the form $\Gamma; \Delta \Longrightarrow Q$ defined inductively by the following equations.

$$\mathrm{rexp}(Q^-) = \{(\cdot; \cdot \Longrightarrow Q^-)\}$$
$$\mathrm{rexp}(A \& B) = \mathrm{rexp}(A) \cup \mathrm{rexp}(B)$$
$$\mathrm{rexp}(\top) = \emptyset$$
$$\mathrm{rexp}(A \multimap B) = \left\{(\Gamma_A, \Gamma_B; \Delta_A, \Delta_B \Longrightarrow Q) : \begin{array}{l} (\Gamma_A; \Delta_A) \in \mathrm{lexp}(A) \text{ and} \\ (\Gamma_B; \Delta_B \Longrightarrow Q) \in \mathrm{rexp}(B) \end{array}\right\}$$
$$\mathrm{rexp}(\forall x.A) = \mathrm{rexp}([u/x]A) \quad \text{for a fresh } u$$

This definition is associated with a key *expansion lemma*.

Lemma 7 (Expansion lemma) For any proposition A:

1. For any Γ, Δ, Ω and γ,
   if for every $(\Gamma'; \Delta') \in lexp(A)$, $\Gamma, \Gamma'; \Delta, \Delta'; \Omega \Longrightarrow \gamma$ is derivable,
   then $\Gamma; \Delta; \Omega \cdot A \Longrightarrow \gamma$.
2. For any Γ, Δ and Ω,
   if for every $(\Gamma'; \Delta' \Longrightarrow Q'^-) \in rexp(A)$, $\Gamma, \Gamma'; \Delta, \Delta'; \Omega \Longrightarrow \cdot; Q'^-$ is derivable,
   then $\Gamma; \Delta; \Omega \Longrightarrow A; \cdot$.

Proof By induction on the structure of A. We present here some of the key cases.

Case of A is positive, say $B \otimes C$, and arguing for $lexp(A)$. Let Γ, Δ, Ω and γ be given, and assume that for every $(\Gamma'; \Delta') \in lexp(B \otimes C)$ it is the case that the sequent $\Gamma, \Gamma'; \Delta, \Delta'; \Omega \Longrightarrow \gamma$ is derivable. Choose such a $(\Gamma'; \Delta') \in lexp(A \otimes B)$. By Definition 6, $(\Gamma'; \Delta')$ has the form $(\Gamma'_B, \Gamma'_C; \Delta'_B, \Delta'_C)$ for which we know that $(\Gamma'_B; \Delta'_B) \in lexp(B)$ and $(\Gamma'_C; \Delta'_C) \in lexp(C)$.

$\Gamma, \Gamma'_C; \Delta; \Omega \cdot B \Longrightarrow \gamma$	i.h. for B, (Γ, Γ'_C), Δ, and Ω
$\Gamma; \Delta; \Omega \cdot B \cdot C \Longrightarrow \gamma$	i.h. for C, Γ, Δ and $(\Omega \cdot B)$
$\Gamma; \Delta; \Omega \cdot B \otimes C \Longrightarrow \gamma$	$\otimes L$

Then we note that this conclusion is independent of the choice of $(\Gamma'; \Delta')$. Other cases of lexp(A) with A being positive have similar arguments.

Case of $A = N^+$ and arguing for lexp(N^+). In this case, any $(\Gamma'; \Delta') \in$ lexp(A) has the form $(\cdot; N^+)$.

$\Gamma; \Delta, N^+; \Omega \Longrightarrow \gamma$	assumption
$\Gamma; \Delta; \Omega \cdot N^+ \Longrightarrow \gamma$	lact

This completes the inventory of cases for lexp.

Case of $A = B\&C$ and arguing for rexp(A). Let Γ, Δ and Ω be given and assume that for every $(\Gamma'; \Delta' \Longrightarrow Q') \in$ lexp($B\&C$), $\Gamma, \Gamma'; \Delta, \Delta'; \Omega \Longrightarrow \cdot; Q'^-$. By Definition 6, lexp($B \otimes C$) = lexp($B$) $\cup$ lexp(C) the outer quantification also holds for each component of the union; *i.e.*, for every $(\Gamma'; \Delta' \Longrightarrow Q') \in$ lexp(B), $\Gamma, \Gamma'; \Delta, \Delta'; \Omega \Longrightarrow \cdot; Q'^-$, and similarly for lexp(C).

$\Gamma; \Delta; \Omega \Longrightarrow B; \cdot$	i.h. on B, Γ, Δ and Ω
$\Gamma; \Delta; \Omega \Longrightarrow C; \cdot$	i.h. on C, Γ, Δ and Ω
$\Gamma; \Delta; \Omega \Longrightarrow B\&C; \cdot$	$\&R$

Other cases for rexp(A) with A being negative have similar arguments.

Case of $A = Q^-$ and arguing for rexp(A). In this case, all $(\Gamma'; \Delta' \Longrightarrow Q'') \in$ rexp(A) have the form $(\cdot; \cdot \Longrightarrow Q'^-)$.

$\Gamma; \Delta; \Omega \Longrightarrow \cdot; Q^-$	assumption
$\Gamma; \Delta; \Omega \Longrightarrow Q^-; \cdot$	ract

This completes the inventory of all cases for rexp(A). □

We use the expansion lemma to establish the key theorem that will give us the identity principle as a corollary.

Theorem 8 *For any proposition A,*

1. *For every* $(\Gamma; \Delta) \in$ lexp(A)*, we can show* $\Gamma; \Delta \gg A$*; and*
2. *For every* $(\Gamma; \Delta \Longrightarrow Q^-) \in$ rexp(A)*, we can show* $\Gamma; \Delta; A \ll Q^-$.

Proof By structural induction on A and the definition of lexp and rexp (Definition 6). In the inductive argument, the case for rexp(Q) where Q is non-atomic can be used in the argument for lexp(A) (and lexp(P) for rexp(A) similarly). This order is well-founded because there are only finitely many phase changes between synchronous and asynchronous subformulas in a given proposition. We show below some of the key cases of the induction.

Case of lexp($A \otimes B$): every $(\Gamma; \Delta) \in \text{lexp}(A \otimes B)$ is of the form $(\Gamma_A, \Gamma_B; \Delta_A, \Delta_B)$ for which $(\Gamma_A; \Delta_A) \in \text{lexp}(A)$ and $(\Gamma_B; \Delta_B) \in \text{lexp}(B)$.

$\Gamma_A; \Delta_A \gg A$	i.h.
$\Gamma_A, \Gamma_B; \Delta_A \gg A$	weakening
$\Gamma_A, \Gamma_B; \Delta_B \gg B$	similarly
$\Gamma_A, \Gamma_B; \Delta_A, \Delta_B \gg A \otimes B$	$\otimes R$.

All inductive cases of lexp are similar.

Case of rexp($A\&B$): let $(\Gamma; \Delta \Longrightarrow Q) \in \text{rexp}(A\&B)$ be given. By Definition 6, we have (without loss of generality), $(\Gamma; \Delta \Longrightarrow Q) \in \text{rexp}(A)$.

$\Gamma; \Delta; A \ll Q^-$	i.h.
$\Gamma; \Delta; A\&B \ll Q^-$	$\&L_1$.

The other inductive cases of *rexp* are similar.

Case of lexp(N^+): There are three sub-cases here.

Subcase N^+ is a positive-biased atom p. By init$^-$, $\cdot; p \gg p$.

Subcase N^+ is a negative-biased atom n.

$\cdot; \cdot; n \ll n$	init$^-$
$\cdot; n; \cdot \Longrightarrow \cdot; n$	focus$^-$
$\cdot; n \gg n$	blur$^-$

Subcase N^+ is non-atomic. Then, by the induction hypothesis (type 2), for every $(\Gamma; \Delta \Longrightarrow Q^-) \in \text{rexp}(N^+)$, the sequent $\Gamma; \Delta; Q \ll Q^-$ is derivable, and so is $\Gamma; \Delta, Q; \cdot \Longrightarrow \cdot; Q^-$ by focus$^-$.

$\cdot; Q^-; \cdot \Longrightarrow \cdot; Q^-$	expansion Lemma 7
$\cdot; Q^-; \cdot \Longrightarrow N^+; \cdot$	act R
$\cdot; Q^- \gg N^+$	blur$^-$

The case of rexp(P^-) is similar. □

Corollary 9 (Identity principle) *For any proposition A, the sequent $\cdot; \cdot; A \Longrightarrow A; \cdot$ is derivable.*

Proof Suppose A of the form Q^-. There are three cases here.

Case A is a negative-biased atom n.

$\cdot; \cdot; n \ll n$	init$^-$
$\cdot; n; \cdot \Longrightarrow \cdot; n$	focus$^-$
$\cdot; \cdot; n \Longrightarrow n; \cdot$	act L and act R

Case A is a positive-biased atom p.

$$\cdot; p \gg p \qquad \text{init}^+$$
$$\cdot; p; \cdot \Longrightarrow \cdot; p \qquad \text{focus}^+$$
$$\cdot; \cdot; p \Longrightarrow p; \cdot \qquad \text{act } L \text{ and act } R$$

Case A is a non-atomic.

For every $(\Gamma; \Delta) \in lexp(A)$, $\Gamma; \Delta \gg A$ — Theorem 8
For every $(\Gamma; \Delta) \in lexp(A)$, $\Gamma; \Delta; \cdot \Longrightarrow \cdot; A$ — focus$^+$

Note that A is positive, so the above focus$^+$ is valid.

$$\cdot; \cdot; A \Longrightarrow \cdot; A \qquad \text{the expansion Lemma (7)}$$
$$\cdot; \cdot; A \Longrightarrow A; \cdot \qquad \text{act}R$$

The case of A being negative has a similar argument. □

This specific statement of the identity principle will not be used in the completeness proof below; instead, we shall use a slightly variant formulation.

Lemma 10 *The following are derivable* (*for arbitrary* A *and* B *and* a)*:*

1. $\cdot; \cdot; A \cdot B \Longrightarrow A \otimes B; \cdot$
2. $\cdot; \cdot; \cdot \Longrightarrow \mathbf{1}; .$
3. $\cdot; \cdot; A \Longrightarrow A \oplus B; \cdot$ *and* $\cdot; \cdot; B \Longrightarrow A \oplus B; \cdot$
4. $A; \cdot; \cdot \Longrightarrow !A; \cdot$
5. $\cdot; \cdot; [u/x]A \Longrightarrow \exists x.\ A; \cdot$ *where* u *is not free in* $\exists x.\ A$
6. $\cdot; \cdot; A\&B \Longrightarrow A; \cdot$ *and* $\cdot; \cdot; A\&B \Longrightarrow B; \cdot$
7. $\cdot; \cdot; A \cdot A \multimap B \Longrightarrow B; \cdot$
8. $\cdot; \cdot; \forall x.\ A \Longrightarrow [u/x]A; \cdot$ *where* u *is not free in* $\forall x.\ A$

Proof Each case is a simple consequence of the identity principle (Corollary 9). The following is a representative case for $A \otimes B$.

$$\cdot; \cdot; A \otimes B \Longrightarrow A \otimes B; \cdot \qquad \text{Corollary 9.}$$

There are two rules that can conclude this sequent: ract or $\otimes L$. In the former case

$$\cdot; \cdot; A \otimes B \Longrightarrow \cdot; A \otimes B \qquad \text{assumption}$$
$$\cdot; \cdot; A \cdot B \Longrightarrow \cdot; A \otimes B \qquad \text{premiss of } \otimes L \text{ (only possible rule)}$$
$$\cdot; \cdot; A \cdot B \Longrightarrow A \otimes B; \cdot \qquad \text{ract}$$

In the latter case, the premiss is already of the required form $\cdot; \cdot; A \cdot B \Longrightarrow A \otimes B; \cdot$ The remaining cases use similar arguments. □

Theorem 11 (Completeness) *If* $\Gamma; \Delta \Longrightarrow C$ *and* Ω *is any serialisation of* Δ, *then* $\Gamma; \cdot; \Omega \Longrightarrow C; \cdot$.

Proof First we show that all ordinary rules are admissible in the focusing system using cut. We then proceed by induction on derivation $\mathcal{D} :: \Gamma; \Delta \Longrightarrow C$, splitting cases on the last applied rule, using cut and Lemmas 2 and 10 as required. The following is a representative case for $\otimes R$:

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Gamma; \Delta \Longrightarrow A \quad \mathcal{D}_2 :: \Gamma; \Delta' \Longrightarrow B}{\Gamma; \Delta, \Delta' \Longrightarrow A \otimes B} \otimes R$$

Let Ω and Ω' be serialisations of Δ and Δ' respectively.

$\Gamma; \cdot; \Omega \Longrightarrow A; \cdot$	i.h. on $\mathcal{D}_1$
$\Gamma; \cdot; \Omega' \Longrightarrow B; \cdot$	i.h. on $\mathcal{D}_2$
$\Gamma; \cdot; A \cdot B \Longrightarrow A \otimes B; \cdot$	Lemma 10 and weakening
$\Gamma; \cdot; \Omega \cdot \Omega' \Longrightarrow A \otimes B; \cdot$	cut twice

Any serialisation of Δ, Δ' is a permutation of $\Omega \cdot \Omega'$. □

As a remark, once we have the cut and the identity principle, the proof of completeness is extremely straightforward. There are other proofs of completeness of focusing calculi in the literature that do not use cut-elimination as a basis. Andreoli's original proof of completeness for a classical focusing calculus in [1] used a number of permutation arguments for rules. Howe's extension of focusing to intuitionistic and linear logics divided each case of Andreoli's permutation argument into a number of lemmas [16]. Each of Howe's lemma actually bears a strong resemblance to one of the commutative cases of cut, though a precise correspondence is hard to state given the dissimilarities of the two calculi. We believe that cut and identity – independent of their use in proving completeness – are sufficiently interesting in and of themselves as they substantiate the logical basis of focusing. Similar notions of cut and cut-admissibility also exist in Ludics [14], though our calculus and Ludics are philosophically dissimilar enough that we cannot simply import the cut-admissibility argument from Ludics. Rather, we view our proof of cut-admissibility as belonging to a different tradition which sometimes goes by the name "structural cut-elimination" [23].

Example The primary benefit of focusing is the ability to generate derived "big step" inference rules: the intermediate results of a focusing or active phase are not important. Andreoli called these rules "bipoles" because they combine two phases with principal propositions of opposite polarities. Each derived rule starts (at the bottom) with a neutral sequent from which a synchronous proposition is selected for focus. This is followed by a sequence of focusing steps until the proposition under focus becomes asynchronous. Then, the active rules are applied, and we eventually obtain a collection of neutral sequents as the leaves of this fragment of the focused derivation. These neutral sequents are then treated as the premisses of the derived rule that produces the neutral sequent with which we started.

We omit a formal presentation of the derived rule calculus; instead, we motivate it with an example. Consider the implication $\mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$[3] in the unrestricted context Γ. Assuming that this implication is required for the proof, there are essentially two ways to use it. The first usage corresponds to a forward reading of

[3]Standing roughly for "quarter and nickel goes to three dimes".

the implication, would allow one to conclude $\mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$, assuming one were able to produce $\mathtt{q} \otimes \mathtt{n}$. This amounts to the following derivation tree:

$$\dfrac{\dfrac{\dfrac{\dfrac{\Gamma ; \Delta_1 ; \cdot \Longrightarrow \cdot ; \mathtt{q}}{\Gamma ; \Delta_1 ; \cdot \Longrightarrow \mathtt{q} ; \cdot}}{\Gamma ; \Delta_1 \gg \mathtt{q}}\,\text{blur}^- \quad \dfrac{\dfrac{\Gamma ; \Delta_2 ; \cdot \Longrightarrow \cdot ; \mathtt{n}}{\Gamma ; \Delta_2 ; \cdot \Longrightarrow \mathtt{n} ; \cdot}}{\Gamma ; \Delta_2 \gg \mathtt{n}}\,\text{blur}^-}{\Gamma ; \Delta_1, \Delta_2 \gg \mathtt{q} \otimes \mathtt{n}}\,\otimes R \quad \dfrac{\dfrac{\Gamma ; \Delta_3, \mathtt{d}, \mathtt{d}, \mathtt{d} ; \cdot \Longrightarrow \cdot ; Q^-}{\Gamma ; \Delta_3 ; \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \Longrightarrow \cdot ; Q^-}\,\otimes L;\text{lact}}{\Gamma ; \Delta_3 ; \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \ll Q^-}\,\text{blur}^+}{\dfrac{\Gamma ; \Delta_1, \Delta_2, \Delta_3 ; \mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \ll Q^-}{\Gamma ; \Delta_1, \Delta_2, \Delta_3 ; \cdot \Longrightarrow \cdot ; Q^-}\,\text{focus}^!}\,\multimap L$$

We assume here that all atoms are negative-biased, so none of the branches of the derivation can be closed off with an init^+. Thus, we obtain the following derived rule (eliding the empty active zones):

$$\frac{\Gamma ; \Delta_1 \Longrightarrow \mathtt{q} \quad \Gamma ; \Delta_2 \Longrightarrow \mathtt{n} \quad \Gamma ; \Delta_3, \mathtt{d}, \mathtt{d}, \mathtt{d} \Longrightarrow Q^-}{\Gamma ; \Delta_1, \Delta_2, \Delta_3 \Longrightarrow Q^-} \tag{1}$$

In particular, if Q^- is precisely $\mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$, then (1) specializes to:

$$\frac{\Gamma ; \Delta_1 \Longrightarrow \mathtt{q} \quad \Gamma ; \Delta_2 \Longrightarrow \mathtt{n}}{\Gamma ; \Delta_1, \Delta_2 \Longrightarrow \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}}$$

which is precisely the forward reading of the implication $\mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$.

The opposite reading of the implication would be a proof that uses $\mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$ to produce a proof that uses $\mathtt{q} \otimes \mathtt{n}$. Somewhat unsurprisingly, this situation results from assuming that all atoms are positive-biased. In this case, we get the following derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma ; \mathtt{q} \gg \mathtt{q}}\,\text{init}^+ \quad \dfrac{}{\Gamma ; \mathtt{n} \gg \mathtt{n}}\,\text{init}^+}{\Gamma ; \mathtt{q}, \mathtt{n} \gg \mathtt{q} \otimes \mathtt{n}}\,\otimes R \quad \dfrac{\dfrac{\Gamma ; \Delta, \mathtt{d}, \mathtt{d}, \mathtt{d} ; \cdot \Longrightarrow \cdot ; Q^-}{\Gamma ; \Delta ; \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \Longrightarrow \cdot ; Q^-}\,\otimes L;\text{lact}}{\Gamma ; \Delta ; \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \ll Q^-}\,\text{blur}^+}{\dfrac{\Gamma ; \mathtt{q}, \mathtt{n}, \Delta ; \mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d} \ll Q^-}{\Gamma ; \mathtt{q}, \mathtt{n}, \Delta \Longrightarrow Q}\,\text{focus}^!}\,\multimap L$$

In this positive-biased case, we can terminate the left branch of the derivation with a pair of "init" rules. This rule forces the linear context in this branch of the proof to contain just the atoms q and n. The derived rule we obtain is, therefore,

$$\frac{\Gamma ; \Delta, \mathtt{d}, \mathtt{d}, \mathtt{d} \Longrightarrow Q}{\Gamma ; \Delta, \mathtt{q}, \mathtt{n} \Longrightarrow Q} \tag{2}$$

There are two key differences to observe between the derived rules (1) and (2). The first is that simply altering the bias of the atoms has a big impact on the kinds of rules that are generated; even if we completely ignore the semantic aspect, the rule (2) is preferable to (1) because it is much easier to use single premiss rules.

The second – and more important – observation is that the rule that was generated for the positive-biased atoms has a stronger and more obvious similarity to the proposition $\mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$ that was under focus. If we view the linear zone as the "state" of a system, then the rule (2) corresponds to transforming a portion of the state by replacing q and n by three ds (reading the rule from bottom to top). If, as is common for linear logic, the unrestricted context Γ contains state transition rules for some encoding of a stateful system, then the derived rules generated by left-biasing allows us to directly observe the evolution of the state of the system by looking at the composition of the linear context.

3 The Focused Inverse Method

In this section we briefly sketch the inverse method using the focusing calculus of the previous section. The construction of the inverse method for linear logic is described in more detail in [7]. To distinguish forward from backward sequents, we shall use a single arrow ($\longrightarrow$), but keep the names of the rules the same. In the forward direction, the primary context management issue concerns rules where the conclusion cannot be simply assembled from the premisses. The backward $\top R$ rule has an arbitrary linear context Δ, and the unrestricted context Γ is also unknown in several rules such as init and $\top R$. For the unrestricted zone, this problem is solved in the usual (non-linear) inverse method by collecting only the needed unrestricted assumptions and remembering that they can be weakened if needed [12]. We adapt the solution to the linear zone, which may either be precisely determined (as in the case for initial sequents) or subject to weakening (as in the case for $\top R$). We therefore differentiate sequents whose linear context can be weakened and those whose can not.

Definition 12 (forward sequents) A *forward sequent* is of the form $\Gamma; [\Delta]_w \longrightarrow \gamma$, with w a Boolean (0 or 1) called the *weak-flag*, and γ being either empty ($\cdot$) or a singleton. The sequent $\Gamma; [\Delta]_w \longrightarrow \gamma$ corresponds to the backward sequent $\Gamma'; \Delta' \Longrightarrow C$ if $\Gamma \subseteq \Gamma'$, $\gamma \subseteq C$; and $\Delta = \Delta'$ if $w = 0$ and $\Delta \subseteq \Delta'$ if $w = 1$. Sequents with $w = 1$ are called *weakly linear* or simply *weak*, and those with $w = 0$ are *strongly linear* or *strong*.

Initial sequents are always strong, since their linear context cannot be weakened. On the other hand, $\top R$ always produces a weak sequent. For binary rules, the unrestricted zones are simply juxtaposed. We can achieve the effect of taking their union by applying the explicit contraction rule (which is absent, but admissible in the backward calculus). For the linear zone we have to distinguish cases based on whether the sequent is weak or not. We write the rules using two operators on the linear context – multiplicative composition ($\times$) and additive composition ($+$).

$$\frac{\Gamma; [\Delta]_w \longrightarrow A \quad \Gamma'; [\Delta']_{w'} \longrightarrow B}{\Gamma, \Gamma'; [\Delta]_w \times [\Delta']_{w'} \longrightarrow A \otimes B} \otimes R \qquad \frac{\Gamma; [\Delta]_w \longrightarrow^{w} A \quad \Gamma'; [\Delta']_{w'} \longrightarrow B}{\Gamma, \Gamma'; [\Delta]_w + [\Delta']_{w'} \longrightarrow A \& B} \& R$$

These compositions are defined as follows: For multiplicative rules, it is enough for one premiss to be weak for the conclusion to be weak; the weak flags are therefore joined with a disjunction ($\vee$). Dually, for additive rules, both premisses must be weak for the conclusion to be weak; in this case the weak flags are joined with a conjunction ($\wedge$).

Definition 13 (context composition) The partial operators $\times$ and $+$ on forward linear contexts are defined as follows: $[\Delta]_w \times [\Delta']_{w'} =_{\text{def}} [\Delta, \Delta']_{w \vee w'}$, and

$$[\Delta]_w + [\Delta']_{w'} =_{\text{def}} \begin{cases} [\Delta]_0 & \text{if } w = 0 \text{ and either } w' = 0 \text{ and } \Delta = \Delta', \\ & \text{or } w' = 1 \text{ and } \Delta' \subseteq \Delta \\ [\Delta']_0 & \text{if } w' = 0,\ w = 1 \text{ and } \Delta \subseteq \Delta' \\ [\Delta \sqcup \Delta']_1 & \text{if } w = w' = 1 \end{cases}$$

Here $\Delta \sqcup \Delta'$ is the multiset union of Δ and Δ'.

In the lifted version of this calculus with free variables, there is no longer a single context represented by $\Delta \sqcup \Delta'$ because two propositions might be equalized by substitution. The approach taken in [7] was to define an additional "context simplification" procedure that iteratively calculates a set of candidates that includes every possible context represented by $\Delta \sqcup \Delta'$ by means of contraction. Many of these candidates are then immediately rejected by subsumption arguments. We refer to [7] for the full set of rules, the completeness theorem, and the lifted version of this forward calculus.

3.1 Focused Forward Search

The sketched calculus in the previous section mentioned only single-step rules. We are interested in doing forward search with derived inference rules generated by means of focusing. We therefore have to slightly generalize the context composition operators into a language of context expressions. In the simplest case, we merely have to add a given proposition to the linear context, irrespective of the weak flag. This happens, for instance, in the "$focus^-$" rule where the focused proposition is transferred to the linear context. We write this adjunction as usual using a comma. In the more general case, however, we have to combine two context expressions additively or multiplicatively depending on the kind of rule they were involved in; for these uses, we appropriate the same syntax we used for the single step compositions in the previous section.

$$\text{(context expressions)} \qquad \mathcal{D} ::= [\Delta]_w \mid \mathcal{D}, A \mid \mathcal{D}_1 + \mathcal{D}_2 \mid \mathcal{D}_1 \times \mathcal{D}_2$$

Context expressions can be *simplified* into forward contexts in a bottom-up procedure. We write $\mathcal{D} \hookrightarrow [\Delta]_w$ to denote that $\mathcal{D}$ simplifies into $[\Delta]_w$; it has the following rules.

$$\frac{}{[\Delta]_w \hookrightarrow [\Delta]_w} \qquad \frac{\mathcal{D} \hookrightarrow [\Delta]_w}{\mathcal{D}, A \hookrightarrow [\Delta, A]_w} \qquad \frac{\mathcal{D}_1 \hookrightarrow [\Delta_1]_{w_1} \quad \mathcal{D}_2 \hookrightarrow [\Delta_2]_{w_2}}{\mathcal{D}_1 + \mathcal{D}_2 \hookrightarrow [\Delta_1]_{w_1} + [\Delta_2]_{w_2}}$$

$$\frac{\mathcal{D}_1 \hookrightarrow [\Delta_1]_{w_1} \quad \mathcal{D}_2 \hookrightarrow [\Delta_2]_{w_2}}{\mathcal{D}_1 \times \mathcal{D}_2 \hookrightarrow [\Delta_1]_{w_1} \times [\Delta_2]_{w_2}}$$

The forward version of backward derived rules can be written with these context expressions in a natural way by allowing unsimplified context expressions in the place of linear contexts in forward sequents. As an example, the negative unrestricted proposition $\mathtt{q} \otimes \mathtt{n} \multimap \mathtt{d} \otimes \mathtt{d} \otimes \mathtt{d}$ has the following derived rule with negative-biased atoms

$$\frac{\Gamma_1; [\Delta_1]_{w_1} \longrightarrow \mathtt{q} \quad \Gamma_2; [\Delta_2]_{w_2} \longrightarrow \mathtt{n} \quad \Gamma_3; [\Delta_3]_{w_3}, \mathtt{d}, \mathtt{d}, \mathtt{d} \longrightarrow Q}{\Gamma_1, \Gamma_2, \Gamma_3; [\Delta_1]_{w_1} \times [\Delta_2]_{w_2} \times [\Delta_3]_{w_3} \longrightarrow Q}$$

After constructing the neutral sequent with a context expression we then simplify it. Note that context simplification is a partial operation, so we may not obtain any conclusions, for example, if the premisses to an additive rule are strong sequents but the linear contexts do not match.

3.2 Focusing in the Inverse Method

The details of the focused inverse method are given in [8]; here we briefly summarize the major differences that arise as a result of focusing bias, that is, allowing both positive and negative atoms. The key calculation as laid out in [8] is of the *frontier literals* of the goal sequent, i.e., those subformulas that are available in neutral sequents to be focused on. For all but the atoms the calculation is the same as before, and for the atoms we make the following modifications.

1. A positive-biased atom is in the frontier if it lies in the boundary of a phase transition from active to focus.
2. A negative-biased atom is in the frontier if it lies in the boundary of a phase transition from active to focus.

We then specialize the inference rules to these frontier literals by computing the derived rules that correspond to giving focus to these literals.

Although the addition of bias gives us different rules for focusing, the use of the rules in the search engine is no different from before. The details of the implementation of the main loop can be found in [7]. The main innovation in our formulation of the inverse method in comparison with other descriptions in the literature is the use of a lazy variant of the OTTER loop that both simplifies the design of the rules and performs well in practice.

3.3 Globalization

When proposing a sequent $\Gamma_g; \Delta_g \Longrightarrow \gamma_g$ as the overall goal to prove, the final unrestricted zone Γ_g is shared in all branches of a proof if it were constructed by backward search. One thus can think of Γ_g as part of the ambient state of the prover, instead of representing it explicitly as part of the current goal. Hence, there is never any need to explicitly record Γ_g or portions of it in the sequents themselves. This gives us the following global and local versions of the focus$^!$ rule in the forward direction.

$$\frac{\Gamma; [\Delta]_w; A \ll Q^- \quad A \in \Gamma_g}{\Gamma; [\Delta]_w \longrightarrow Q^-}\text{delete} \qquad \frac{\Gamma; [\Delta]_w; A \ll Q^- \quad A \notin \Gamma_g}{\Gamma, A; [\Delta]_w \longrightarrow Q^-}\text{focus}^!$$

Globalization thus corresponds to a choice of whether to add the constructed principal proposition of a derived rule into the unrestricted zone or not, depending on whether or not it is part of the unrestricted zone in the goal sequent.

4 The Horn Fragment

In complex specifications that employ linearity, there are often significant sub-specifications that lie in the Horn fragment. Unfortunately, the straightforward inverse method is quite inefficient on Horn propositions, something already noticed by Tammet [27]. His prover switches between hyperresolution for Horn and near-Horn propositions and the inverse method for other propositions.

With focusing, this *ad hoc* strategy selection becomes entirely unnecessary. The focused inverse method for intuitionistic linear logic, when applied to a classical, non-linear Horn proposition, will exactly behave as classical hyperresolution or SLD resolution depending on the focusing bias of the atomic propositions. This remarkable property provides further evidence for the power of focusing as a technique for forward reasoning. In the next two sections we shall describe this correspondence in more detail.

A Horn clause has the form $\neg a_1, \ldots, \neg a_n, a$ where the a_i and a are atomic predicates over their free variables. This can easily be generalized to include conjunction and truth, but we restrict our attention to this simple clausal form, as theories with conjunction and truth can be simplified into this form. A Horn theory Ψ is just a set of Horn clauses, and a Horn query is of the form $\Psi \vdash g$ where g is a ground atomic "goal" proposition.[4] In the following section we use a simple translation $(-)^o$ of these Horn clauses into linear logic where $\neg a_1, \ldots, \neg a_n, a$ containing the free variables $\overrightarrow{x}$ is translated into $\forall \overrightarrow{x}.a_1 \multimap \cdots \multimap a_n \multimap a$, and the query $\Psi \vdash g$ is translated as $(\Psi)^o; [\cdot]_0 \longrightarrow g$.

4.1 Hyperresolution

The hyperresolution strategy for the Horn query $\Psi \vdash g$ is just forward reasoning with the following rule (for $n \geq 1$):

$$\frac{a_1' \quad \cdots \quad a_n'}{\theta a} \quad \begin{cases} \text{where } \neg a_1, \ldots, \neg a_n, a \in \Psi;\ \rho_1, \ldots, \rho_n \text{ are renaming substitutions;} \\ \text{and } \theta = \mathrm{mgu}(\langle \rho_1 a_1', \ldots, \rho_n a_n' \rangle, \langle a_1, \ldots, a_n \rangle) \end{cases}$$

The procedure begins with the collection of unit clauses in Ψ and $\neg g$ as the initial set of facts, and succeeds if the empty fact (contradiction) is generated. Because every clause in the theory has a positive literal, the only way an empty fact can be generated is if it proves the fact g itself (note that g is ground). Because this proof starts from the unit clauses and derives newer facts by interpreting the Horn clauses forwards, it is a "bottom-up" variant of the usual Prolog-style logic programming.

[4] Queries with more general goals can be compiled to this form by adding an extra clause to the theory.

Consider the goal sequent in the translation $(\Psi)^o; [\cdot]_0 \longrightarrow g$ where the atoms are all negative-biased. The frontier is every clause $\forall \vec{x}\,.p_1 \multimap \cdots \multimap p_n \multimap p \in (\Psi)^o$. Focusing on one such clause gives the following abstract derivation in the forward direction:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\Gamma_1\,;[\Delta_1]_{w_1} \longrightarrow \cdot\,; a_1}{\Gamma_1\,;[\Delta_1]_{w_1}\,;\cdot \longrightarrow a_1\,;\cdot}}{\Gamma_1\,;[\Delta_1]_{w_1} \gg a_1} \quad \cdots \quad \dfrac{\dfrac{\Gamma_n\,;[\Delta_n]_{w_n} \longrightarrow \cdot\,; a_n}{\Gamma_n\,;[\Delta_n]_{w_n}\,;\cdot \longrightarrow a_n\,;\cdot}}{\Gamma_n\,;[\Delta_n]_{w_n} \gg a_n} \quad \dfrac{}{\Gamma\,;[\cdot]_0\,; a \ll a}\,\text{init}^-}{\Gamma_1,\ldots,\Gamma_n\,;[\Delta_1]_{w_1}\times\cdots\times[\Delta_n]_{w_n}\,; a_1 \multimap \cdots \multimap a_n \multimap a \ll a}\,{\multimap}L}{\Gamma_1,\ldots,\Gamma_n\,;[\Delta_1]_{w_1}\times\cdots\times[\Delta_n]_{w_n}\,;\forall\vec{x}.\,a_1 \multimap \cdots \multimap a_n \multimap a \ll a}\,\forall L}{\Gamma_1,\ldots,\Gamma_n\,;[\Delta_1]_{w_1}\times\cdots\times[\Delta_n]_{w_n} \longrightarrow \cdot\,; a}\,\text{delete}$$

If we use the shorthand $\Gamma; [\Delta]_w \longrightarrow Q^-$ for the neutral sequent $\Gamma; [\Delta]_w; \cdot \longrightarrow \cdot; Q^-$, the above derived rule is, therefore:

$$\frac{\Gamma_1; \Delta_1 \longrightarrow a_1 \quad \cdots \quad \Gamma_n; [\Delta_n]_{w_n} \longrightarrow a_n}{\Gamma_1, \ldots, \Gamma_n; [\Delta_1]_{w_1} \times \cdots \times [\Delta_n]_{w_n} \longrightarrow a}$$

In the case where $n = 0$, i.e., the clause in the Horn theory was a unit clause a, we obtain an initial sequent of the form $\cdot; [\cdot]_0 \longrightarrow a$. As this clause has an empty left hand side, and none of the derived rules add elements to the left, we can make an immediate observation (Lemma 14) that gives us an isomorphism of rules (Theorem 15).

Lemma 14 *Every sequent generated in the proof of the goal* $(\Psi)^o; [\cdot]_0 \longrightarrow g$ *has an empty left hand side.*

Theorem 15 (Isomorphism of rules) *For every clause* $\neg a_1, ..., \neg a_n, a \in \Psi$ *there is a derived rule*

$$\frac{\Gamma_1; [\Delta_1]_{w_1} \longrightarrow a_1 \quad \cdots \quad \Gamma_n; [\Delta_n]_{w_n} \longrightarrow a_n}{\Gamma_1, \ldots, \Gamma_n; [\Delta_1]_{w_1} \times \cdots \times [\Delta_n]_{w_n} \longrightarrow a}$$

generated for the proof of the goal sequent $(\Psi)^0; [\cdot]_0 \longrightarrow g$ *for a fresh goal literal* g *and only negative-biased atoms.*

Proof (sketch) Note that only the translations of the Horn clauses are on the frontier. The result follows by a straightforward induction over the structure of a Horn clause and the rules of the forward focusing calculus. We omit the details of this rather easy proof that has already been illustrated above. □

These facts let us establish an isomorphism between hyperresolution and negative-biased focused derivations.

Theorem 16 *Every hyperresolution derivation for the Horn query* $\Psi \vdash g$ *has an isomorphic focused derivation for the goal sequent* $(\Psi)^o; [\cdot]_0 \longrightarrow g$ *with negative-biased atoms.*

Proof (sketch) For every fact a' generated by the hyperresolution strategy, we have a corresponding fact $\cdot; [\cdot]_0 \longrightarrow a'$ in the focused derivation (up to a renaming of the free variables). When matching these sequents for consideration as input for a derived rule corresponding to the Horn clause $\neg a_1, \ldots, \neg a_n, a$, we calculate the simultaneous mgu of all the a_i and a'_i for a Horn clause, which is precisely the operation also performed in the hyperresolution rule. The required isomorphism then follows from Theorem 15. □

4.2 SLD Resolution

SLD Resolution [18] is a variant of linear resolution that is complete for Horn theories and is the basic reasoning mechanism in Prolog-like logic programming languages. It is sometimes called "top-down" or "goal-directed" logic programming because it starts from the goal literal and reasons backwards to the unit clauses. The procedure is as follows: for the Horn query $\Psi \vdash g$, we start with just the initial clause g, and then perform forward search using the following rule (using Ξ to stand for clauses).

$$\frac{\Xi, b}{\theta(\Xi, \rho a_1, \rho a_2, \ldots, \rho a_n)} \quad \begin{cases} \text{where } \neg a_1, \ldots, \neg a_n, a \in \Psi; \rho \text{ is a renaming subst.;} \\ \text{and } \theta = \text{mgu}(\rho a, b) \end{cases}$$

When $n = 0$, i.e., for unit clauses in the Horn theory, this rule corresponds to simply deleting the member of the input clause that was unifiable with the unit clause (and applying the resulting substitution to the rest of the clauses). The search procedure succeeds when it is able to derive the empty clause.

To show how SLD resolution is modeled by our focusing system, we reuse the translation from before, but this time all atoms are given a positive polarity. The derivation that corresponds to focusing on the translation of the Horn clause $\neg a_1, \ldots, \neg a_n, a$ is:

$$\dfrac{\dfrac{\dfrac{}{\cdot\,;\, a_1 \gg a_1}\text{init}^+ \quad \cdots \quad \dfrac{}{\cdot\,;\, a_n \gg a_n}\text{init}^+ \quad \dfrac{\dfrac{\Gamma\,;\,[\Delta]_w, a \longrightarrow Q}{\Gamma\,;\,[\Delta]_w\,;\, a \longrightarrow \cdot\,;\, Q^-}}{\Gamma\,;\,[\Delta]_w\,;\, a \ll \cdot\,;\, Q^-}}{\Gamma\,;\,[\Delta]_w, a_1, \ldots, a_n\,;\, a_1 \multimap \cdots a_n \multimap a \ll \cdot\,;\, Q^-}\multimap L}{\Gamma\,;\,[\Delta]_w, a_1, \ldots, a_n \longrightarrow Q}\text{delete}$$

In other words, the derived rule is:

$$\frac{\Gamma; [\Delta, a]_w \longrightarrow Q}{\Gamma; [\Delta, a_1, \ldots, a_n]_w \longrightarrow Q}$$

The frontier of the goal $(\Psi)^0; [\cdot]_0 \longrightarrow g$ in the positive-biased setting contains every member of $(\Psi)^0$, so we obtain one such derived rule for each clause in the Horn theory. The frontier contains, in addition, the positive atom g; assuming there is a negative instance of g somewhere in the theory, we thus generate a single initial sequent, $\cdot; [g]_0 \longrightarrow g$. We immediately observe that:

Lemma 17 *Every sequent generated in the focused derivation of* $(\Psi)^0; [\cdot]_0 \longrightarrow g$ *is of the form* $\cdot; [\Delta]_0 \longrightarrow g$.

Theorem 18 (Isomorphism of rules) *For every clause* $\neg a_1, ..., \neg a_n, a \in \Psi$*, there is a derived rule*

$$\frac{\Gamma; [\Delta, a]_w \longrightarrow Q}{\Gamma; [\Delta, a_1, \ldots, a_n]_w \longrightarrow Q}$$

created for the goal sequent $(\Psi)^0; [\cdot]_0 \longrightarrow g$ *for some goal literal* g *and only positive-biased atoms.*

Proof (sketch) Note that only the translations of the clauses and the goal literal g itself are in the frontier. For g, we get just a single initial sequent $\cdot; [g]_0 \longrightarrow g$. For the translation of the clauses, we use a simple induction on the structure of the clauses and the rules of the forward focusing calculus. Again, we omit the rather easy proof that has been illustrated above. □

Theorem 19 *Every SLD resolution derivation for the Horn query* $\Psi \vdash g$ *has an isomorphic focused derivation for the goal sequent* $(\Psi)^o; [\cdot]_0 \longrightarrow g$ *with positive-biased atoms.*

Proof (sketch) Very similar argument as in Theorem 16, except we note that in the matching conditions in the derived rules we rename the input sequents, whereas in the SLD resolution case we rename the Horn clause itself. However, this renaming is merely an artifact of the procedure and does not itself alter the derivation. □

Although the derivations are isomorphic, the focused derivations may not be as efficient as the SLD resolution in practice because of the need to rename (i.e., copy) the premisses as part of the matching conditions of a derived rule – premisses might contain many more components than the Horn clause itself.

To summarize, given set biases on the atomic propositions, we are able to model either hyperresolution (forward-chaining) or a SLD-resolution (backward-chaining) in forward search in the inverse method. If we look at backward search – starting from the goal sequent and using the rules of Fig. 2 – then again it is clear that using negative-biased atoms gives us SLD-resolution.[5] One interesting case is backward search with positive-biased atoms. For the purely propositional case, it is very easy to see that the resulting search strategy would be hyperresolution. In the first-order case, we conjecture we can recover hyperresolution by introducing parametric assumptions, but an analysis of this beyond the scope of this paper. With this small

[5]In fact, negative-biased backward focusing can be taken as a definition of top-down logic programming.

caveat, one obtains the following diagram, where forward search refers to the focused inverse method and backward search to focused goal-directed search.

	forward search	**backward search**
negative-biased	hyperresolution	SLD-resolution
positive-biased	SLD-resolution	hyperresolution

4.3 Example: Fibonacci Numbers

Here we give a very brief example of the use of biases in a simple Horn problem that nevertheless has important computational features: computing Fibonacci numbers.[6] Depending on whether certain rules are used forwards or backwards, the running time can be linear or exponential. The Horn clauses are as follows, containing two predicates `fib` and `sum` and unary natural numbers encoded with `z` (zero) and `s` (successor).

$\mathtt{fib}(\mathtt{z}, \mathtt{s}\,\mathtt{z}).$
$\mathtt{fib}(\mathtt{s}\,\mathtt{z}, \mathtt{s}\,\mathtt{z}).$
$\forall x, y, z, n.\ \neg\mathtt{fib}(n, x), \neg\mathtt{fib}(\mathtt{s}\,n, y), \neg\mathtt{sum}(x, y, z), \mathtt{fib}(\mathtt{s}(\mathtt{s}\,n), z).$

$\forall x.\ \mathtt{sum}(\mathtt{z}, x, x).$
$\forall x, y, z.\ \neg\mathtt{sum}(x, y, z), \mathtt{sum}(\mathtt{s}\,x, y, \mathtt{s}\,z).$

If `fib` is positive biased, we get the following derived rule for the third clause:

$$\frac{\Gamma_1; [\Delta, \mathtt{fib}(s(s\,n), z)]_{w_1} \longrightarrow Q \quad \Gamma_2; [\Delta_2]_{w_2}, \longrightarrow \mathtt{sum}(x, y, z)}{\Gamma; [\Delta_1]_{w_1} \times [\Delta_2]_{w_2}, \mathtt{fib}(n, s), \mathtt{fib}(sn, y) \longrightarrow Q}$$

When this rule is applied again to the conclusion, the $\mathtt{fib}(\mathtt{s}\,n, y)$ will produce another instance of $\mathtt{fib}(n, x')$ (for some x'), which is a variant of a resource already present in the context. Thus, each case of the Fibonacci function will be reproven, giving a proof of exponential length. This is the backward reading (also known as top-down logic programming) of the Horn clauses (with no detection of shared derivations).

If both `sum` and `fib` are negative biased, then the derived rule for the third clause is as follows:

$$\frac{\Gamma_1; [\Delta_1]_{w_1} \longrightarrow \mathtt{fib}(n, x) \quad \Gamma_2; [\Delta_2]_{w_2} \longrightarrow \mathtt{fib}(\mathtt{s}\,n, y) \quad \Gamma_3; [\Delta_3]_{w_3} \longrightarrow \mathtt{sum}(x, y, z)}{\Gamma_1, \Gamma_2, \Gamma_3; [\Delta_1]_{w_1} \times [\Delta_2]_{w_2} \times [\Delta_3]_{w_3} \longrightarrow \mathtt{fib}(\mathtt{s}(\mathtt{s}\,n), z)}$$

Here, we observe that for every n, a conclusion with $\mathtt{fib}(n, -)$ is constructed exactly once. This is, therefore, the forward reading (also known as bottom-up logic programming) of the Horn clauses.

An interesting feature to consider is the assignment of biases to the `sum` predicate. If it is negative biased, the fifth clause above has the following rule:

$$\frac{\Gamma_1; [\Delta]_w \longrightarrow \mathtt{sum}(x, y, z)}{\Gamma_1; [\Delta]_w \longrightarrow \mathtt{sum}(\mathtt{s}\,x, y, \mathtt{s}\,z)}$$

[6]This example was suggested to the first author by Dale Miller.

This has very poor performance because the rule can be iterated to generate a fresh sequent (the fourth Horn clause above will seed the initial database with the sequent $\cdot; [\mathtt{sum}(\mathtt{z}, n, n)]_0 \longrightarrow \mathtt{sum}(\mathtt{z}, n, n)$ to start the iteration). However, if it is given a positive bias, then we obtain the rule:

$$\frac{\Gamma_1; [\Delta, \mathtt{sum}(\mathtt{s}\,x, y, \mathtt{s}\,z)]_w \longrightarrow Q}{\Gamma_1; [\Delta]_w, \mathtt{sum}(x, y, z) \longrightarrow Q}$$

This rule can only be applied a finite number of times because the terms in the conclusion get smaller in each step. We thus observe that the best assignment of biases for this problem is to give a negative bias to `fib` to promote sharing of sub-derivations, and a positive bias to `sum` to make the search space finite, thus performing a hybrid search overall in the theory. Precisely, the derived rule for `fib` then is:

$$\frac{\Gamma_1; [\Delta_1]_{w_1} \longrightarrow \mathtt{fib}(n, x) \quad \Gamma_2; [\Delta_2]_{w_2} \longrightarrow \mathtt{fib}(\mathtt{s}\,n, y)}{\Gamma_1, \Gamma_2; [\Delta_1]_{w_1} \times [\Delta_2]_{w_2}, \mathtt{sum}(x, y, z) \longrightarrow \mathtt{fib}(\mathtt{s}(\mathtt{s}\,n), z)}$$

5 Experiments

5.1 Propositional Linear Logic

The first class of experiments we performed were on propositional linear logic. We implemented several minor variants of an inverse method prover for propositional linear logic.[7] The propositional fragment is the only fragment where we can compare with external provers, as we are not aware of any first order linear logic provers besides our own. The external prover we compared against is Tammet's Gandalf "nonclassical" distribution (version 0.2), compiled using a packaged version of the Hobbit Scheme compiler. This classical linear logic prover comes in two flavors: resolution (**Gr**) and tableau (**Gt**). Neither version incorporates focusing or globalization, and we did not attempt to bound the search for either prover. Other provers such as LinTAP [21] and llprover [28] fail to prove all but the simplest problems, so we did not do any serious comparisons against them. Our experiments were all run on a 3.4GHz Pentium 4 machine with 1MB L1 cache and 1GB main memory; our provers were compiled using MLTon version 20060213 using the default optimization flags; all times indicated are wall-clock times in seconds and includes the GC time; × denotes unprovability within a time limit of 1 hour. In the following tables, iters refers to number of iterations of the lazy OTTER loop, gen the number of generated sequents, and subs the number of subsumed sequents.

Stateful system encodings In these examples, we encoded the state transition rules for stateful systems such as a change machine, a Blocks World problem with a fixed number of blocks, a few sample Petri nets. For the Blocks World example, we also compared a version that uses the CLF monad [4] and one without.

[7] Available from http://www.cs.cmu.edu/~kaustuv/research.html.

name	negative-biased				positive-biased				Gt	Gr
	iters	gen	subs	time	iters	gen	subs	time	time	time
`blocks`	20	43	18	0.001	12	84	61	0.001	×	×
`blocks-clf`	27	65	26	0.002	5	24	7	<**0.001**	N/A	N/A
`change`	16	22	7	0.001	11	20	6	0.001	0.63	0.31
`petri-1`	23	38	23	**0.001**	284	1099	921	0.062	×	7.08
`petri-2`	57	133	105	**0.003**	393	1654	1433	0.068	×	7.13

Graph exploration algorithms In these examples we encode algorithms for calculating Euler or Hamiltonian tours on graphs as linear theorem proving problems. The problems have an equal balance of proofs (i.e., a tour exists) and refutations (i.e., no tour exists).

name	negative-biased				positive-biased			
	iters	gen	subs	time	iters	gen	subs	time
`euler-1`	6291	11853	5565	9.010	6291	11853	5565	**8.570**
`euler-2`	15640	34329	18689	152.12	15640	34329	18689	**145.9**
`euler-3`	64360	159194	94834	3043.35	64360	159194	94834	**2938.55**
`hamilton`	708	911	185	0.11	165	178	0	<**0.001**

The Euler tour computation uses a symmetric algorithm, so both backward and forward chaining generate the same facts, though, interestingly, a positive-biased search performs slightly better than the negative-biased system. For the Hamiltonian tour examples, the positive-biased search is vastly superior.

Affine logic encoding Linearity is often too stringent a requirement for situations where we simply need *affine* logic, i.e., where every hypothesis is consumed *at most* once. Affine logic can be embedded into linear logic by translating every affine arrow $A \to B$ as either $A \multimap B \otimes \top$ or $A \,\&\, \mathbf{1} \multimap B$. Of course, one might select complex encodings; for example choosing $A \,\&\, !(0 \multimap X) \multimap B$ (for some arbitrary fresh proposition X) instead of $A \,\&\, \mathbf{1} \multimap B$. Even though the two translations are equivalent, the prover performs poorly on the former. The Gandalf provers **Gt** and **Gr** fail on these examples.

encoding	negative-biased				positive-biased			
	iters	gen	subs	time	iters	gen	subs	time
$A \multimap B \otimes \top$	38	108	73	0.003	34	107	73	**0.002**
$A \,\&\, \mathbf{1} \multimap B$	252	1103	828	0.098	62	229	126	**0.019**
$A \,\&\, \mathbf{1}(\mathbf{0} \multimap X) \multimap B$	264	7099	6793	2.028	235	841	578	**0.042**

Quantified Boolean formulas In these examples we used two variants of the algorithm from [20] for encoding QBFs in linear logic. The first variant uses exponentials to encode reusable "copy" rules; this variant performs very well in practice, so the table below collates the results of all the example QBFs in one entry. The second variant maps to the multiplicative-additive fragment of linear logic without exponentials. This variant produces problems that are considerably harder, so we have divided the problems in three sets in increasing order of complexity.

encodings	**negative-biased**				**positive-biased**			
	iters	gen	subs	time	iters	gen	subs	time
`qbf-exp`	1508	1722	140	**0.13**	7948	17610	9590	2.69
`qbf-nonexp-1`	1457	5590	4067	**0.54**	1581	4352	2612	0.58
`qbf-nonexp-2`	15267	517551	502174	368.92	9469	49777	37716	**29.55**
`qbf-nonexp-3`	28556	990196	961494	2807.64	21233	89542	115917	**308.24**

For these examples, when the number of iterations is low (i.e., the problems are simple), the negative-biased search appears to perform better than the positive-biased system. However, as the problems get harder, the positive-biased system becomes dominant.

5.2 First-Order Linear Logic

We have also implemented a first prover for linear logic. Experiments with an early version of the first were documented in [7]. Since then we have made several improvements to the prover, including a complete re-implementation of the focused rule generation engine, and also increased our collection of sample problems.

First stateful systems The first experiments were with first encodings of various stateful systems. We selected a first Blocks World encoding (both with and without the CLF monad), Dijkstra's Urn Game, and an AI planning problem for a certain board game. The positive-biased system performs consistently better than the negative-biased system for these problems.

problem	**negative-biased**				**positive-biased**			
	iters	gen	subs	time	iters	gen	subs	time
`blocks`	45	424	317	0.12	26	387	337	**0.04**
`blocks-clf`	64	697	412	0.264	15	81	69	**0.006**
`urn`	29	72	27	0.24	13	58	55	**0.11**
`board`	349	7021	3138	3.26	166	5296	1752	**0.88**

Purely intuitionistic problems Unfortunately, we are unable to compare our implementation with any other linear provers; to the best of our knowledge, our prover is the only first linear prover in existence. We therefore ran our prover on some problems drawn from the SICS benchmark [26]. These intuitionistic problems were translated into linear logic in two different ways– the first uses Girard's original encoding of classical logic in classical linear logic where every subformula is affixed with the exponential, and the second is a focus-preserving encoding as described in [8]. We also compared our prover with *Sandstorm*, a focusing inverse method theorem prover for intuitionistic logic implemented by students at CMU. The focus-preserving translation is always better than the Girard-translation; however, the complexity of linear logic, particularly the significant complexity of linear contraction, makes it uncompetitive with the intuitionistic prover.

problem	**negative-biased**				**positive-biased**				**SS**
	iters	gen	subs	time	iters	gen	subs	time	time
`SICS1-gir`	360	1948	1394	1.312	368	2897	2181	0.6	0.04
`SICS1-foc`	56	365	313	0.056	64	496	415	0.04	
`SICS2-gir`	3035	16391	11732	11.04	3460	27192	20389	5.856	0.06
`SICS2-foc`	489	3133	2688	0.472	616	4672	3902	0.376	
`SICS3-gir`	20958	1131823	810085	762.312	12924	1015552	761517	218.712	1.12
`SICS3-foc`	3377	21659	18646	33.096	2300	17464	14969	23.296	
`SICS4-gir`	×	×	×	×	×	×	×	×	3.89
`SICS4-foc`	8896	57056	49047	87.184	6144	46818	39993	62.24	

Horn examples from TPTP For our last example, we selected 20 non-trivial Horn problems from the TPTP version 3.1.1. The selection of problems was not systematic, but we did not constrain our selection to any particular section of the TPTP. We used the translation described in Section 4. We omit the list of selected problems.

negative-biased				**positive-biased**			
iters	gen	subs	time	iters	gen	subs	time
4911	314640	287004	**462.859**	6289	704482	526207	638.818

For Horn problems, the negative-biased system, which models hyperresolution, performs better than the positive-biased system, which models SLD resolution. This observation is not unprecedented – the Gandalf system switches to a Hyperresolution strategy for Horn theories [27]. The likely reason is that in the positive-biased system, unlike in SLD resolution system, the derived rule renames the input sequent rather than the rule itself.

6 Conclusion

We have presented an improvement of the focusing inverse method that exploits the flexibility in assigning polarity to atoms, which we call *bias*. This strictly generalizes both hyperresolution and SLD resolution on (classical) Horn clauses to all of intuitionistic linear logic. This strategy shows significant improvement on a number of example problems.

One important avenue of future work pertains to the nature of polarity assignments to the atomic propositions. Andreoli's initial observation in [1] was that the synchronous or asynchronous nature of the atoms may be assessed differently in disjoint multiplicative branches of a proof. This is more general than the fixed global assignment of polarities in our system, so it is worthwhile to consider an extension with variable assignment in our calculus.

The main open question raised by Section 4 is whether the observation that focusing generalises hyperresolution and SLD resolution on the Horn fragment can be extended to a fuller logic. This question is naturally meaningless for intuitionistic logic because hyperresolution is a classical strategy. Focusing for purely classical proof search is interesting, but because all propositional connectives can be treated both synchronously and asynchronously, the interest comes from dual interpretations of classical proofs. In essence, classical logic is too permissive, and it is only when interpreted in more refined logics that interesting properties emerge. For classical linear logic, which does not have this unbridled permissiveness, the connection between biased focusing and resolution is currently open. We conjecture that a suitable adaptation of the focusing calculus for classical linear logic will turn out to give an explanation for hyperresolution for the full classical linear logic.

Another important item of future work would be a detailed analysis of connections with a bottom-up logic programming interpreter for the LO fragment of classical linear logic [3]. This fragment, which is in fact affine, has the property that the unrestricted context remains constant throughout a derivation, and incorporates focusing at least partially via a back-chaining rule. It seems plausible that our prover might simulate their interpreter when LO specifications are appropriately translated into intuitionistic linear logic, similar to the translation of classical Horn clauses.

From a user's perspective, a better characterization of bias assignment is necessary. As shown in Section 4.3, a non-uniform assignment of biases to atoms can significantly improve the search performance over uniform assignments. It is possible that the correct assignment of biases for a given theory can be derived by a *magic set* analysis [24, 25]; Pientka has conjectured further that this bias assignment amounts to using the mode information in the theory to narrow the derived inference rules (Pientka, personal communication, June 2007).

Appendix: Cut-Admissibility Proof

We now present the details of the proof of the cut-admissibility theorem (Theorem 5). Recall that we have two input derivations $\mathcal{D}$ and $\mathcal{E}$ that we perform an induction over, and the inductive hypothesis can be used whenever

(a) The cut proposition is strictly smaller; or
(b) The cut proposition remains the same, but the inductive hypothesis is used for higher numbered cuts to justify a lower numbered cut (that is, a type 5 for a type 5 cut, etc.); or
(c) A preservative cut (see Appendix 6) is used to justify any of the above cuts; or

(d) The cut proposition and $\mathcal{E}$ remain the same, and $\mathcal{D}$ is similar to a strictly smaller first derivation; or
(e) The cut proposition and $\mathcal{D}$ remain the same, and $\mathcal{E}$ is similar to a strictly smaller second derivation.

We now distinguish various kinds of situations which arise in the course of the proof. For each type, proof cases turn out to be very similar so we usually show only a representative case or two.

Appendix 1: Principal Cuts

The same proposition is introduced in the final rule of both $\mathcal{D}$ and $\mathcal{E}$.

Case $\otimes$:

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Gamma; \Delta_1, \gg A \quad \mathcal{D}_2 :: \Gamma; \Delta_2 \gg B}{\Gamma; \Delta_1 \Delta_2 \gg A \otimes B} \otimes R$$

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma ; \Delta' ; \Omega \cdot A \cdot B \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta' ; \Omega \cdot A \otimes B \cdot \Omega' \Longrightarrow \gamma} \otimes L$$

$\Gamma ; \Delta_2, \ \Delta' ; \Omega \ \cdot \ A \ \cdot \ \Omega' \Longrightarrow \gamma$ — cut on $\mathcal{D}_2$ and $\mathcal{E}'$

$\Gamma ; \Delta_1, \ \Delta_2, \ \Delta' ; \Omega \ \cdot \ \Omega' \Longrightarrow \gamma$ — cut on $\mathcal{D}_1$ and above

Case $\mathbf{1}$:

$$\mathcal{D} = \frac{}{\Gamma; \cdot ; \cdot \gg \mathbf{1}} \mathbf{1}R \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma ; \Delta' ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta' ; \Omega \cdot \mathbf{1} \cdot \Omega' \Longrightarrow \gamma} \mathbf{1}L$$

Here $\mathcal{F} = \mathcal{E}'$.

Case $\oplus$:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma ; \Delta \gg A}{\Gamma ; \Delta \gg A \oplus B} \oplus R_1$$

$$\mathcal{E} = \frac{\mathcal{E}_1 :: \Gamma ; \Delta' ; \Omega \cdot A \cdot \Omega' \Longrightarrow \gamma \quad \mathcal{E}_2 :: \Gamma ; \Delta' ; \Omega \cdot B \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta' ; \Omega \cdot A \oplus B \cdot \Omega' \Longrightarrow \gamma} \oplus L$$

$\Gamma ; \Delta, \Delta' ; \Omega \Longrightarrow \gamma$ — cut on $\mathcal{D}'$ and $\mathcal{E}_1$

The case of $plus R_2$ is similar.

Case $\mathbf{0}$: there are no principal cuts for $\mathbf{0}$.

Case !:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma ; \cdot ; \cdot \Longrightarrow A; \cdot}{\Gamma ; \cdot ; \gg !A} !R \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma, A ; \Delta' ; \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma ; \Delta' ; \Omega \cdot ! \ A \cdot \Omega' \Longrightarrow \gamma} !L$$

$\Gamma ; \Delta' ; \Omega \ . \ \Omega\,' \Longrightarrow \gamma$ — cut on $\mathcal{D}'$ and $\mathcal{E}'$

Case ∃:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\, ;\, \Delta \gg [t/x]A}{\Gamma\, ;\, \Delta \gg \exists x.A}\ \exists R \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\, ;\, \Delta'\, ;\, \Omega \cdot [u/x]A \cdot \Omega' \Longrightarrow \gamma}{\Gamma\, ;\, \Delta'\, ;\, \Omega \cdot \exists x.A \cdot \Omega' \Longrightarrow \gamma}\ \exists L^u$$

$$\Gamma\, ;\, \Delta, \Delta'\, ;\, \Omega \cdot \Omega' \Longrightarrow \gamma \qquad \text{cut on } \mathcal{D}' \text{ and } [t/u]\mathcal{E}'$$

Case ⊸:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\, ;\, \Delta\, ;\, \Omega \cdot A \Longrightarrow B;\, \cdot}{\Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow A \multimap B;}\ \multimap R \quad \mathcal{E} = \frac{\mathcal{E}_1 :: \Gamma\, ;\, \Delta_1'\, ;\, B \ll Q^- \quad \mathcal{E}_2 :: \Gamma\, ;\, \Delta_2' \gg A}{\Gamma\, ;\, \Delta_1', \Delta_2'\, ;\, A \multimap B \ll Q^-}\ \multimap L$$

$$\Gamma\, ;\, \Delta_2', \Delta\, ;\, \Omega \Longrightarrow B;\, \cdot \qquad \text{cut on } \mathcal{E}_2 \text{ and } \mathcal{D}'$$
$$\Gamma\, ;\, \Delta_1', \Delta_2', \Delta\, ;\, \Omega \Longrightarrow \cdot\, ;\, Q^- \qquad \text{cut on above and } \mathcal{E}_1$$

Case &:

$$\mathcal{D} = \frac{\mathcal{D}_1 :: \Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow A\, ;\, \cdot \quad \mathcal{D}_2 :: \Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow B\, ;\, \cdot}{\Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow A \& B;}\ \& R$$

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\, ;\, \Delta'\, ;\, A \ll Q^-}{\Gamma\, ;\, \Delta'\, ;\, A \,\&\, B \ll Q^-}\ \& L_1$$

$$\Gamma ; \Delta, \Delta' ; \Omega \Longrightarrow \cdot\, ;\, Q^- \qquad \text{cut on } \mathcal{D}_1 \text{ and } \mathcal{E}'$$

The case for $\& L_2$ is similar.

Case ⊤: there are no principal cuts for ⊤.

Case ∀:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow [u/x]A\, ;\, \cdot}{\Gamma\, ;\, \Delta\, ; \Omega \Longrightarrow \forall x.A\, ;\, \cdot}\ \forall R^u \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\, ;\, \Delta'\, ;\, [t/x]A \ll Q^-}{\Gamma\, ;\, \Delta'\, ;\, \forall x.A \ll Q^-}\ \forall L$$

$$\Gamma\, ;\, \Delta, \Delta'\, ;\, \Omega \Longrightarrow \cdot\, ; Q^- \qquad \text{cut on } [t/u]\mathcal{D}' \text{ and } \mathcal{E}'.$$

Appendix 2: Focus Cuts

Here, the last rule in $\mathcal{D}$ or $\mathcal{E}$ gives focus to the cut proposition.

Case $\mathcal{E}$ ends in f*ocus*$^-$. Therefore, the cut proposition is negative, N.

Subcase The cut proposition is active in $\mathcal{D}$.

$$\mathcal{D} :: \Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow N\, ;\, \cdot \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\, ;\, \Delta'\, ;\, N \ll Q^-}{\Gamma\, ;\, \Delta', N'\, ;\, \cdot \Longrightarrow \cdot\, ;\, Q^-}\ \text{focus}^-$$

$$\Gamma\, ;\, \Delta, \Delta'\, ;\, \Omega \Longrightarrow \cdot\, ;\, Q^- \qquad \text{cut on } \mathcal{D} \text{ and } \mathcal{E}'$$

Subcase The cut proposition is passive in $\mathcal{D}$. By the occurrence restriction, it has to be a negative-biased atom.

$$\mathcal{D} :: \Gamma\, ;\, \Delta\, ;\, \Omega \Longrightarrow \cdot\, ;\, n \quad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma\, ;\, \cdot\, ;\, n \ll n}{\Gamma ; n ; \cdot \Longrightarrow \cdot\, ;\, n}\ \text{focus}^-$$

Here $\mathcal{F} = \mathcal{E}$.

Case $\mathcal{D}$ ends in focus$^+$. Therefore, the cut proposition is positive, P.

Subcase The cut proposition is active in $\mathcal{E}$.

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, \Delta \gg P}{\Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, P}\ \text{focus}^+ \quad \mathcal{E} :: \Gamma \,;\, \Delta' \,;\, \Omega \cdot P \cdot \Omega' \Longrightarrow \gamma$$

$$\Gamma \,;\, \Delta,\ \Delta' \,;\, \Omega \cdot \Omega' \Longrightarrow \gamma \qquad \text{cut on } \mathcal{D}' \text{ and } \mathcal{E}$$

Subcase The cut proposition is passive in $\mathcal{E}$. By the occurrence restriction, it has to be a positive-biased atom.

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, p \gg p}{\Gamma \,;\, p \,;\, \cdot \Longrightarrow \cdot \,;\, p}\ \text{focus}^+ \quad \mathcal{E} :: \Gamma \,;\, \Delta', p ; \Omega \Longrightarrow \gamma$$

Here $\mathcal{F} = \mathcal{E}$.

Subcase The cut proposition cannot be in the unrestricted context in $\mathcal{E}$ because the linear context in $\mathcal{D}$ is not empty.

Case $\mathcal{E}$ ends in focus$^!$.

$$\mathcal{D} :: \Gamma ; \cdot ; \cdot \Longrightarrow \begin{cases} \cdot \,; A \\ A \,;\, \cdot \end{cases} \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma, A ; \Delta \,;\, A \ll Q^-}{\Gamma \,;\, A \,;\, \Delta \,; \Longrightarrow \cdot \,;\, Q^-}\ \text{focus}^!$$

$$\Gamma, A ; \cdot \,;\, \cdot \Longrightarrow \begin{cases} \cdot \,; A \\ A ; \cdot \end{cases} \qquad \text{weakening}$$

That is, all sequents in $\mathcal{D}$ are weakened, but the structure of $\mathcal{D}$ itself is unchanged.

$$\Gamma\ , A \,;\, \Delta, \Delta' ; \cdot \Longrightarrow \cdot \,; Q^- \qquad \text{cut on above and } \mathcal{E}'$$
$$\Gamma \,;\, \Delta, \Delta' ; \cdot \Longrightarrow \cdot \,; Q^- \qquad \text{cut on } \mathcal{D} \text{ and above.}$$

Appendix 3: Blur Cuts

Here the last rule in $\mathcal{D}$ or $\mathcal{E}$ blurs focus from the cut proposition.

Case $\mathcal{E}$ ends in blur$^+$. Therefore, the cut proposition is positive, P.

$$\mathcal{D} :: \Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow \begin{cases} \cdot \,; P \\ P ; \cdot \end{cases} \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, P \Longrightarrow \cdot \,;\, Q^-}{\Gamma \,;\, \Delta' \,;\, P \ll Q^-}\ \text{blur}^+$$

$$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \Longrightarrow \cdot ; Q^- \qquad \text{cut on } \mathcal{D} \text{ and } \mathcal{E}'$$

Case $\mathcal{D}$ ends in blur$^-$, so the cut proposition is negative, N. For example:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow N \,;\, \cdot}{\Gamma \,;\, \Delta \gg N}\ \text{blur}^- \qquad \mathcal{E} :: \Gamma \,;\, \Delta' \,;\, \Omega \cdot N \cdot \Omega' \Longrightarrow \gamma$$

$$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot \Omega' \Longrightarrow \gamma \qquad \text{cut on } \mathcal{D}' \text{ and } \mathcal{E}$$

Similar arguments hold for other forms of $\mathcal{E}$ such as $\mathcal{E} :: \Gamma \,;\, \Delta', N \,;\, \Omega \Longrightarrow \gamma$, and also for $\mathcal{E} :: \Gamma, N \,;\, \Delta' \,;\, \Omega \Longrightarrow \gamma$ (in which case $\Delta = \cdot$).

Appendix 4: Critical Pairs

These cuts of the specific form

$$\mathcal{D} :: \Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, A \quad \text{and} \quad \mathcal{E} :: \Gamma \,;\, \Delta', A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$$

Clearly, by the occurrence restrictions on active sequents, A has to be atomic. These cuts are important because the induction switches sides depending on the bias of the atom.

1. A is a negative-biased atom n. Here we proceed by induction on $\mathcal{E}$.

   (1) $\mathcal{E} = \dfrac{\dfrac{}{\Gamma \,;\, \cdot \,;\, n \ll n}\text{init}^-}{\Gamma ; n \,;\, \cdot \Longrightarrow \cdot \,;\, n}\ \text{focus}^-$

   In this case, $\mathcal{F} = \mathcal{D} :: \Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, n$.

   (2) $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta', n \gg Q}{\Gamma \,;\, \Delta', n \,;\, \cdot \Longrightarrow \cdot ; Q}\text{focus}^+$

   $\Gamma \,;\, \Delta, \Delta' \gg Q$ — preservative cut 1 on $\mathcal{D}$ and $\mathcal{E}'$

   $\Gamma \,;\, \Delta, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q$ — focus$^+$

   (3) $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma ; \Delta', n \,;\, N \gg Q^-}{\Gamma \,;\, \Delta', N, n \,;\, \cdot \Longrightarrow \cdot ; Q^-}\ \text{focus}^-$

   $\Gamma \,;\, \Delta, \Delta', n \,;\, N \ll Q^-$ — preservative cut 1 on $\mathcal{D}$ and $\mathcal{E}'$

   $\Gamma \,;\, \Delta, \Delta', n, N \,;\, \cdot \Longrightarrow \cdot ; Q^-$ — focus$^-$

   (4) $\mathcal{E}$ ends in focus$^!$. This is similar to the previous case.

2. p is a positive-biased atom. Here we proceed by induction on $\mathcal{D}$.

   (1) $\mathcal{D} = \dfrac{\dfrac{}{\Gamma \,;\, p \gg p \,;}\text{init}^+}{\Gamma \,;\, p \,;\, \cdot \Longrightarrow \cdot \,;\, p}\ \text{focus}^+$

   Here $\mathcal{F} = \mathcal{E} :: \Gamma \,;\, \Delta', p \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$.

   (2) $\mathcal{D} = \dfrac{\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, N \ll p}{\Gamma \,;\, \Delta, N \,;\, \cdot \Longrightarrow \cdot \,;\, p}\ \text{focus}^-$

   $\Gamma \,;\, \Delta, \Delta' \,;\, N \ll Q^-$ — preservative cut 2 on $\mathcal{D}'$ and $\mathcal{E}$

   $\Gamma \,;\, \Delta, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ — focus$^-$

   (3) $\mathcal{D}$ ends in focus$^!$. This is similar to the previous case.

Appendix 5: Commutative Cuts

The next kind of cuts are cuts that do not have any focused proposition in the conclusion, and the induction proceeds by commuting the cut in the derivation $\mathcal{D}$ or $\mathcal{E}$. In these cuts, therefore, the cut-proposition has to be a side-proposition in one of the derivations $\mathcal{D}$ or $\mathcal{E}$. We shall lay out the cases by enumerating the possibilities for $\mathcal{D}$ before those of $\mathcal{E}$.

1. $\mathcal{D} :: \Gamma \,;\, \Delta \gg A$

   (a) $\mathcal{E} :: \Gamma \,;\, \Delta' \,;\, \Omega \cdot A \cdot \Omega' \Longrightarrow \gamma$.

(1) $\mathcal{E}$ ends in a right-active rule, such as

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta'\ ;\ \Omega \cdot A \cdot \Omega' \cdot D \Longrightarrow E\ ;\ \cdot}{\Gamma\ ;\ \Delta' ; \Omega \cdot A \cdot \Omega' \Longrightarrow D \multimap E\ ;\ \cdot} \multimap R$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot \Omega' \cdot D \Longrightarrow E\ ;\ \cdot$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot \Omega' \Longrightarrow D \multimap E\ ;\ \cdot$ — $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule where A is not principal, such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta'\ ;\ \Omega_1 \cdot D \cdot E \cdot \Omega_2 \cdot A \cdot \Omega' \Longrightarrow \gamma}{\Gamma\ ;\ \Delta'\ ;\ \Omega_1 \cdot D \otimes E \cdot \Omega_2 \cdot A \cdot \Omega' \Longrightarrow \gamma} \otimes L$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega_1 \cdot D \cdot E \cdot \Omega_2\ \cdot\ \Omega' \Longrightarrow \gamma$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega_1 \cdot D \otimes E \cdot \Omega_2\ \cdot\ \Omega' \Longrightarrow \gamma$ — $\otimes L$

For the remainder of the cases, $\mathcal{E}$ ends in a left-active rule where A is principal.

(3) $A = N^+$ and $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma\ ;\ \Delta', N^+\ ;\ \Omega \cdot \Omega' \Longrightarrow \gamma}{\Gamma\ ;\ \Delta'\ ;\ \Omega \cdot N^+ \cdot \Omega' \Longrightarrow \gamma}$ act$L$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot \Omega' \cdot D \Longrightarrow \gamma$ — cut on $\mathcal{D}$ and $\mathcal{E}'$

(4) $A = P$ is the principal in the last rule of $\mathcal{E} :: \Gamma\ ;\ \Delta'\ ;\ \Omega \cdot P \cdot \Omega' \Longrightarrow \gamma$. In this case we have a principal cut P is also principal in $\mathcal{D}$.

(b) $\mathcal{E} :: \Gamma\ ;\ \Delta', A\ ;\ \Omega \Longrightarrow \gamma$.

(1) $\mathcal{E}$ ends in a right-active rule, such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta', A\ ;\ \Omega \cdot D \Longrightarrow E\ ;\ \cdot}{\Gamma\ ;\ \Delta', A\ ;\ \Omega \Longrightarrow D \multimap E\ ;\ \cdot} \multimap R$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot D \Longrightarrow E\ ;\ \cdot$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \Longrightarrow D \multimap E\ ;\ \cdot$ — $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule, such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta', A\ ;\ \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow \gamma}{\Gamma\ ;\ \Delta', A\ ;\ \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow \gamma} \otimes L$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow \gamma$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow \gamma$ — $\otimes L$

(3) The previous two cases take care of the active rules. We now have to account for the neutral case $\mathcal{E} :: \Gamma\ ;\ \Delta', A\ ;\ \cdot \Longrightarrow \cdot\ ;\ Q^-$.

<u>Case</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma\ ;\ \Delta', A \gg Q}{\Gamma\ ;\ \Delta', A\ ;\ \cdot \Longrightarrow \cdot\ ;\ Q}$ focus$^+$

$\Gamma\ ;\ \Delta, \Delta' \gg Q$ — preservative cut 1 on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \cdot \Longrightarrow \cdot\ ;\ Q$ — focus$^+$

<u>Case</u> $\mathcal{E}$ ends with left-focus on A. By the occurrence restriction, A is either negative or a positive-biased atom. The latter case $(A = p)$ is ruled out because f*ocus*$^-$ only grants focus to

negative propositions. In the former case, i.e., for $A = N$, we have:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta', N \ll Q^-}{\Gamma \,;\, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}\text{focus}^-$$

$\Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow N \,;\, \cdot$ only possible premiss of $\mathcal{D}$ (using blur$^+$)
$\Gamma \,;\Delta, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ cut on above and $\mathcal{E}'$ (principal cut)

<u>*Case*</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta', A \,;\, N \ll Q^-}{\Gamma \,;\, \Delta', N, A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^-$

$\Gamma \,;\, \Delta, \Delta' \,;\, N \ll Q^-$ preservative cut 1 on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^-$

<u>*Case*</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, C \,;\, \Delta', A \,;\, C \ll Q^-}{\Gamma, C \,;\, \Delta', A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^!$

$\Gamma, C \,;\, \Delta \gg A$ weakening $\mathcal{D}$
$\Gamma, C \,;\, \Delta, \Delta' \,;\, C \ll Q^-$ preservative cut 1 on above and $\mathcal{E}'$
$\Gamma, C \,;v\Delta, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^!$

(c) $\mathcal{E} :: \Gamma, A \,;\, \Delta' \,;\, \Omega \Longrightarrow \gamma$. In this case $\Delta = \cdot$, i.e., $\mathcal{D} :: \Gamma \,;\, \cdot \gg A$.

(1) $\mathcal{E}$ ends in a right-active rule, such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma, A \,;\, \Delta' \,;\, \Omega \cdot D \Longrightarrow E \,;\, \cdot}{\Gamma, A \,;\, \Delta' \,;\, \Omega \Longrightarrow D \multimap E \,;\, \cdot} \multimap R$$

$\Gamma \,;\, \Delta' \,;\, \Omega \cdot D \Longrightarrow E \,;\, \cdot$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta' \,;\, \Omega \Longrightarrow D \multimap E \,;\, \cdot$ $\otimes L$

(2) $\mathcal{E}$ ends in a left-active rule, such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma, A \,;\, \Delta' \,;\, \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow \gamma}{\Gamma, A \,;\, \Delta' \,;\, \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow \gamma} \otimes L$$

$\Gamma \,;\, \Delta' \,;\, \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta' \,;\, \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow \gamma$ $\otimes L$

(3) This leaves just the neutral cases $\mathcal{E} :: \Gamma, A \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$.

<u>*Case*</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, A \,;\, \Delta', \gg Q}{\Gamma, A \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot ; Q}$ focus$^+$

$\Gamma \,;\, \Delta' \gg Q$ preservative cut 2 on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q$ focus$^+$

<u>*Case*</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, A \,;\, \Delta' \,;\, N \ll Q^-}{\Gamma, A \,;\, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^-$

$\Gamma \,;\, \Delta' \,;\, N \ll Q^-$ preservative cut 2 on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^-$

<u>Case</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, C, A \,;\, \Delta' \,;\, C \ll Q^-}{\Gamma, C, A \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^!$

$\Gamma, C \,;\, \Delta \gg A$ — weakening $\mathcal{D}$
$\Gamma, C \,;\, \Delta' \,;\, C \ll Q^-$ — preservative cut 2 on above and $\mathcal{E}'$
$\Gamma, C \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ — focus$^!$

<u>Case</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, A \,;\, \Delta' \,;\, A \ll Q^-}{\Gamma, A \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^!$

Because A is focused on the left tin $\mathcal{E}'$ and the right in $\mathcal{D}$, one of the two must break focus. First consider the latter case, i.e., for $A = N$

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, \cdot \,;\, \cdot \Longrightarrow N \,;\, \cdot}{\Gamma \,;\, \cdot \gg N} \text{ blur}^+$$

In this case we are in a smaller cut of type 5 (see case 2 below) after weakening $\mathcal{D}' :: \Gamma, N \,;\, \cdot \,;\, \cdot \Longrightarrow N \,;\, \cdot$. In the other case, i.e., for $A = P$,

$$\mathcal{D}' = \frac{\mathcal{E}'' :: \Gamma, P \,;\, \Delta \,;\, P \Longrightarrow \cdot \,;\, Q^-}{\Gamma, P \,;\, \Delta \,;\, P \ll Q^-} \text{ blur}^-$$

P is principal in both $\mathcal{E}''$ and $\mathcal{D}$, so we treat it as a principal cut.

2. $\mathcal{D} :: \Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow A \,;\, \cdot$.
   (a) $\mathcal{E} :: \Gamma \,;\, \Delta' \,;\, A \ll Q^-$
      (1) $A = P$ and $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, P \Longrightarrow \cdot \,;\, Q^-}{\Gamma \,;\, \Delta' \,;\, P \ll Q^-}$ blur$^+$

      $\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \Longrightarrow \cdot \,;\, Q^-$ — cut on $\mathcal{D}$ and $\mathcal{E}'$

      (2) $A = n$ and $\mathcal{E} = \dfrac{}{\Gamma \,;\, \cdot \,;\, n \ll n}$ init$^-$

      $\Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow \cdot \,;\, n$ — inversion on $\mathcal{D}$

      (3) A is non-atomic and negative. Then, there is a similar derivation $\mathcal{D}' \approx \mathcal{D}$ for which the last rule in $\mathcal{D}'$ has A as a principal proposition. Then A is principal in both $\mathcal{D}'$ and $\mathcal{E}$, so we have a principal cut.
   (b) $\mathcal{E} :: \Gamma \,;\, \Delta' \,;\, \Omega' \cdot A \cdot \Omega'' \Longrightarrow \gamma$.
      (1) $\mathcal{E}$ ends in a right-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, \Omega' \cdot A \cdot \Omega'' \cdot D \Longrightarrow E \,;\, \cdot}{\Gamma \,;\, \Delta' \,;\, \Omega' \cdot A \cdot \Omega'' \Longrightarrow D \multimap E \,;\, \cdot} \multimap R$$

      $\Gamma \,;\, \Delta, \Delta' \,;\, \Omega' \cdot \Omega \cdot \Omega'' \cdot D \Longrightarrow E \,;\, \cdot$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
      $\Gamma \,;\, \Delta, \Delta' \,;\, \Omega' \cdot \Omega \cdot \Omega'' \Longrightarrow D \multimap E \,;\, \cdot$ — $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, \Omega_1' \cdot D \cdot E \cdot \Omega_2' \cdot A \cdot \Omega'' \Longrightarrow \gamma}{\Gamma \,;\, \Delta' \,;\, \Omega_1' \cdot D \otimes E \cdot \Omega_2' \cdot A \cdot \Omega'' \Longrightarrow \gamma} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1' \cdot D \cdot E \cdot \Omega_2' \cdot \Omega \cdot \Omega'' \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta,\ \Delta' ;\, \Omega_1' \cdot D \otimes E \cdot \Omega_2' \cdot \Omega \cdot \Omega'' \Longrightarrow \gamma$ $\otimes L$

We are left with the cases for which the last rule in $\mathcal{E}$ was on A.

(3) $A = N^+$ and $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta', N^+ \,;\, \Omega' \cdot \Omega'' \Longrightarrow \gamma}{\Gamma \,;\, \Delta' \,;\, \Omega' \cdot N^+ \cdot \Omega'' \Longrightarrow \gamma}$ act L

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega' \cdot \Omega'' \cdot \Omega \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$

(4) A is a non-atomic positive proposition P, and principal in the last rule of $\mathcal{E} :: \Gamma \,;\, \Delta' \,;\, \Omega \cdot P \cdot \Omega' \Longrightarrow \gamma$. In this case we find a similar $\mathcal{E}' \approx \mathcal{E}$ where the rule for P is delayed as long as possible, then proceed by induction on $\mathcal{E}'$. All cases will be inductive steps of the forms 2(b)i or 2(b)ii above except for the case of the form $\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, P \Longrightarrow \cdot \,;\, Q^-$. Now we induct on $\mathcal{D}$.

<u>Case</u> $\mathcal{D}$ ends in a left-active rule such as

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow P \,;\, \cdot}{\Gamma \,;\, \Delta \,;\, \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow P \,;\, \cdot} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot D \cdot E \cdot \Omega' \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}'$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot D \otimes E \cdot \Omega' \Longrightarrow \cdot \,;\, Q^-$ $\otimes L$

<u>Case</u> $\mathcal{D} = \dfrac{\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow \cdot \,;\, P}{\Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow P \,;\, \cdot}$ act R

Here we have several possibilities:

<u>Subcase</u> $\mathcal{D}'$ ends in a left-active rule, such as:

$$\mathcal{D}' = \frac{\mathcal{D}'' :: \Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot \,;\, P}{\Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot \,;\, P} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}''$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 . D \otimes E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ $\otimes L$

This takes care of all left-active rules in $\mathcal{D}'$, so we just have to account for the focus rules.

<u>Subcase</u> $\mathcal{D}' = \dfrac{\mathcal{D}'' :: \Gamma \,;\, \Delta \gg P}{\Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, P}$ focus$^+$

P is principal in both $\mathcal{D}''$ and $\mathcal{E}'$, which is a principal cut.

<u>Subcase</u> $\mathcal{D}' = \dfrac{\mathcal{D}'' :: \Gamma \,;\, \Delta \,;\, N \ll P}{\Gamma \,;\, \Delta, N \,;\, \cdot \Longrightarrow \cdot \,;\, P}$ focus$^-$

$\Gamma \,;\, \Delta, \Delta' \,;\, N^s \ll Q^-$ preservative cut 6 on $\mathcal{D}''$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta', N^s \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^-$

<u>Subcase</u> $\mathcal{D}' = \dfrac{\mathcal{D}'' :: \Gamma, C \,;\, \Delta \,;\, C \ll P}{\Gamma, C \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, P}$ focus$^!$

$\Gamma, C \,;\, \Delta' \,;\, P \Longrightarrow \cdot \,;\, Q^-$ weakening $\mathcal{E}'$
$\Gamma, C \,;\, \Delta, \Delta' \,;\, C \ll Q^-$ preservative cut 2 on $\mathcal{D}''$ and above
$\Gamma, C \,;\, \Delta, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^!$

(c) $\mathcal{E} :: \Gamma \,;\, \Delta', A \,;\, \Omega' \Longrightarrow \gamma$.

(1) $\mathcal{E}$ ends in a right-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta', A \,;\, \Omega' \cdot D \Longrightarrow E \,;\, \cdot}{\Gamma \,;\, \Delta', A \,;\, \Omega' \Longrightarrow D \multimap E \,;\, \cdot} \multimap R$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot \Omega' \cdot D \Longrightarrow E \,;\, \cdot$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot \Omega' \Longrightarrow D \multimap E \,;\, \cdot$ $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma \,;\, \Delta', A \,;\, \Omega_1' \cdot D \cdot E \cdot \Omega_2' \Longrightarrow \gamma}{\Gamma \,;\, \Delta', A \,;\, \Omega_1' \cdot D \otimes E \cdot \Omega_2' \Longrightarrow \gamma} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot \Omega_1' \cdot D \cdot E \cdot \Omega_2' \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \cdot \Omega_1' \cdot D \otimes E \cdot \Omega_2' \Longrightarrow \gamma$ $\otimes L$

(3) The above two cases take care of all the active rules that $\mathcal{E}$ ends with. This leaves the neutral case $\mathcal{E} :: \Gamma \,;\, \Delta', A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$.

<u>Case</u> $A = N$ and $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, N \ll Q}{\Gamma \,;\, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q}$ focus$^-$

Here we commute into $\mathcal{D}$. If $\mathcal{D}$ ends with a left-active rule:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow N \,;\, \cdot}{\Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega' \Longrightarrow N \,;\, \cdot} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}'$ and $\mathcal{E}$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ $\otimes L$

Otherwise, $\mathcal{D}$ ends with a right-active rule on N. Then, N is principal in both $\mathcal{D}$ and $\mathcal{E}'$, so we have a principal cut.

<u>Case</u> $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta', A \,;\, N \ll Q^-}{\Gamma \,;\, \Delta', N, A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^-$

We find a similar $\mathcal{D}' \approx \mathcal{D}$ where the rules on A are delayed as far as possible, then proceed by induction on the structure of $\mathcal{D}'$.

If $\mathcal{D}'$ ends in a left-active rule, such as:

$$\mathcal{D}' = \frac{\mathcal{D}'' :: \Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow A \,;\, \cdot}{\Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \otimes E.\Omega_2 \Longrightarrow A \,;\, \cdot} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}''$ and $\mathcal{E}$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ $\otimes L$.

This only leaves $\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow A \,;\, \cdot$. If A is positive, then the only possible premiss of $\mathcal{D}'$ is $\Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow \cdot \,;\, A$, so together with $\mathcal{E}$ we have a critical pair.
Otherwise, we may assume that A is a negative proposition M, thereby gaining access to preservative cut 1.

$\Gamma \,;\, \Delta, \Delta' \,;\, N \ll Q^-$ preservative cut on $\mathcal{D}'$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta', N \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^-$

Case $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma, C \,;\, \Delta', A \,;\, C \ll Q^-}{\Gamma, C \,;\, \Delta', A \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-}$ focus$^!$

This is similar to the previous case, except we have to weaken the derivation $\mathcal{D}$ first.

Case $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta', A \gg Q}{\Gamma \,;\, \Delta', A \,;\, \cdot \Longrightarrow \cdot \,;\, Q}$ focus$^+$

We find a similar $\mathcal{D}' \approx \mathcal{D}$ where the rules on A are delayed as far as possible, then proceed by induction on the structure of $\mathcal{D}'$.

If $\mathcal{D}'$ ends in a left-active rule such as

$$\mathcal{D}' = \frac{\mathcal{D}'' :: \Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow A \,;\, \cdot}{\Gamma \,;\, \Delta \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow A \,;\, \cdot} \otimes L$$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}''$ and $\mathcal{E}$
$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot \,;\, Q^-$ $\otimes L$.

This leaves $\mathcal{D}' :: \Gamma \,;\, \Delta \,;\, \cdot \Longrightarrow A \,;\, \cdot$. If A is positive, then the only possible premiss of $\mathcal{D}'$ is $\Gamma \,;\, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, A$, which together with $\mathcal{E}$ is a critical pair.

Otherwise, when A is negative, we can use a preservative cut 1.

$\Gamma \,;\, \Delta, \Delta' \gg Q$ preservative cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \,;\, \Delta, \Delta' \,;\, \cdot \Longrightarrow \cdot \,;\, Q^-$ focus$^+$ or focus$^{+'}$

3. $A = P^-$ and $\mathcal{D} :: \Gamma \,;\, \Delta \,;\, \Omega \Longrightarrow \cdot \,;\, P^-$.
   (a) $\mathcal{E} :: \Gamma \,;\, \Delta' \,;\, P^- \ll Q^-$
      (1) $P^- = n$ and $\mathcal{E} = \dfrac{}{\Gamma \,;\, \cdot \,;\, n \ll n}$ init$^-$
      Here $\mathcal{F} = \mathcal{D}$.
      (2) P^- is non-atomic, i.e., $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma \,;\, \Delta' \,;\, P \Longrightarrow \cdot \,;\, Q^-}{\Gamma \,;\, \Delta' \,;\, P \ll Q^-}$ blur$^+$

$\Gamma \,;\, \Delta, \Delta' \,;\, \Omega \Longrightarrow \cdot \,;\, Q^-$ cut on $\mathcal{D}$ and $\mathcal{E}'$

(b) $\mathcal{E} :: \Gamma\,;\Delta'\,;\Omega'\cdot P^-\cdot\Omega'' \Longrightarrow \gamma$.

(1) $\mathcal{E}$ ends in a right-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\Delta'\,;\Omega'\cdot P^-\cdot\Omega''\cdot D \Longrightarrow E\,;\cdot}{\Gamma\,;\Delta'\,;\Omega'\cdot P^-\cdot\Omega'' \Longrightarrow D \multimap E\,;\cdot}\multimap R$$

$\Gamma\,;\Delta,\Delta'\,;\Omega'\cdot\Omega\cdot\Omega''\cdot D \Longrightarrow E\,;\cdot$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\,;\Delta,\Delta'\,;\Omega'\cdot\Omega\cdot\Omega'' \Longrightarrow D\multimap E\,;\cdot$ $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\,;\Delta'\,;\Omega_1'\cdot D\cdot E\cdot\Omega_2'\cdot P^-\cdot\Omega'' \Longrightarrow \gamma}{\Gamma\,;\Delta'\,;\Omega_1'\cdot D\otimes E\cdot\Omega_2'\cdot P^-\cdot\Omega'' \Longrightarrow \gamma}\otimes L$$

$\Gamma\,;\Delta,\Delta'\,;\Omega_1'\cdot D\cdot E\cdot\Omega_2'\cdot\Omega\cdot\Omega'' \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\,;\Delta,\Delta'\,;\Omega_1'\cdot D\otimes E\cdot\Omega_2'\cdot\Omega\cdot\Omega'' \Longrightarrow \gamma$ $\otimes L$

(3) $P^- = n$ and $\mathcal{E} = \dfrac{\mathcal{E}' :: \Gamma\,;\Delta', n\,;\Omega'\cdot\Omega'' \Longrightarrow \gamma}{\Gamma\,;\Delta'\,;\Omega'\cdot n\cdot\Omega'' \Longrightarrow \gamma}\text{act } L$

$\Gamma\,;\Delta,\Delta'\,;\Omega'\cdot\Omega''\cdot\Omega \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$

(4) P is non-atomic and principal in $\mathcal{E} :: \Gamma\,;\Delta'\,;\Omega\cdot P\cdot\Omega' \Longrightarrow \gamma$. In this case we find a similar $\mathcal{E}' \approx \mathcal{E}$ where the rule for P is delayed as long as possible, then proceed by induction on $\mathcal{E}'$. All cases will be inductive steps of the forms 3(b)i or 3(b)ii above except for the case of the form $\mathcal{E}' :: \Gamma\,;\Delta'\,;P \Longrightarrow \cdot\,;Q^-$. Now we induct on $\mathcal{D}$.

<u>Case</u> $\mathcal{D}$ ends in a left-active rule, such as:

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma\,;\Delta\,;\Omega_1\cdot D\cdot E\cdot\Omega_2 \Longrightarrow \cdot\,;P}{\Gamma\,;\Delta\,;\Omega_1\cdot D\otimes E\cdot\Omega_2 \Longrightarrow \cdot\,;P}\otimes L$$

$\Gamma\,;\Delta,\Delta'\,;\Omega_1\cdot D\cdot E\cdot\Omega_2 \Longrightarrow \cdot\,;Q^-$ cut on $\mathcal{D}'$ and $\mathcal{E}'$
$\Gamma\,;\Delta,\Delta'\,;\Omega_1\cdot D\otimes E\cdot\Omega_2 \Longrightarrow \cdot\,;Q^-$ $\otimes L$

This takes care of everything in Ω, so we just need to consider the cases where $\mathcal{D} :: \Gamma\,;\Delta\,;\cdot \Longrightarrow \cdot\,;P$.

<u>Case</u> $\mathcal{D}$ ends in focus$^+$. Then, P is principal in both $\mathcal{D}$ and $\mathcal{E}'$, so we have a principal cut.

<u>Case</u> $\mathcal{D} = \dfrac{\mathcal{D}' :: \Gamma\,;\Delta\,;N \ll P}{\Gamma\,;\Delta, N\,;\cdot \Longrightarrow \cdot\,;P}\text{focus}^-$

$\Gamma\,;\Delta,\Delta'\,;N \ll Q^-$ preservative cut 2 on $\mathcal{D}'$ and $\mathcal{E}'$
$\Gamma\,;\Delta,\Delta', N\,;\cdot \Longrightarrow \cdot\,;Q^-$ focus$^-$

<u>Case</u> $\mathcal{D} = \dfrac{\mathcal{D}' :: \Gamma, C\,;\Delta\,;C \ll P}{\Gamma, C\,;\Delta\,;\cdot \Longrightarrow \cdot\,;P}\text{focus}^!$

$\Gamma, C\,;\Delta'\,;P \Longrightarrow \cdot\,;Q^-$ weakening $\mathcal{E}'$
$\Gamma, C\,;\Delta,\Delta'\,;C \ll Q^-$ preservative cut 2 on $\mathcal{D}'$ and above
$\Gamma, C\,;\Delta,\Delta'\,;\cdot \Longrightarrow \cdot\,;Q^-$ focus$^!$

(c) $\mathcal{E} :: \Gamma\ ;\ \Delta', P^-\ ;\ \Omega' \Longrightarrow \gamma$. By the occurrence restriction, P^- is of the form n.

(1) $\mathcal{E}$ ends in a right-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta', n\ ;\ \Omega' \cdot D \Longrightarrow E\ ;\ \cdot}{\Gamma\ ;\ \Delta', n\ ;\ \Omega' \Longrightarrow D \multimap E\ ;\ \cdot} \multimap R$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega' \cdot D \Longrightarrow E\ ;\ \cdot$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega' \Longrightarrow D \multimap E\ ;\ \cdot$ $\multimap R$

(2) $\mathcal{E}$ ends in a left-active rule such as:

$$\mathcal{E} = \frac{\mathcal{E}' :: \Gamma\ ;\ \Delta', n\ ;\ \Omega'_1 \cdot D \cdot E \cdot \Omega'_2 \Longrightarrow \gamma}{\Gamma\ ;\ \Delta', n\ ;\ \Omega'_1 \cdot D \otimes E \cdot \Omega'_2 \Longrightarrow \gamma} \otimes L$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega'_1 \cdot D \cdot E \cdot \Omega'_2 \cdot \Omega \Longrightarrow \gamma$ cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega'_1 \cdot D \otimes E \cdot \Omega'_2 \cdot \Omega \Longrightarrow \gamma$ $\otimes L$

(3) $\mathcal{E} :: \Gamma\ ;\ \Delta', n\ ;\ \cdot \Longrightarrow \cdot\ ;\ Q^-$. Here we induct on $\mathcal{D}$.

(1) $\mathcal{D}$ ends in a left-active rule such as:

$$\mathcal{D} = \frac{\mathcal{D}' ::\ \Gamma;\ \Delta\ ;\ \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot\ ;\ n}{\Gamma\ ;\ \Delta\ ;\ \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot\ ;\ n} \otimes L$$

$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega_1 \cdot D \cdot E \cdot \Omega_2 \Longrightarrow \cdot\ ;\ Q^-$ cut on $\mathcal{D}'$ and $\mathcal{E}$
$\Gamma\ ;\ \Delta, \Delta'\ ;\ \Omega_1 \cdot D \otimes E \cdot \Omega_2 \Longrightarrow \cdot\ ;\ Q^-$ $\otimes L$

(2) The above case takes care of all the active rules at the end of $\mathcal{D}$. We are left with just the neutral $\mathcal{D} :: \Gamma\ ;\ \Delta\ ;\ \cdot \Longrightarrow \cdot\ ;\ n$. This is a critical pair.

Appendix 6: Preservative Cuts

The only remaining kinds of cuts are those that preserve focus. Each kind of preservative cut will be shown to reduce to other strictly smaller cuts.

1. If $\mathcal{D} :: \Gamma\ ;\ \Delta \gg A$ and
   (a) $\mathcal{E} :: \Gamma\ ;\ \Delta', A\ ;\ C \ll Q^-$, then $\Gamma\ ;\ \Delta, \Delta'\ ;\ C \ll Q^-$.
   (b) $\mathcal{E} :: \Gamma\ ;\ \Delta', A \gg C$, then $\Gamma\ ;\ \Delta, \Delta' \gg C$.
2. If $\mathcal{D} :: \Gamma\ ;\ \cdot \gg A$ and
   (a) $\mathcal{E} :: \Gamma, A\ ;\ \Delta\ ;\ C \ll Q^-$, then $\Gamma\ ;\ \Delta\ ;\ C \ll Q^-$.
   (b) $\mathcal{E} :: \Gamma, A\ ;\ \Delta \gg C$, then $\Gamma\ ;\ \Delta \gg C$.
3. If $\mathcal{D} :: \Gamma\ ;\ \Delta\ ;\ \cdot \Longrightarrow \begin{cases} \cdot\ ;\ N \\ N\ ;\ \cdot \end{cases}$ and
   (a) $\mathcal{E} :: \Gamma\ ;\ \Delta', N\ ;\ C \ll Q^-$, then $\Gamma\ ;\ \Delta, \Delta'\ ;\ C \ll Q^-$.
   (b) $\mathcal{E} :: \Gamma\ ;\ \Delta', N \gg C$, then $\Gamma\ ;\ \Delta, \Delta' \gg C$.
4. If $\mathcal{D} :: \Gamma\ ;\ \cdot\ ;\ \cdot \Longrightarrow \begin{cases} \cdot\ ;\ A \\ A\ ;\ \cdot \end{cases}$ and
   (a) $\mathcal{E} :: \Gamma, A\ ;\ \Delta\ ;\ C \ll Q^-$, then $\Gamma\ ;\ \Delta\ ;\ C \ll Q^-$.
   (b) $\mathcal{E} :: \Gamma, A\ ;\ \Delta \gg C$, then $\Gamma\ ;\ \Delta \gg C$.

For these four types of preservative cuts, we proceed by induction on $\mathcal{E}$.

(1) Focus is blurred in the last rule in $\mathcal{E}$. The following is a representative case.

$$\mathcal{D} :: \Gamma \mathbin{;} \Delta \gg A \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma \mathbin{;} \Delta', A \mathbin{;} \cdot \Longrightarrow N \mathbin{;} \cdot}{\Gamma \mathbin{;} \Delta', A \gg N}\,\text{blur}^-$$

$\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} \cdot \Longrightarrow N \mathbin{;} \cdot$ — cut on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \mathbin{;} \Delta, \Delta' \gg N$ — blur$^-$

(2) Focus is maintained in the last rule in $\mathcal{E}$. The following is a representative case.

$$\mathcal{D} :: \Gamma \mathbin{;} \Delta \gg A \qquad \mathcal{E} = \frac{\mathcal{E}' :: \Gamma \mathbin{;} \Delta', A \gg D}{\Gamma \mathbin{;} \Delta', A \gg D \oplus E}\,\oplus R_1$$

$\Gamma \mathbin{;} \Delta, \Delta' \gg D$ — preservative cut 1 on $\mathcal{D}$ and $\mathcal{E}'$
$\Gamma \mathbin{;} \Delta, \Delta' \gg D \oplus E$ — $\oplus R_1$

This leaves just the init$^+$ and init$^-$ rules.

(3) The cut proposition $A = p$ and $\mathcal{E} = \frac{}{\Gamma \mathbin{;} p \gg p}$ init$^+$. The latter two cases 3 and 4 do not apply. Case 2 also does not apply because we need p in the linear zone. The only case is 1, for which $\Delta = p$. The required conclusion is thus $\Gamma ; p \gg p$, which is obvious from init$^+$.

(4) $\mathcal{E} = \frac{}{\Gamma, A \mathbin{;} p \gg p}$ init$^+$. In this case Δ must be $\cdot$, which limits us to cases 2 and 4. In either case, the conclusion $\Gamma \mathbin{;} p \gg p$ is true by init$^+$.

(5) $\mathcal{E} = \frac{}{\Gamma, A \mathbin{;} n \ll n}$ init$^-$. The argument is nearly identical to the previous case. This is the only possible case in which $\mathcal{E}$ can end in init$^-$, as the cut proposition cannot be in focus in $\mathcal{E}$.

We also have a few symmetric cases for the preservative cuts.

5. If $\mathcal{D} :: \Gamma \mathbin{;} \Delta \mathbin{;} C \ll A$ and $\mathcal{E} :: \Gamma \mathbin{;} \Delta' \mathbin{;} A \ll Q^-$ then $\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} C \ll Q^-$.
6. If $\mathcal{D} :: \Gamma \mathbin{;} \Delta \mathbin{;} C \ll P$ and $\mathcal{E} :: \left.\begin{array}{l}\Gamma \mathbin{;} \Delta' \mathbin{;} P \\ \Gamma \mathbin{;} \Delta', P \mathbin{;} \cdot\end{array}\right\} \Longrightarrow \cdot \mathbin{;} Q^-$ then $\Gamma ; \Delta, \Delta' \mathbin{;} C \ll Q^-$.

Here, we proceed by induction on $\mathcal{D}$.

(1) Focus is blurred in the last rule in $\mathcal{D}$. The following is a representative case.

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \mathbin{;} \Delta \mathbin{;} P \Longrightarrow \cdot \mathbin{;} A}{\Gamma \mathbin{;} \Delta \mathbin{;} P \ll A}\,\text{blur}^+ \qquad \mathcal{E} = \Gamma \mathbin{;} \Delta' \mathbin{;} A \ll Q^-$$

$\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} P \Longrightarrow \cdot Q$ — cut on $\mathcal{D}'$ and $\mathcal{E}$
$\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} P \ll Q^-$ — blur$^+$

(2) Focus is maintained in the last rule in $\mathcal{D}$. The following is a representative case.

$$\mathcal{D} = \frac{\mathcal{D}' :: \Gamma \mathbin{;} \Delta \mathbin{;} D \ll A}{\Gamma \mathbin{;} \Delta \mathbin{;} D \& E \ll A}\& L_i \qquad \mathcal{E} = \Gamma \mathbin{;} \Delta' \mathbin{;} A \ll Q^-$$

$\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} D \ll Q^-$ — preservative cut 5 on $\mathcal{D}'$ and $\mathcal{E}$
$\Gamma \mathbin{;} \Delta, \Delta' \mathbin{;} D \& E \ll Q^-$ — $\& L_1$

(3) $\mathcal{D} = \frac{}{\Gamma \mathbin{;} n \ll n}$ init$^-$. Then, case 6 is not applicable, and in case 5, $\Delta = \Delta' = \cdot$ and $A = C = Q = n$. The required conclusion, $\Gamma \mathbin{;} \cdot \mathbin{;} n \ll n$ thus follows from init$^-$.

□

References

1. Andreoli, J.-M.: Logic programming with focusing proofs in linear logic. J. Log. Comput. **2**(3), 297–347 (1992)
2. Andreoli, J.-M.: Focussing and proof construction. Ann. Pure Appl. Logic **107**, 131–163 (2001)
3. Bozzano, M., Delzanno, G., Martelli, M.: Model checking linear logic specifications. TPLP **4**(4–6), 573–619 (2004)
4. Cervesato, I., Pfenning, F., Walker, D., Watkins, K.: A concurrent logical framework I & II. Technical report CMU-CS-02-101 and 102, Department of Computer Science, Carnegie Mellon University (2002) (revised May 2003)
5. Chang, B.-Y.E., Chaudhuri, K., Pfenning, F.: A judgmental analysis of linear logic. Technical report CMU-CS-03-131R, Carnegie Mellon University (December 2003)
6. Chaudhuri, K.: The focused inverse method for linear logic. Ph.D. thesis, Carnegie Mellon University (December 2006) (Also available as Technical Report CMU-CS-06-161)
7. Chaudhuri, K., Pfenning, F.: A focusing inverse method theorem prover for first-order linear logic. In: Proceedings of CADE-20, pp. 69–83. Tallinn, Estonia, Springer-Verlag LNAI-3632 (July 2005)
8. Chaudhuri, K., Pfenning F.: Focusing the inverse method for linear logic. In: Ong, L. (ed.) Proceedings of CSL 2005, pp. 200–215. Oxford, UK, Springer-Verlag LNCS-3634 (August 2005)
9. Chaudhuri, K., Pfenning, F., Price, G.: A logical characterization of forward and backward chaining in the inverse method. In: Furbach, U., Shankar, N. (eds.) Proceedings of the 3rd International Joint Conference on Automated Reasoning (IJCAR'06), Seattle, WA, Springer-Verlag LNCS 4130 (August 2006)
10. Danos, V., Joinet, J.-B., Schellinx, H.: LKQ and LKT: sequent calculi for second order logic based upon dual linear decompositions of classical implication. In: Girard, J.-Y., Lafont, Y. (eds.) Proceedings of the Workshop on Linear Logic, London Mathematical Society Lecture Notes vol. 222, pp. 211–224. Cambridge University Press (1995)
11. Danos, V., Joinet, J.-B., Schellinx, H.: A new deconstructive logic: linear logic. J. Symb. Log. **62**(3), 755–807 (1997)
12. Degtyarev, A., Voronkov, A.: The inverse method. In: Robinson, J. A., Voronkov, A. (eds.) Handbook of Automated Reasoning, pp. 179–272. MIT Press (September 2001)
13. Gentzen, G.: Untersuchungen über das logische Schließen. Math. Z. **39**, 176–210, 405–431 (1935) (English translation in Szabo, M. E., (ed.) The collected papers of Gerhard Gentzen, pp. 68–131, North-Holland 1969)
14. Girard, J.-Y.: Locus solum: from the rules of logic to the logic of rules. Math. Struct. Comput. Sci. **11**, 301–506 (2001)
15. Hodas, J. S., Miller, D.: Logic programming in a fragment of intuitionistic linear logic. Inf. Comput. **110**(2), 327–365 (1994)
16. Howe, J.M.: Proof search issues in some non-classical logics. PhD thesis, University of St. Andrews (September 1998)
17. Jagadeesan, R., Nadathur, G., Saraswat, V.: Testing concurrent sytems: an interpretation of intuitionistic logic. In: Proceedings of the 25th Annual Conference on Foundations of Software Technology and Theoretical Computer Science (FSTTCS), pp. 517–528. Springer-Verlag LNCS 3821 (2005)
18. Kowalski, R., Kuehner, D.: Linear resolution with selection function. Artif. Intell. **2**, 227–260 (1971)
19. Liang, C., Miller, D.: Focusing and polarization in intuitionistic logic. Computer Science Logic 451–465 (2007)
20. Lincoln, P., Mitchell, J. C., Scedrov, A., Shankar, N.: Decision problems for propositional linear logic. Ann. Pure Appl. Logic **56**, 239–311 (1992)
21. Mantel, H., Otten, J.: LinTAP: a tableau prover for linear logic. In: Murray, A. (ed.) International Conference TABLEAUX'99, pp. 217–231, New York, Springer-Verlag LNAI 1617 (June 1999)
22. Miller, D.: A multiple-conclusion meta-logic. In: Abramsky, S. (ed.) Ninth Annual Symposium on Logic in Computer Science, pp. 272–281, Paris, France, IEEE Computer Society Press (July 1994)
23. Pfenning, F.: Structural cut elimination in linear logic. Technical report CMU-CS-94-222, Carnegie Mellon University (December 1994)

24. Ramakrishnan, R., Ullman, J.D.: A survey of deductive database systems. J. Log. Program. **23**(2), 125–145 (1995)
25. Rohmer, J., Lescoeur, R., Kerisit, J.: The Alexander method: a technique for the processing of recursive atoms in deductive databases. New Gener. Comput. **4**, 522–588 (1986)
26. Sahlin, D., Franzén, T., Haridi, S.: An intuitionistic predicate logic theorem prover. J. Log. Comput. **2**(5), 619–656 (1992)
27. Tammet, T.: Resolution, inverse method and the sequent calculus. In: Proceedings of KGC'97, pp. 65–83. Springer-Verlag LNCS 1289 (1997)
28. Tamura, N.: A Linear Logic Prover (llprover). At: http://bach.istc.kobe-u.ac.jp/llprover (1999)
29. Watkins, K., Cervesato, I., Pfenning, F., Walker, D.: A concurrent logical framework I: judgments and properties. Technical report CMU-CS-02-101, Carnegie Mellon University (March 2002)
30. Zeilberger, N.: On the unity of duality. In: Ann. Pure Appl. Log., Special Issue on "Classical Logic and Computation." (2008) (to appear)