Focusing the Inverse Method for Linear Logic

Kaustuv Chaudhuri

Carnegie Mellon University
5000 Forbes Avenue, Pittsburgh, PA 15213, USA
E-mail: kaustuv@cs.cmu.edu

Abstract

Focusing is traditionally seen as a means of reducing inessential non-determinism in backward-reasoning strategies such as uniform proof-search or tableaux systems. In this paper we construct a form of focused derivations for propositional linear logic that is appropriate for forward reasoning in the inverse method. We show that the focused inverse method conservatively generalises the classical hyperresolution strategy for Horn-theories, and demonstrate through a practical implementation that the focused inverse method is considerably faster than the non-focused version.

1. Introduction

Linear logic [7] turns eighteen this year. As it has matured and discovered its talents, it has become important to automate linear reasoning; yet despite some early work on resolution for (classical) linear logic [15], the field has remained surprisingly dormant. In his influential paper on focusing in linear logic [1], Andreoli distinguished two kinds of non-deterministic choices for proof-search: an *essential* kind that reveals the basic choices in a proof, and an *inessential* kind that is a by-product of the narrow scope of logical connectives and inference rules. To prove $A \oplus (B \oplus C)$, for instance, is to prove either A or B or C; the fact that B and C were grouped together is irrelevant as $(A \oplus B) \oplus C$ is of identical nature. The usual focusing prescription is given in terms of backward search by dividing it into two phases. Asynchronous connectives, which have invertible rules, are eagerly decomposed in the *active phase* of search. Synchronous connectives with non-invertible rules are eagerly decomposed under focus in the *focal phase*. In each phase, a sequence of decompositions of the same nature are taken together as an atomic step. It is therefore a fairly natural fit for backward reasoning [2].

Focusing is not as obviously applicable in the forward direction where propositions are constructed rather than decomposed. Forward search starts with initial facts that are likely to contribute to a proof of the goal, and forward rules are applied to already derived facts to generate new facts. (These rules are naturally limited to subformulas of the goal sequent, as forward search without a goal direction is nonsensical.) Every derived fact contains an associated proof that can be locally and independently manipulated. Backward search, in contrast, cannot construct any proofs until the search is complete, and disjoint branches might interfere with each other because of global (logic) variables or consumption of resources. This global interference is often a source of complexity in backward reasoning: resource-management, in particular, is arguably the critical factor in backward search in linear logic. In the forward direction, it turns out that the defining resource-management problem in the backward direction simply does not exist! Forward reasoning is therefore of more than casual interest in the linear setting. (Initial experiments have shown that, without focusing, forward search in linear logic is often quicker than a Tableaux-based strategy; see sec. 6.2.) How to adapt focusing to forward search? Backward focusing corresponds to atomic decomposition of a connective; analogously, forward focusing must correspond to atomic composition of a collection of forward steps. We formalise this intuition in this paper.

We begin by sketching the non-focusing forward sequent calculus for linear logic, which is a contribution of this paper. The design of the calculus is guided by a "newness" principle: the conclusion of a forward rule is never subsumed by any of its premisses, for such a rule is entirely wasted work. This condition turns out to be surprisingly subtle. We then reconstruct backward focusing from first-principles with an eye towards its eventual adaptation in the forward direction, and give a novel completeness proof using structural cut-elimination. We then construct a forward version of the focusing calculus around the concept of derived inference rules. These derived rules are based on a relational interpretation the principal propositions themselves. We end with a discussion of the inverse method that uses these derived rules, show that it is a generalisation of the well-known hyperresolution strategy for Horn-theories, and give a few experimental results.

judgemental rules

$$\frac{}{\Gamma \,;\, p \Longrightarrow p}\ \text{init} \qquad \frac{\Gamma, A \,;\, \Delta, A \Longrightarrow C}{\Gamma, A \,;\, \Delta \Longrightarrow C}\ \text{copy}$$

multiplicative

$$\frac{\Gamma \,;\, \Delta_1 \Longrightarrow A \quad \Gamma \,;\, \Delta_2 \Longrightarrow B}{\Gamma \,;\, \Delta_1, \Delta_2 \Longrightarrow A \otimes B}\ \otimes R \qquad \frac{}{\Gamma \,;\, \cdot \Longrightarrow \mathbf{1}}\ \mathbf{1}R$$

$$\frac{\Gamma \,;\, \Delta, A, B \Longrightarrow C}{\Gamma \,;\, \Delta, A \otimes B \Longrightarrow C}\ \otimes L \qquad \frac{\Gamma \,;\, \Delta \Longrightarrow C}{\Gamma \,;\, \Delta, \mathbf{1} \Longrightarrow C}\ \mathbf{1}L$$

$$\frac{\Gamma \,;\, \Delta, A \Longrightarrow B}{\Gamma \,;\, \Delta \Longrightarrow A \multimap B}\ \multimap R \qquad \frac{\Gamma \,;\, \Delta_1, B \Longrightarrow C \quad \Gamma \,;\, \Delta_2 \Longrightarrow A}{\Gamma \,;\, \Delta_1, \Delta_2, A \multimap B \Longrightarrow C}\ \multimap L$$

additive

$$\frac{\Gamma \,;\, \Delta \Longrightarrow A \quad \Gamma \,;\, \Delta \Longrightarrow B}{\Gamma \,;\, \Delta \Longrightarrow A \,\&\, B}\ \& R \qquad \frac{}{\Gamma \,;\, \Delta \Longrightarrow \top}\ \top R \qquad \frac{\Gamma \,;\, \Delta, A_i \Longrightarrow C}{\Gamma \,;\, \Delta, A_1 \,\&\, A_2 \Longrightarrow C}\ \& L_i$$

exponential

$$\frac{\Gamma \,;\, \cdot \Longrightarrow A}{\Gamma \,;\, \cdot \Longrightarrow\, !A}\ !R \qquad \frac{\Gamma, A \,;\, \Delta \Longrightarrow C}{\Gamma \,;\, \Delta, !A \Longrightarrow C}\ !L$$

Figure 1. Backward linear sequent calculus

2. Backward linear sequent calculus

We concentrate on a propositional fragment of intuitionistic linear logic containing $\{\otimes, \mathbf{1}, \multimap, \&, \top, !\}$, leaving out $\oplus$ and $\mathbf{0}$ to simplify the presentation. This fragment is undecidable [13] and complex enough to exhibit interesting resource management problems in both directions. Our sequent calculus is a fragment of JILL [3]. It has dyadic two-sided sequents of the form $\Gamma \,;\, \Delta \Longrightarrow C$, where the Δ holds the linear resources, and Γ the unrestricted resources. Capital letters stand for propositions and lowercase letters for atomic propositions; a superscripted asterisk indicates a non-atomic proposition. The sequent calculus has the following desirable properties: it lacks weakening and contraction rules for the unrestricted context (they are admissible), has a simple structural cut-admissibility proof, and is based on a well-understood judgemental foundation. The calculus has exactly two judgemental rules: one to infer initial sequents, which are axiomatic, and another to copy an unrestricted resource into the linear context. The full list of rules is in fig. 1.

Lemma 2.1 (structural properties).

1. *If* $\Gamma \,;\, \Delta \Longrightarrow C$ *then* $\Gamma' \,;\, \Delta \Longrightarrow C$ *for any* $\Gamma' \supseteq \Gamma$.
2. *If* $\Gamma \,;\, \Delta \Longrightarrow C$ *then* $\Gamma' \,;\, \Delta' \Longrightarrow C$ *for any permutation* Γ' *of* Γ *and* Δ' *of* Δ.

Lemma 2.2 (cut).

1. *If* $\Gamma \,;\, \Delta \Longrightarrow A$ *and* $\Gamma \,;\, \Delta', A \Longrightarrow C$*, then* $\Gamma \,;\, \Delta, \Delta' \Longrightarrow C$.
2. *If* $\Gamma \,;\, \cdot \Longrightarrow A$ *and* $\Gamma, A \,;\, \Delta \Longrightarrow C$*, then* $\Gamma \,;\, \Delta \Longrightarrow C$.

For the fairly standard proofs, see [3]. The most important consequence of cut being admissible is that all rules are analytic, i.e., the calculus has a *subformula property*. The subformula property can in fact be established in a strong form.

Definition 2.3 (subformulas). *A* decorated formula *is a tuple* $\langle A, s, w \rangle$ *where* A *is a proposition,* s *is a* sign *(+ or −) and* w *is a* weight *(! or ·). The* subformula relation $\leq$ *is the smallest reflexive and transitive relation between decorated subformulas satisfying the following equations.*

$$\langle A, s, ! \rangle \leq \langle !A, s, w \rangle$$

$$\langle A, s', \cdot \rangle \leq \langle A \multimap B, s, w \rangle \quad \langle B, s, \cdot \rangle \leq \langle A \multimap B, s, w \rangle$$

$$\langle A_i, s, \cdot \rangle \leq \langle A_1 * A_2, s, w \rangle \qquad * \in \{\otimes, \&\}, i \in \{1, 2\}$$

where s' *is the opposite of* s*. Decorations and the subformula relation are lifted to (multi)sets in the obvious way. If the weight is not relevant, it is left out.*

Theorem 2.4 (strong subformula property). *In any sequent* $\Gamma' \,;\, \Delta' \Longrightarrow C'$ *used in a proof of* $\Gamma \,;\, \Delta \Longrightarrow C$*:*

$$\langle \Gamma', -, ! \rangle \cup \langle \Delta', - \rangle \cup \{\langle C', + \rangle\} \leq \langle \Gamma, -, ! \rangle \cup \langle \Delta, -, \cdot \rangle \cup \{\langle C, +, \cdot \rangle\}$$

The copy rule can be restricted to the !-weighted or *heavy* subformulas of the goal without affecting completeness.

3. Forward linear sequent calculus

Two kinds of non-determinism in backward search are well-known and independent of linearity: conjunctive (which subgoal is active) and disjunctive (which rule to apply) [9]. The resource-nature of linear hypotheses gives rise to additional non-determinism during search. Simplest of resource non-determinism kinds is multiplicative non-determinism, caused by binary multiplicative rules ($\otimes R$ and $\multimap L$), where the linear zone of the conclusion has to be distributed exactly into the premisses. There are an exponential number of possible splits of the linear zone, which is clearly an undesirable branching factor for backward search. Backward search strategies postpone this split in one of two ways: the Lolli-method which gives an input-output interpretation of the search procedure [11], and the Lygon-method which accumulates global consumption constraints for an external Boolean constraint-solver [10]. Interestingly, multiplicative non-determinism is entirely absent in a forward reading of multiplicative rules: the linear context in the conclusion is formed simply by adjoining those of the premisses. On the multiplicative-exponential fragment, for example, forward search has no resource-management issues at all.

[UNPUBLISHED DRAFT – compiled January 21, 2005 at 1:19 p.m.– submitted to LICS'05]

judgemental rules

$$\frac{}{\cdot\,;\,p \longrightarrow^{0} p}\ \text{init} \qquad \frac{\Gamma\,;\,\Delta, A \longrightarrow^{w} C}{\Gamma \cup \{A\}\,;\,\Delta \longrightarrow^{w} C}\ \text{copy}$$

multiplicative connectives

$$\frac{\Gamma\,;\,\Delta \longrightarrow^{w} A \quad \Gamma'\,;\,\Delta' \longrightarrow^{w'} B}{\Gamma \cup \Gamma'\,;\,\Delta, \Delta' \longrightarrow^{w \vee w'} A \otimes B}\ \otimes R \qquad \frac{\Gamma\,;\,\Delta, A, B \longrightarrow^{w} C}{\Gamma\,;\,\Delta, A \otimes B \longrightarrow^{w} C}\ \otimes L$$

$$\frac{\Gamma\,;\,\Delta, A_i \longrightarrow^{1} C \quad (A_j \notin \Delta)}{\Gamma\,;\,\Delta, A_1 \otimes A_2 \longrightarrow^{1} C}\ \otimes L_i \qquad (i, j) \in \{(1, 2), (2, 1)\}$$

$$\frac{}{\cdot\,;\,\cdot \longrightarrow^{0} \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Gamma\,;\,\Delta \longrightarrow^{0} C}{\Gamma\,;\,\Delta, \mathbf{1} \longrightarrow^{0} C}\ \mathbf{1}L$$

$$\frac{\Gamma\,;\,\Delta, A \longrightarrow^{w} B}{\Gamma\,;\,\Delta \longrightarrow^{w} A \multimap B}\ \multimap R \qquad \frac{\Gamma\,;\,\Delta \longrightarrow^{1} B \quad (A \notin \Delta)}{\Gamma\,;\,\Delta \longrightarrow^{1} A \multimap B}\ \multimap R'$$

$$\frac{\Gamma\,;\,\Delta, B \longrightarrow^{w} C \quad \Gamma'\,;\,\Delta' \longrightarrow^{w'} A}{\Gamma \cup \Gamma'\,;\,\Delta, \Delta', A \multimap B \longrightarrow^{w \vee w'} C}\ \multimap L$$

additive connectives

$$\frac{\Gamma\,;\,\Delta \longrightarrow^{w} A \quad \Gamma'\,;\,\Delta' \longrightarrow^{w'} B \quad (\Delta/w \approx \Delta'/w')}{\Gamma \cup \Gamma'\,;\,\Delta \sqcup \Delta' \longrightarrow^{w \wedge w'} A \,\&\, B}\ \&R$$

$$\frac{}{\cdot\,;\,\cdot \longrightarrow^{1} \top}\ \top R \qquad \frac{\Gamma\,;\,\Delta, A_i \longrightarrow^{w} C}{\Gamma\,;\,\Delta, A_1 \,\&\, A_2 \longrightarrow^{w} C}\ \&L_i \qquad i \in \{1, 2\}$$

exponentials

$$\frac{\Gamma\,;\,\cdot \longrightarrow^{w} A}{\Gamma\,;\,\cdot \longrightarrow^{0}\, !A}\ !R \qquad \frac{\Gamma, A\,;\,\Delta \longrightarrow^{w} C}{\Gamma\,;\,\Delta, !A \longrightarrow^{w} C}\ !L$$

$$\frac{\Gamma\,;\,\Delta \longrightarrow^{0} C \quad (A \notin \Gamma)}{\Gamma\,;\,\Delta, !A \longrightarrow^{0} C}\ !L'$$

Figure 2. Forward linear sequent calculus

To distinguish forward from backward sequents, we use a single arrow ($\longrightarrow$). The primary context-management issue concerns rules where the conclusion cannot be simply assembled from the premisses. The backward $\top R$ rule, for instance, has an arbitrary linear zone; the unrestricted zones in axiomatic rules are arbitrary. For the unrestricted zone, this problem is solved in the usual (non-linear) inverse method by assembling the contents lazily [5]. We adapt the solution to the linear zones also. Sequents with lazily constructed linear zones must, conceptually at least, admit a form of weakening, but not every sequent can allow this weakening. The initial sequent $\cdot\,;\,p \longrightarrow p$, for instance, has no sound weakened form. We differentiate the two kinds of sequents with a Boolean weak-flag.

Definition 3.1 (forward sequents).

1. *A* forward sequent *is of the form* $\Gamma\,;\,\Delta \longrightarrow^{w} C$. Γ *contains the unrestricted resources,* Δ *holds the linear resources, and* w *is a Boolean flag.*
2. *The correspondence* $\trianglelefteq$ *between forward and backward sequents is governed by the following conditions:*

$$\Gamma\,;\,\Delta \longrightarrow^{0} C \trianglelefteq \Gamma'\,;\,\Delta \Longrightarrow C \quad \textit{if}\ \Gamma \subseteq \Gamma'$$
$$\Gamma\,;\,\Delta \longrightarrow^{1} C \trianglelefteq \Gamma'\,;\,\Delta' \Longrightarrow C \quad \textit{if}\ \Gamma \subseteq \Gamma' \textit{ and } \Delta \subseteq \Delta'$$

Sequents with $w = 0$ *are called* weakly linear *or simply* weak, *and those with* $w = 1$ *are* strongly linear *or* strong.

Any backward sequent that corresponds to a strong forward sequent also corresponds to a (possibly smaller) weak sequent. Thus, whenever a strong sequent is needed in a forward rule, a weak sequent can be used instead. Similarly, any weak sequent we derive should subsume all larger sequents (strong or weak).

Definition 3.2 (subsumption). *The* subsumption relation $\leq$ *between forward sequents is the smallest reflexive and transitive relation satisfying:*

$$\left.\begin{aligned}&\big(\Gamma\,;\,\Delta \longrightarrow^{0} C\big) \leq \big(\Gamma'\,;\,\Delta \longrightarrow^{0} C\big)\\ &\big(\Gamma\,;\,\Delta \longrightarrow^{1} C\big) \leq \big(\Gamma'\,;\,\Delta' \longrightarrow^{1} C\big)\end{aligned}\right\}\ \textit{where}\ \Gamma \subseteq \Gamma'\ \&\ \Delta \subseteq \Delta'.$$

Note that strong sequents never subsume weak sequents.

The contexts in axiomatic sequents are left blank. Initial sequents cannot be weak, but $\top R$ produces a weak sequent. Weak sequents model affine logic: this is familiar from embeddings into linear logic that translate affine implications $A \rightarrow B$ as $A \multimap (B \otimes \top)$. The collection of inference rules for the forward calculus is in fig. 2. The following property can be established by inspection.

Property 3.3 (newness). *In any forward rule with conclusion* s *and premisses* $\{s_i\}_{i\in 1\ldots n}$, $s_i \not\leq s$ *for every* $i \in \{1 \ldots n\}$. ◻

For binary rules, the unrestricted zones in the two premisses cannot be known to be equal; however, as the premisses stand for arbitrary weakened forms of the unrestricted zone, it is enough for the conclusion to contain the least upper bound (union) of the input contexts. For binary multiplicative rules like $\otimes R$, the conclusion is weak if either of the premisses is weak; thus, the weak-flag of the conclusion is a Boolean-or of those of the premisses. This composition fails for additive rules where the linear zones in the premisses are not guaranteed to be identical. For this rule to apply, the linear zone of a weak premiss must be included in the linear zone of the other strong premiss. If both premisses are strong, their linear zones must be equal, and if both are weak, then the linear zone of the premiss is the least upper bound of the two input zones. For multisets this upper bound ($\sqcup$) must respect multiplicity: if A occurs n times in Δ and m times in Δ', then it occurs $\max(n, m)$ times in $\Delta \sqcup \Delta'$.

[UNPUBLISHED DRAFT – compiled January 21, 2005 at 1:19 p.m.– submitted to LICS'05]

Definition 3.4 (additive compatibility). *Given two forward sequents* $\Gamma\,;\Delta \longrightarrow^{w} C$ *and* $\Gamma'\,;\Delta' \longrightarrow^{w'} C'$*, their additive zones* Δ *and* Δ' *are* additively compatible*, written* $\Delta/w \approx \Delta'/w'$*, if the following hold:*

$$\begin{array}{llll} \Delta/0 \approx \Delta'/0 & \textit{if } \Delta = \Delta' & \Delta/0 \approx \Delta'/1 & \textit{if } \Delta' \subseteq \Delta \\ \Delta/1 \approx \Delta'/1 & \textit{always} & \Delta/1 \approx \Delta'/0 & \textit{if } \Delta \subseteq \Delta' \end{array}$$

Most unary rules are oblivious to the weakening decoration, which simply survives from the premiss to the conclusion. The exception is $!R$, for which it is unsound to have a weak conclusion; there is no derivation of $\cdot\,;\top \Longrightarrow\, !\top$, for example.

The situation for left rules is more complex. Consider the usual $\otimes L$ rule, without any decorations.

$$\frac{\Gamma\,;\Delta, A, B \longrightarrow C}{\Gamma\,;\Delta, A \otimes B \longrightarrow C}$$

If the premiss is strong, then the rule makes perfect sense with the conclusion also being strong. If the premiss is weak, then neither operand should be required to be present in the linear zone; however, the following rule would violate lem. 3.3:

$$\frac{\Gamma\,;\Delta \longrightarrow^{1} C}{\Gamma\,;\Delta, A \otimes B \longrightarrow^{1} C}\ .$$

We should only apply a $\otimes L$ rule when at least one operand is present in the premiss; this suggests the pair:

$$\frac{\Gamma\,;\Delta, A \longrightarrow^{1} C}{\Gamma\,;\Delta, A \otimes B \longrightarrow^{1} C}\ \otimes L_1 \qquad \frac{\Gamma\,;\Delta, B \longrightarrow^{1} C}{\Gamma\,;\Delta, A \otimes B \longrightarrow^{1} C}\ \otimes L_2.$$

Now we have erred too far in the other direction, as the following inference is foolish:

$$\frac{\Gamma\,;\Delta, A, B \longrightarrow^{1} C}{\Gamma\,;\Delta, A, A \otimes B \longrightarrow^{1} C}\ \otimes L_2.$$

We might as well have consumed both A and B to form the conclusion, and obtained a stronger result. The sensible strategy is: when A and B are both present, they must *both* be consumed. Otherwise, only apply the rule when one operand is present in a weak sequent. A similar observation can be made about all such rules: there is one weakness-agnostic form, and some possible refined forms to account for weakness.

Theorem 3.5 (soundness).
1. *If* $\Gamma\,;\Delta \longrightarrow^{0} C$*, then* $\Gamma\,;\Delta \Longrightarrow C$*.*
2. *If* $\Gamma\,;\Delta \longrightarrow^{1} C$*, then* $\Gamma\,;\Delta' \Longrightarrow C$ *for any* $\Delta' \supseteq \Delta$*.*

Proof (sketch). By induction on the structure of the forward derivations. Case 2 is needed because a weak sequent may be used as a premiss of a rule such as &R that can have a strong conclusion. □

For the completeness theorem we note that the forward calculus infers a possibly stronger form of the goal sequent.

Theorem 3.6 (completeness). *If* $\Gamma\,;\Delta \Longrightarrow C$*, then:*

1. *either* $\Gamma'\,;\Delta \longrightarrow^{0} C$*;*
2. *or* $\Gamma'\,;\Delta' \longrightarrow^{1} C$ *for some* $\Delta' \subseteq \Delta$

for some $\Gamma' \subseteq \Gamma$*.*

Proof (sketch). By induction on the structure of the backward derivation. Weak sequents (case 2) require simple case-analyses of the presence of the copied proposition. □

4. Focused derivations

Search using the backward calculus can always apply invertible rules eagerly in any order as there always exists a proof that goes through the premisses of the invertible rule. Andreoli pointed out in [1] that a similar and dual feature exists for non-invertible rules also: it is enough for completeness to apply a sequence of non-invertible rules eagerly in an atomic operation, as long as the corresponding connectives are of the same *synchronous* nature. For instance, to infer a negative p_1 & (p_2 & p_3), there are three different possible proofs, one for each p_i; these three choices present an essential non-determinism in search. There is never a need to pause with p_2 & p_3 and consider applying a rule on a different proposition; such a loss of “focus” on p_2 & p_3 represents an *inessential* non-determinism during proof search. A backward focused proof thus has two phases. In the *active phase* all possible rules are applied in an arbitrary order to asynchronous propositions. When only synchronous propositions remain, one proposition is selected and a *focused phase* for that proposition begun; non-invertible rules are then eagerly (and non-deterministically) applied to decompose that proposition into asynchronous propositions. The proof then again enters the active phase.

In classical linear logic the synchronous or asynchronous nature of a given connective is identical to its polarity; the negative connectives (&, $\top$, ⅋, $\bot$, $\forall$) are asynchronous, and the positive connectives ($\otimes$, **1**, $\oplus$, **0**, $\exists$) are synchronous. The nature of intuitionistic connectives, though, must be derived without an appeal to polarity, which is alien to the constructive and judgemental philosophy underlying the logic. We derive this nature by examining the rules and phases of search: an asynchronous connective is one for which decomposition is complete in the active phase; a synchronous connective is one for decomposition is complete in the focused phase. This definition happens to coincide with polarities for classical linear logic, but is decidedly external. The conjunction from intuitionistic (non-linear) logic, for instance, is nominally of negative polarity but can be seen as both synchronous and asynchronous by our definition. In classical (non-linear) logic, *every* propositional connective is both synchronous and asynchronous (see sec. 4.2). It is certainly possible to construct connectives that are neutral, i.e., of neither nature; a simple example is the connective +, with $A + B$ defined as $A \oplus B \oplus (A \,\&\, B)$. Atomic propositions too require careful thought: Andreoli observed in [1] that it is sufficient to assign arbitrarily a synchronous or asynchronous nature to the atoms as long as duality is preserved; we instead view all atoms as synchronous, as explained below.

As our backward linear sequent calculus is two-sided, we have left- and right- synchronous and asynchronous connectives. For non-atomic propositions a left-synchronous connective is right-asynchronous, and a left-asynchronous connective right-synchronous; this appears to be universal in well-behaved logics. We define the following notations:

meta-variable	meaning
P	left-synchronous (&, $\top$, $\multimap$, p)
Q	right-synchronous ($\otimes$, **1**, !, p)
L	left-asynchronous ($\otimes$, **1**, !)
R	right-asynchronous (&, $\top$, $\multimap$)

(In this separation of classes ! is counted as right-synchronous, but it is more correct to view it as a composition of a pair of synchronous and asynchronous micro-connectives, as discussed below.)

The backward focusing calculus consists of three kinds of sequents; *right-focal sequents* of the form $\Gamma ; \Delta \gg A$ (A under focus), *left-focal sequents* of the form $\Gamma ; \Delta ; A \ll Q$, and *active sequents* of the form $\Gamma ; \Delta ; \Omega \Longrightarrow C$. Γ indicates the unrestricted zone as usual, Δ contains *only* left-synchronous propositions, and Ω is an ordered sequence of propositions (of arbitrary nature). The active phase is entirely deterministic: it starts on the right side of the active sequent, decomposing it until it becomes right-synchronous, i.e., of the form $\Gamma ; \Delta ; \Omega \Longrightarrow Q$. Then the propositions in Ω are decomposed in order from right to left. The order of Ω is used solely to identify the operating end of the active zone in proof search, and is not of any deep logical consequence. Eventually the sequent is reduced to the form $\Gamma ; \Delta ; \cdot \Longrightarrow Q$, which we call *neutral sequents*.

A focusing phase is launched from a neutral sequent by selecting a proposition from Γ, Δ or the right hand side. The focused formula is decomposed if it is synchronous; otherwise, if it happens to be asynchronous, the focus is blurred in order to enter the active phase. Two focusing rules require special mention. If the left-focal formula is an atom, then the sequent is initial iff the linear zone Δ is empty *and* the right hand side matches the focused formula; this gives the focused version of the "init" rule. If an atom has right-focus, however, it is not enough to simply check that the left matches the right, as there might be some pending decompositions; consider eg. $\cdot ; p \,\&\, q \gg q$. Focus is therefore blurred in this case, and we correspondingly disallow a right atom in a neutral sequent from gaining focus. The other important rule is $!R$, for which the focus is immediately blurred; to illustrate the reason, consider the following diagram:

$$\dfrac{\dfrac{\dfrac{A \otimes B ; \cdot \gg A \quad A \otimes B ; \cdot \gg B}{A \otimes B ; \cdot \gg A \otimes B}\,\otimes R}{A \otimes B ; \cdot \gg !(A \otimes B)}\; !\,R^*}{A \otimes B ; \cdot ; \cdot \Longrightarrow !(A \otimes B)}\;\mathrm{rf}$$

Neither premiss is derivable because it was incorrect to maintain focus in $!\,R^*$; instead, we should have first decomposed $A \otimes B$ on the left. The $!\,R$ rule therefore blurs focus; we can think of this as a composition of two micro-rules:

$$\dfrac{\dfrac{\Gamma ; \Delta ; \cdot \Longrightarrow A}{\Gamma \Longrightarrow A\ \mathit{valid}}\; !\,R_u}{\Gamma ; \cdot \gg !\,A}\; !\,R_l$$

The lower half $!R_l$ changes the judgement from linear truth to validity (categorical truth); it therefore blurs focus, as only the truth judgement can have focus. The upper half $!R_u$ is asynchronous: it is just the categorical definition of validity. Girard has made a similar observation about exponentials: they are composed of one micro-connective to change polarity, and another to model a given behaviour [8, Page 114]. A similar observation can be made about other modal operators like ? of JILL [3] or the concurrency monad of CLF [16]: all of them cause a phase change.

The full set of rules is in fig. 3. Soundness of this calculus is rather an obvious property— forget the distinction between Δ and Ω, elide the focus and blur rules, and the original backward calculus appears. Completeness is not as trivial.

4.1. Completeness via cut-admissibility

We show the completeness of the focusing calculus by interpreting every backward sequent as an active sequent in the focusing calculus, then showing that the backward rules are admissible in the focusing calculus. This proof relies on cut-admissibility in the focusing calculus. Because a non-atomic left-synchronous proposition is right-asynchronous, a left-focal sequent needs to match only an active sequent in a cut; similarly for right-synchronous propositions. Active sequents should match other active sequents, however. Cuts are focus-destroying, as they generally require commutations spanning phase boundaries; the products of a cut are therefore active. This is sufficient for our purposes as we intend to interpret non-focusing sequents as active sequents.

The proof requires two key lemmas: the first notes that permuting the ordered context doesn't affect provability, as the ordered context does not mirror any deep non-commutativity in the logic. This lemma thus allows cutting formulas from anywhere inside the ordered context, and also to re-order the context when needed.

Lemma 4.1. *If $\Gamma ; \Delta ; \Omega \Longrightarrow C$, then $\Gamma ; \Delta ; \Omega' \Longrightarrow C$ for any permutation Ω' of Ω.* □

$\boxed{\Gamma ; \Delta \gg A \qquad \text{right-focal}}$

$$\frac{\Gamma ; \Delta_1 \gg A \quad \Gamma ; \Delta_2 \gg B}{\Gamma ; \Delta_1, \Delta_2 \gg A \otimes B}\ \otimes R \qquad \frac{}{\Gamma ; \cdot \gg \mathbf{1}}\ \mathbf{1}R \qquad \frac{\Gamma ; \cdot ; \cdot \Longrightarrow A}{\Gamma ; \cdot \gg {!A}}\ {!R}$$

$\boxed{\Gamma ; \Delta ; A \ll Q \qquad \text{left-focal}}$

$$\frac{}{\Gamma ; \cdot ; p \ll p}\ \text{init} \qquad \frac{\Gamma ; \Delta ; A_i \ll C}{\Gamma ; \Delta ; A_1 \,\&\, A_2 \ll C}\ \&L_i \qquad i \in \{1, 2\}$$

$$\frac{\Gamma ; \Delta_1 ; B \ll C \quad \Gamma ; \Delta_2 \gg A}{\Gamma ; \Delta_1, \Delta_2 ; A \multimap B \ll C}\ \multimap R$$

$\boxed{\text{focus}}$

$$\frac{\Gamma ; \Delta ; P \ll Q}{\Gamma ; \Delta, P ; \cdot \Longrightarrow Q}\ \text{lf} \qquad \frac{\Gamma, A ; \Delta ; A \ll Q}{\Gamma, A ; \Delta ; \cdot \Longrightarrow Q}\ \text{copy} \qquad \frac{\Gamma ; \Delta \gg Q^*}{\Gamma ; \Delta ; \cdot \Longrightarrow Q^*}\ \text{rf}$$

$\boxed{\Gamma ; \Delta ; \Omega \Longrightarrow R \qquad \text{right-active}}$

$$\frac{\Gamma ; \Delta ; \Omega \Longrightarrow A \quad \Gamma ; \Delta ; \Omega \Longrightarrow B}{\Gamma ; \Delta ; \Omega \Longrightarrow A \,\&\, B}\ \&R \qquad \frac{}{\Gamma ; \Delta ; \Omega \Longrightarrow \top}\ \top R$$

$$\frac{\Gamma ; \Delta ; \Omega \cdot A \Longrightarrow B}{\Gamma ; \Delta ; \Omega \Longrightarrow A \multimap B}\ \multimap R$$

$\boxed{\Gamma ; \Delta ; \Omega \Longrightarrow Q \qquad \text{left-active}}$

$$\frac{\Gamma ; \Delta ; \Omega \cdot A \cdot B \Longrightarrow Q}{\Gamma ; \Delta ; \Omega \cdot A \otimes B \Longrightarrow Q}\ \otimes L \qquad \frac{\Gamma ; \Delta ; \Omega \Longrightarrow Q}{\Gamma ; \Delta ; \Omega \cdot \mathbf{1} \Longrightarrow Q}\ \mathbf{1}L$$

$$\frac{\Gamma, A ; \Delta ; \Omega \Longrightarrow Q}{\Gamma ; \Delta ; \Omega \cdot {!A} \Longrightarrow Q}\ {!L} \qquad \frac{\Gamma ; \Delta, P ; \Omega \Longrightarrow Q}{\Gamma ; \Delta ; \Omega \cdot P \Longrightarrow Q}\ \text{act}$$

$\boxed{\text{blur}}$

$$\frac{\Gamma ; \Delta ; L \Longrightarrow Q}{\Gamma ; \Delta ; L \ll Q}\ \text{lb} \qquad \frac{\Gamma ; \Delta ; \cdot \Longrightarrow R}{\Gamma ; \Delta \gg R}\ \text{rb} \qquad \frac{\Gamma ; \Delta ; \cdot \Longrightarrow p}{\Gamma ; \Delta \gg p}\ \text{rb}'$$

Figure 3. Backward linear focusing calculus

The other lemma shows that left-focus rules are admissible in the active phase, and is vital in showing that focused cuts can be left- or right-commuted.

Lemma 4.2. *The following active rules are admissible.*

$$\frac{\Gamma ; \Delta_1 ; \Omega_1 \Longrightarrow Q_1 \quad \Gamma ; \Delta_2 ; \Omega_2 \Longrightarrow Q_2}{\Gamma ; \Delta_1, \Delta_2 ; \Omega_1 \cdot \Omega_2 \Longrightarrow Q_1 \otimes Q_2} \qquad \frac{}{\Gamma ; \cdot ; \cdot \Longrightarrow \mathbf{1}}$$

$$\frac{\Gamma ; \cdot ; \cdot \Longrightarrow Q}{\Gamma ; \cdot ; \cdot \Longrightarrow {!Q}} \qquad \frac{}{\Gamma ; p ; \cdot \Longrightarrow p} \qquad \frac{\Gamma ; \Delta, P_i ; \Omega \Longrightarrow C}{\Gamma ; \Delta, P_1 \,\&\, P_2 ; \Omega \Longrightarrow C}$$

$$\frac{\Gamma ; \Delta, P ; \Omega \Longrightarrow C \quad \Gamma ; \Delta' \gg A}{\Gamma ; \Delta, \Delta', A \multimap P ; \Omega \Longrightarrow C} \qquad \frac{\Gamma ; \Delta, B \ll C \quad \Gamma ; \Delta' ; \Omega \Longrightarrow P}{\Gamma ; \Delta, \Delta', P \multimap B ; \Omega \Longrightarrow C}$$

Note that they correspond exactly to the focal rules. □

Theorem 4.3 (cut). *If*

1. $\Gamma ; \Delta \gg A$ *and* $\Gamma ; \Delta' ; \Omega \cdot A \Longrightarrow C$ *then* $\Gamma ; \Delta, \Delta' ; \Omega \Longrightarrow C$.
2. $\Gamma ; \cdot \gg A$ *and* $\Gamma, A ; \Delta ; \Omega \Longrightarrow C$ *then* $\Gamma ; \Delta ; \Omega \Longrightarrow C$.
3. $\Gamma ; \Delta ; A \ll Q$ *and* $\Gamma ; \Delta' ; \Omega \Longrightarrow A$ *then* $\Gamma ; \Delta, \Delta' ; \Omega \Longrightarrow Q$.
4. $\Gamma ; \Delta ; \Omega \Longrightarrow A$ *and:*
   (a) $\Gamma ; \Delta ; \Omega' \cdot A \Longrightarrow C$ *then* $\Gamma ; \Delta, \Delta' ; \Omega' \cdot \Omega \Longrightarrow C$.
   (b) $\Gamma ; \Delta', A ; \Omega' \Longrightarrow C$ *then* $\Gamma ; \Delta, \Delta' ; \Omega' \cdot \Omega \Longrightarrow C$.
5. *If* $\Gamma ; \cdot ; \cdot \Longrightarrow A$ *and* $\Gamma, A ; \Delta' ; \Omega' \Longrightarrow C$ *then* $\Gamma ; \Delta' ; \Omega' \Longrightarrow C$.

The proof is by lexicographic induction on the given derivations. The argument is lengthy rather than complex, and is an adaptation of similar structural cut-admissibility proofs in eg. [3].

Lemma 4.4. $\Gamma ; \cdot ; A \Longrightarrow A$ *for any* A *and* Γ. □

Theorem 4.5 (completeness). *If* $\Gamma ; \Delta \Longrightarrow C$ *and* Ω *is any serialisation of* Δ*, then* $\Gamma ; \cdot ; \Omega \Longrightarrow C$.

Proof (sketch). First show that all ordinary rules are admissible in the focusing system using cut. Proceed by induction on derivation $\mathcal{D} :: \Gamma ; \Delta \Longrightarrow C$, splitting cases on the last applied rule, using cut and lemmas 4.1 and 4.4 as required. □

4.2. Comparison with non-linear focusing

There have been many proposed embeddings of ordinary (non-linear) logics into linear logic using the exponential operator [7, 3] that translate sub-formulas uniformly. These translations do not preserve the focusing properties of the source logic, as the exponentials can blur the focus too early. It is possible though to give a focusing-aware translation that is faithful to the focusing system of the source logic. As an example, consider the basic (disjunction-free) intuitionistic propositional logic with connectives $\{\wedge, \mathtt{t}, \supset\}$. The focusing system for this logic treats $\wedge$ as both synchronous and asynchronous, but $\supset$ is left-synchronous. The rules are as follows:

$$\frac{}{\Gamma ; p \ll_I p} \qquad \frac{\Gamma ; A_i \ll_I Q}{\Gamma ; A_1 \wedge A_2 \ll_I Q} \qquad \frac{\Gamma ; B \ll_I Q \quad \Gamma \gg_I A}{\Gamma ; A \supset B \ll_I Q}$$

$$\frac{\Gamma \gg_I A \quad \Gamma \gg_I B}{\Gamma \gg_I A \wedge B} \qquad \frac{\Gamma ; \Omega \Longrightarrow_I A \quad \Gamma ; \Omega \Longrightarrow_I B}{\Gamma ; \Omega \Longrightarrow_I A \wedge B}$$

$$\frac{}{\Gamma \gg_I \mathtt{t}} \qquad \frac{}{\Gamma ; \Omega \Longrightarrow_I \mathtt{t}} \qquad \frac{\Gamma, A ; \Omega \Longrightarrow_I Q}{\Gamma ; \Omega \cdot A \Longrightarrow_I Q}$$

$$\frac{\Gamma ; \Omega \cdot A \Longrightarrow_I B}{\Gamma ; \Omega \Longrightarrow_I A \supset B} \qquad \frac{\Gamma ; \Omega \cdot A \cdot B \Longrightarrow_I Q}{\Gamma ; \Omega \cdot A \wedge B \Longrightarrow_I Q}$$

$$\frac{\Gamma \gg_I Q^*}{\Gamma ; \cdot \Longrightarrow_I Q^*} \qquad \frac{\Gamma ; A \ll_I Q}{\Gamma, A ; \cdot \Longrightarrow_I Q} \qquad \frac{\Gamma ; \cdot \Longrightarrow_I R}{\Gamma \gg_I R} \qquad \frac{\Gamma ; L \Longrightarrow_I Q}{\Gamma ; L \ll_I Q}$$

We intend to translate signed intuitionistic formulas to signed linear formulas in a way that preserves the focusing structure of proofs. The translation is modal with two phases: A (active) and F (focal). A positive focal (and negative active) $\wedge$

is translated as ⊗, and the duals as &. For every use of the act rule, the corresponding translation phase affixes an exponential; the phase-transitions in the image of the translation exactly mirror those in the source.

$$F(p)^- = p \qquad F(p)^+ = p$$
$$A(p)^- = !\,p \qquad A(p)^+ = p$$

$$F(A \wedge B)^- = F(A)^- \mathbin{\&} F(B)^- \qquad F(A \wedge B)^+ = F(A)^+ \otimes F(B)^+$$
$$A(A \wedge B)^- = A(A)^- \otimes A(B)^- \qquad A(A \wedge B)^+ = A(A)^+ \mathbin{\&} A(B)^+$$

$$F(\top)^- = \top \qquad F(\top)^+ = \mathbf{1}$$
$$A(\top)^- = \mathbf{1} \qquad A(\top)^+ = \top$$

$$F(A \supset B)^- = F(A)^+ \multimap F(B)^- \qquad F(A \supset B)^+ = A(A \supset B)^+$$
$$A(A \supset B)^- = !\,F(A \supset B)^- \qquad A(A \supset B)^+ = A(A)^- \multimap A(B)^+$$

The reverse translation, written $-^o$, is trivial: simply erase all !s and rewrite & and ⊗ as ∧. The faithfulness of the translations can be established as a pair of soundness and completeness theorems, provable by simple structural induction.

Theorem 4.6. *Soundness:*

1. *If* $\Gamma\,;\cdot \gg A$ *then* $\Gamma^o \gg_I A^o$.
2. *If* $\Gamma\,;\cdot\,; A \ll Q$ *then* $\Gamma^o\,; A^o \ll_I Q^o$.
3. *If* $\Gamma\,;\cdot\,;\Omega \Longrightarrow C$ *then* $\Gamma^o\,;\Omega^o \Longrightarrow_I C^o$.

Completeness:

1. *If* $\Gamma \gg_I A$ *then* $F(\Gamma)^-\,;\cdot \gg F(A)^+$.
2. *If* $\Gamma\,; A \ll_I Q$ *then* $F(\Gamma)^-\,; F(A)^- \ll F(Q)^+$.
3. *If* $\Gamma\,;\Omega \Longrightarrow_I Q$ *then* $F(\Gamma)^-\,;\cdot\,; A(\Omega)^- \Longrightarrow F(Q)^+$.
4. *If* $\Gamma\,;\Omega \Longrightarrow_I R$ *then* $F(\Gamma)^-\,;\cdot\,; A(\Omega)^- \Longrightarrow A(R)^+$. □

An important feature of this translation is that only negative atoms and implications are !-affixed; this mirrors a similar observation by Dyckhoff that the ordinary intuitionistic logic has a contraction-free sequent calculus that only needs to duplicate negative atoms and implications [6].

5. Forward focusing calculus

We now construct the forward version of the focusing calculus. Intermediate sequents in the eager active and focusing phases must not be stored in the database of facts, which should contain just the neutral sequents at the phase boundaries. The focusing construction is not entirely analogous to that of sec. 3, as the optional presence of resources are manifestations of a global property (weakness) rather than a local feature of the principal formula (synchronous or asynchronous connectives). We proceed by constructing derived rules for neutral sequents that make the intermediate focal and active sequents irrelevant. To illustrate, the negative proposition $p \mathbin{\&} q \multimap r \mathbin{\&} (s \otimes t)$ (written P) has the following pair of derived rules:

$$\frac{\begin{array}{c}\Gamma\,;\Delta, r \Longrightarrow C\\ \Gamma\,;\Delta' \Longrightarrow p \quad \Gamma\,;\Delta' \Longrightarrow q\end{array}}{\Gamma\,;\Delta,\Delta', P \Longrightarrow C} \qquad \frac{\begin{array}{c}\Gamma\,;\Delta, s, t \Longrightarrow C\\ \Gamma\,;\Delta' \Longrightarrow p \quad \Gamma\,;\Delta' \Longrightarrow q\end{array}}{\Gamma\,;\Delta,\Delta', P \Longrightarrow C}$$

In the rest of the paper we assume that goal sequents are neutral; any given sequent can be neutralised by running the active phase.

5.1. A calculus of derived inferences

For any given proposition, we are interested in constructing a derived inference for the proposition corresponding to a single pair of focusing and inverse phases. We proceed by interpreting the proposition itself as the rules that it embodies. Every proposition is viewed as a relation from sequences of forward sequents (corresponding to the premisses) and forward sequents (corresponding to conclusions of the derived rules). We write these relations in an applicative style–$R[x] = y$ for $(x, y) \in R$–and use a double-headed arrow ($\twoheadrightarrow$) for sequents in this derived rule calculus.

The derived rule for positive subformulas is:

$$\frac{s_1 \quad s_2 \quad \cdots \quad s_n}{\mathsf{foc}^+(A)[s_1 \cdot s_2 \cdots s_n]}\ \langle A, +, \cdot\rangle$$

Each s_i is a forward sequent in the derived-rules calculus, and $\mathsf{foc}^+(A)$ is a relation between sequences of sequents and sequents. Similarly, for negative propositions, we have two rules corresponding to whether the principal proposition is a heavy subformula or not.

$$\frac{s_1 \quad \cdots \quad s_n \quad (\mathsf{foc}^-(A)[s_1 \cdots s_n] = \Gamma\,;\Delta \twoheadrightarrow^w C)}{\Gamma \cup \{A\}\,;\Delta \twoheadrightarrow^w C}\ \langle A, -, !\rangle$$

$$\frac{s_1 \quad \cdots \quad s_n \quad (\mathsf{foc}^-(A)[s_1 \cdots s_n] = \Gamma\,;\Delta \twoheadrightarrow^w C)}{\Gamma\,;\Delta, A \twoheadrightarrow^w C}\ \langle A, -, \cdot\rangle$$

We write Σ for a sequence of forward sequents. The equations governing the relations foc^+ and foc^- and the auxiliary relation act, are shown in fig. 4. They make use of the following (partial) combinators on sequents:

Definition 5.1 (sequent combinators).

$$(\Gamma\,;\Delta \twoheadrightarrow^w A) \otimes (\Gamma'\,;\Delta' \twoheadrightarrow^{w'} B) = \Gamma \cup \Gamma'\,;\Delta,\Delta' \twoheadrightarrow^{w \vee w'} A \otimes B$$
$$!(\Gamma\,;\cdot \twoheadrightarrow^w A) = \Gamma\,;\cdot \twoheadrightarrow^0 !\,A$$
$$(\Gamma\,;\Delta \twoheadrightarrow^w A) \multimap (\Gamma'\,;\Delta' \twoheadrightarrow^w C) = \Gamma \cup \Gamma'\,;\Delta,\Delta' \twoheadrightarrow^{w \vee w'} C$$
$$(\Gamma\,;\Delta \twoheadrightarrow^w A) \mathbin{\&} (\Gamma'\,;\Delta' \twoheadrightarrow^{w'} B) =$$
$$\Gamma \cup \Gamma'\,;\Delta \sqcup \Delta' \twoheadrightarrow^{w \wedge w'} A \mathbin{\&} B \quad \text{if } \Delta/w \approx \Delta'/w'$$

In the definitions in fig. 4, multiple possibilities on the right indicate a non-deterministic choice in the following sense: if either of the right hand sides computes a sequent, then the left hand side computes the same sequent. As a simple illustration, the following is one possible derived rule for $\langle p \mathbin{\&} q \multimap r \mathbin{\&} (s \otimes t), -, \cdot\rangle$, written out in the usual style:

right focus

$$\mathsf{foc}^+(A \otimes B)[\Sigma \cdot \Sigma'] = \mathsf{foc}^+(A)[\Sigma] \otimes \mathsf{foc}^+(B)[\Sigma']$$
$$\mathsf{foc}^+(\mathbf{1})[\cdot] = \cdot \,;\, \cdot \twoheadrightarrow^0 \mathbf{1}$$
$$\mathsf{foc}^+(!\,A)[\Sigma] = !\,\mathsf{foc}^+(A)[\Sigma]$$
$$\mathsf{foc}^+(R)[\Sigma] = \mathsf{act}(\cdot\,;\cdot\,;\cdot \Longrightarrow R)[\Sigma]$$
$$\mathsf{foc}^+(p)[\Sigma] = \mathsf{act}(\cdot\,;\cdot\,;\cdot \Longrightarrow p)[\Sigma]$$

left focus

$$\mathsf{foc}^-(A \,\&\, B)[\Sigma] \begin{cases} = \mathsf{foc}^-(A)[\Sigma] \\ = \mathsf{foc}^-(B)[\Sigma] \end{cases}$$
$$\mathsf{foc}^-(A \multimap B)[\Sigma \cdot \Sigma'] = \mathsf{foc}^+(A)[\Sigma] \multimap \mathsf{foc}^-(B)[\Sigma']$$
$$\mathsf{foc}^-(p)[\cdot] = \cdot \,;\, p \twoheadrightarrow^0 p \quad \text{if } \langle p, +, \cdot\rangle \text{ is a subformula}$$
$$\mathsf{foc}^-(L)[\Sigma] = \mathsf{act}(\cdot\,;\cdot\,; L \Longrightarrow \cdot)[\Sigma]$$

active (ξ is of the form $\cdot$ or Q)

$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow A \,\&\, B)[\Sigma\cdot\Sigma'] = \mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow A)[\Sigma] \,\&\, \mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow B)[\Sigma']$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow \top)[\cdot] = \Gamma\,;\Delta \twoheadrightarrow^1 \top$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow A \multimap B)[\Sigma] \begin{cases} = \mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot A \Longrightarrow B)[\Sigma] \\ = \mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow B)[\Sigma] \quad \text{if result is weak} \end{cases}$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot A\otimes B \Longrightarrow \xi)[\Sigma] \begin{cases} = \mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot A\cdot B \Longrightarrow \xi)[\Sigma] \\ = \mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot A \Longrightarrow \xi)[\Sigma] \quad \text{if result is weak} \\ = \mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot B \Longrightarrow \xi)[\Sigma] \quad \text{if result is weak} \end{cases}$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot\mathbf{1} \Longrightarrow \xi)[\Sigma] = \mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow \xi)[\Sigma] \quad \text{if result is strong}$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot !\,A \Longrightarrow \xi)[\Sigma] \begin{cases} = \mathsf{act}(\Gamma, A\,;\Delta\,;\Omega \Longrightarrow \xi)[\Sigma] \\ = \mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow \xi)[\Sigma] \quad \text{if result is strong} \end{cases}$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\Omega\cdot P \Longrightarrow \xi)[\Sigma] = \mathsf{act}(\Gamma\,;\Delta, P\,;\Omega \Longrightarrow \xi)[\Sigma]$$
$$\mathsf{act}(\Gamma\,;\Delta\,;\cdot \Longrightarrow \xi)[\Gamma,\Gamma'\,;\Delta,\Delta' \twoheadrightarrow^w Q] = \Gamma'\,;\Delta' \twoheadrightarrow^w Q \quad \text{if } \xi = \cdot \text{ or } Q$$

Figure 4. interpreting propositions as rules

$$\frac{\Gamma_1\,;\Delta_1 \longrightarrow^{w_1} p \quad \Gamma_2\,;\Delta_2 \longrightarrow^{w_2} q \quad (\Delta_1/w_2 \approx \Delta_2/w_2) \qquad \Gamma_3\,;\Delta_3, s \longrightarrow^1 C \quad (t \notin \Delta_3)}{\Gamma_1 \cup \Gamma_2 \cup \Gamma_3\,; (\Delta_1 \sqcup \Delta_2), \Delta_3, p \,\&\, q \multimap r \,\&\, (s \otimes t) \longrightarrow^{(w_1 \wedge w_2)\vee w_3} C}$$

Define $\overline{\Gamma\,;\Delta \longrightarrow^w C}$ as $\Gamma\,;\Delta \twoheadrightarrow^w C$.

Lemma 5.2. *If Σ are derivable and:*

1. $\mathsf{foc}^+(A)[\overline{\Sigma}] = \Gamma\,;\Delta \twoheadrightarrow^w A$, *then* $\Gamma\,;\Delta \longrightarrow^w A$.
2. $\mathsf{foc}^-(A)[\overline{\Sigma}] = \Gamma\,;\Delta \twoheadrightarrow^w C$, *then* $\Gamma\,;\Delta, A \longrightarrow^w C$.
3. $\mathsf{act}(\Gamma\,;\Delta\,;\Omega \Longrightarrow \xi)[\overline{\Sigma}] = \Gamma'\,;\Delta' \twoheadrightarrow^w C$, *then* $\Gamma,\Gamma'\,;\Delta,\Delta' \longrightarrow^w C$.

Proof. Structural induction on the given derivations, using the definitions of foc^+, foc^- and act. □

Corollary 5.3 (soundness). *If $\Gamma\,;\Delta \twoheadrightarrow^w C$ then $\Gamma\,;\Delta \longrightarrow^w C$.*

For the completeness theorem, we cannot simply show that any sequent in the backward focusing calculus has a stronger form in the forward direction, as the foc and act relations "discard" intermediate values. Indeed, the only sequents that are explicitly written down in the derived rules calculus are neutral sequents. We generalise the induction hypotheses to account for these intermediate results.

Lemma 5.4.
If $\mathsf{act}(\Gamma_1, \Gamma_2\,;\Delta_1, \Delta_2\,;\Omega \Longrightarrow \xi)[\Sigma] = \Gamma\,;\Delta \twoheadrightarrow^w C$,
then $\mathsf{act}(\Gamma_1\,;\Delta_1\,;\Omega \Longrightarrow \xi)[\Sigma] = \Gamma, \Gamma_2\,;\Delta, \Delta_2 \twoheadrightarrow^w C$. □

Lemma 5.5. *Using the convention that Γ' is some subset of Γ, Δ' some subset of Δ and Ω' some subset of Ω, if*

1. $\Gamma\,;\Delta \gg A$ *then for some derivable* Σ
   (a) *either* $\mathsf{foc}^+(A)[\Sigma] = \Gamma'\,;\Delta \twoheadrightarrow^0 A$;
   (b) *or* $\mathsf{foc}^+(A)[\Sigma] = \Gamma'\,;\Delta' \twoheadrightarrow^1 A$.
2. $\Gamma\,;\Delta\,; A \ll Q$, *then for some derivable* Σ
   (a) *either* $\mathsf{foc}^-(A)[\Sigma] = \Gamma'\,;\Delta \longrightarrow^0 Q$;
   (b) *or* $\mathsf{foc}^-(A)[\Sigma] = \Gamma'\,;\Delta' \longrightarrow^1 Q$.
3. $\Gamma\,;\Delta\,;\Omega \Longrightarrow C$, *then for some derivable Σ and $\xi = \cdot$ or C:*
   (a) *either* $\mathsf{act}(\Gamma'\,;\Delta\,;\Omega \Longrightarrow \xi)[\Sigma] = \cdot\,;\cdot \twoheadrightarrow^0 C$;
   (b) *or* $\mathsf{act}(\Gamma'\,;\Delta'\,;\Omega' \Longrightarrow \xi)[\Sigma] = \cdot\,;\cdot \twoheadrightarrow^1 C$

Proof. Adaptation of the proof of thm. 3.6, using lem. 5.4 for the active and blur rules. □

Corollary 5.6 (completeness). *If $\Gamma\,;\Delta\,;\cdot \Longrightarrow Q$, then:*

(a) *either* $\Gamma'\,;\Delta \longrightarrow^0 Q$;
(b) *or* $\Gamma'\,;\Delta' \longrightarrow^1 Q$

for some $\Gamma' \subseteq \Gamma$ and $\Delta' \subseteq \Delta$. □

Property 5.7 (newness). *In any derived rule with conclusion s and premisses $\{s_i\}_{i\in 1\ldots n}$, $s_i \not\leq s$ for every $i \in \{1\ldots n\}$.*

Proof. By the definitions of foc and act, and inspection. □

[UNPUBLISHED DRAFT – compiled January 21, 2005 at 1:19 p.m.– submitted to LICS'05]

6. The focusing inverse method

What remains is to implement a search strategy that uses the forward calculus. The primary issue in the forward direction is what propositions to generate rules for. As the calculus of derived rules has only neutral sequents as premisses and conclusions, we need only generate rules for propositions that occur in neutral sequents; we call them *frontier propositions*. To find the frontier propositions in a goal sequent, we simply abstractly replay the focusing and active phases to identify the phase transitions. Each transition from an active to a focal phase produces a frontier proposition. Formally, we define two generating functions, f (focal) and a (active), from signed propositions to multisets of frontier propositions. None of the logical constants are in the frontier as we never need to construct explicit rules for them, as the conclusions of rules such as $\top R$ and $\mathbf{1}R$ are easy to predict. Similarly we do not count a negative focused atom in the frontier as we know that the conclusion of the init rule needs to have the form $\Gamma\,;\cdot\,; p \ll p$.

$$
\begin{array}{ll}
f(p)^- = \emptyset & f(p)^+ = a(p)^\pm = \{p\} \\
f(A \otimes B)^- = a(A \otimes B)^- & f(A \otimes B)^+ = f(A)^+, f(B)^+ \\
a(A \otimes B)^- = a(A)^-, a(B)^- & a(A \otimes B)^+ = f(A \otimes B)^+, A \otimes B \\[4pt]
f(A \mathbin{\&} B)^- = f(A)^-, f(B)^- & f(A \mathbin{\&} B)^+ = a(A \mathbin{\&} B)^+ \\
a(A \mathbin{\&} B)^- = f(A \mathbin{\&} B)^-, A \mathbin{\&} B & a(A \mathbin{\&} B)^+ = a(A)^+, a(B)^+ \\[4pt]
f(A \multimap B)^- = f(A)^+, f(B)^- & f(A \multimap B)^+ = a(A \multimap B)^+ \\
a(A \multimap B)^- = f(A \multimap B)^-, A \multimap B & a(A \multimap B)^+ = a(A)^-, a(B)^+ \\[4pt]
f(!\,A)^- = a(!\,A)^- & f(!\,A)^+ = a(A)^+ \\
a(!\,A)^- = a(A)^- & a(!\,A)^+ = f(A)^+, !\,A \\[4pt]
f(\mathbf{1})^\pm = a(\mathbf{1})^\pm = \emptyset & f(\top)^\pm = a(\top)^\pm = \emptyset
\end{array}
$$

For example, $f(p \mathbin{\&} q \multimap r \mathbin{\&} (s \otimes t))^- = p, q, s, t$.

Definition 6.1 (frontier). *Given a goal $\Gamma\,;\Delta \Longrightarrow Q$, its frontier contains:*

i. all (top-level) propositions in Γ, Δ, Q;
ii. for any $A \in \Gamma, \Delta$, the collection $f(A)^-$; and
iii. the collection $f(Q)^+$.

Lemma 6.2 (neutral subformula property). *In any backward focused proof, all neutral sequents consist only of frontier propositions of the goal sequent.* □

In the preparatory phase for the inverse method, we calculate the frontier propositions of the goal sequent. There is no need to generate initial sequents separately, as the executions of negative atoms in the frontier directly give us the necessary initial sequents.

During the search procedure, each rule is applied to sequents selected from the current database, and if the rule applies successfully then we get a new sequent, which is then considered for insertion in the database. It is possible (and common) that a generated sequent is actually subsumed by some sequent already in the database (forward subsumption). It is also possible (though less common) for a new sequent to be stronger than some sequents already in the database. In this case, the old weaker sequents are no longer considered for new derivations (backward subsumption). The general design of the main loop of the prover and the argument for its completeness are fairly standard [5, 15]; many optimisations are possible, but they are outside the scope of this paper.

6.1. Generalising hyperresolution

Hyperresolution is a complete strategy for classical logic [4, 14] that in practice gives an efficient search procedure for Horn and near-Horn fragments [15]. We concentrate on the following intuitionistic (non-linear) Horn-fragment:

$$
\begin{array}{rl}
\text{(goals)} & G ::= p \mid G_1 \wedge G_2 \mid \mathtt{t} \\
\text{(clauses)} & D ::= p \mid G \supset D \mid D_1 \wedge D_2 \mid \mathtt{t} \\
\text{(theories)} & \Psi ::= \cdot \mid \Psi, D
\end{array}
$$

Definition 6.3 (hyperresolution strategy). *Let $\hat{D}$ represent the (curried) clausal form of D. The* hyperresolution strategy *for the Horn-sequent $\Psi \Longrightarrow_h G$ is a proof of G starting from assumptions of the form $\hat{D}$ for every $D \in \Psi$, and rules:*

$$\frac{G_1 \quad G_2 \quad \cdots \quad G_n}{G}\ \text{hyper}_{\hat{D}}$$

where $G_1 \supset \cdots \supset G_m \supset G$ is a clausal form of some $D \in \Psi$.

Definition 6.4 (translation). *The translation $(-)^h$ of formulas in the Horn fragment to linear logic is as follows:*

$$(p)^h = p \qquad (\mathtt{t})^h = \mathbf{1} \qquad (G_1 \wedge G_2)^h = (G_1)^h \otimes (G_2)^h$$
$$(D_1 \wedge D_2)^h = (D_1)^h \otimes (D_2)^h \qquad (G \supset D)^h = (G)^h \multimap (D)^h.$$

It is easy to see that the frontier propositions of $(\Psi)^h\,;\cdot \Longrightarrow (G)^h$ are the positive atoms, every $(D)^h \in (\Psi)^h$ and $(G)^h$.

Theorem 6.5. *If $\Psi \Longrightarrow_h G$, then $(\Psi')^h\,;\cdot \longrightarrow\!\!\!\!\!\rightarrow^0 (G)^h$ for some $\Psi' \subseteq \Psi$.*

Proof (sketch). Consequence of thm. 5.6. It is clear by a simple examination the foc and act relations that for every $D \in \Psi$ such that $\hat{D} = G_1 \supset \cdots \supset G_n$, the $\left\langle (D)^h, -, ! \right\rangle$ rule is of the form:

$$\frac{\Gamma_1\,;\Delta_1 \longrightarrow\!\!\!\!\!\rightarrow^0 (G_1)^h \quad \cdots \quad \Gamma_n\,;\Delta_n \longrightarrow\!\!\!\!\!\rightarrow^0 (G_n)^h}{\Gamma\,;\Delta \longrightarrow\!\!\!\!\!\rightarrow^0 (G)^h}$$

where each Γ_i and $\Gamma \subseteq (\Psi)^h$. As the initial sequents have empty linear zones (all negative frontier propositions are in $(\Psi)^h$), they are empty in all derived sequents, the similarity to $\text{hyper}_{\hat{D}}$ is obvious. □

Test	NF	F	Gt	Gr
`blocks-world`	0.43 s	≤ 0.01 s	13.51 s	0.03 s
`change`	13.56 s	≤ 0.01 s	—	2.14 s
`affine`	46.81 m	4.63 s	—	—
`qbf1`	0.03 s	0.01 s	0.03 s	0.02 s
`qbf2`	1.60 s	0.03 s	—	42.34 s
`qbf4`	≈ 35 m	0.53 s	—	—

NF = Non-focusing, F = focusing, Gt = Gandalf-tableaux, Gr = Gandalf-resolution
All measurements are wall-clock times; "—" denotes unsuccessful proof within ≈ ten hours.

Table 1. some test problems.

6.2. Some experimental results

We have implemented an expanded version of the forward focusing calculus as a certifying inverse method prover for intuitionistic linear logic, including the missing linear connectives. Table 1 contains a running-time comparison of the focusing prover against a non-focusing version of the prover (directly implementing the calculus of sec. 3), and Tammet's Gandalf "nonclassical" distribution that includes a pair of (non-certifying) provers for classical linear logic, one using a refinement of Mints' resolution system for classical linear logic [14, 15], and the other using a Tableaux-based strategy. Neither of these provers incorporates focusing. The test problems ranged from simple stateful encodings such as blocks-world or change-machines, to more complex problems such as encoding of affine logic problems, and translations of various quantified Boolean formulas using the algorithm in [13]. Focusing was faster in every case, with an average speedup of about three orders of magnitude over the non-focusing version.

7. Conclusion

We have presented a design for a focused forward reasoning calculus that is essentially bi-directional: it uses the (backward) focusing calculus to derive inference rules, which it then applies in the forward direction by keeping track of weakness. There are two interesting questions to ask about this calculus: first, does it give a decision procedure for propositional affine logic? It is known to be decidable, but the proof is fairly non-trivial [12]. Second, does it wring the maximum possible juice out of the focusing and active phases? For instance, $(!A) \otimes (!B) \equiv\ !(A \mathbin{\&} B)$, but they have very different behaviour: the former continues the focusing phase (on the right), while the latter blurs focus immediately. A principled use of such equivalences might allow enlarging the focus and active phases.

Acknowledgement: We thank Frank Pfenning and Kevin Watkins for many illuminating discussions.

References

[1] J.-M. Andreoli. Logic programming with focusing proofs in linear logic. *Journal of Logic and Computation*, 2(3):297–347, 1992.
[2] J.-M. Andreoli. Focussing and proof construction. *Annals of Pure and Applied Logic*, 107:131–163, 2001.
[3] B.-Y. E. Chang, K. Chaudhuri, and F. Pfenning. A judgmental analysis of linear logic. Technical Report CMU-CS-03-131R, Carnegie Mellon University, December 2003.
[4] C. L. Chang and R. C. T. Lee. *Symbolic logic and mechanical theorem proving*. Academic Press, 1973.
[5] A. Degtyarev and A. Voronkov. *Handbook of Automated Reasoning*, chapter The Inverse Method, pages 179–272. MIT Press, September 2001.
[6] R. Dyckhoff. Contraction-free sequent calculi for intuitionistic logic. *Journal of Symbolic Logic*, (57):795–807, 1992.
[7] J.-Y. Girard. Linear logic. *Theoretical Computer Science*, 50:1–102, 1987.
[8] J.-Y. Girard. Locus solum: from the rules of logic to the logic of rules. *Mathematical Structures in Computer Science*, 11:301–506, 2001.
[9] R. Hähnle. Tableaux and related methods. In *Handbook of Automated Reasoning*, volume 1, pages 100–178. Elsevier and MIT Press, 2001.
[10] J. Harland and D. J. Pym. Resource-distribution via boolean constraints. In W. McCune, editor, *Proceedings of CADE-14*, pages 222–236. Springer-Verlag LNAI 1249, July 1997.
[11] J. S. Hodas and D. Miller. Logic programming in a fragment of intuitionistic linear logic. *Information and Computation*, 110(2):327–365, 1994.
[12] A. P. Kopylov. Decidability of linear affine logic. *Information and Computation*, 164(1):173–198, January 2001.
[13] P. Lincoln, J. C. Mitchell, A. Scedrov, and N. Shankar. Decision problems for propositional linear logic. *Annals of Pure and Applied Logic*, 56:239–311, 1992.
[14] G. Mints. Resolution calculus for the first order linear logic. *Journal of Logic, Language and Information*, 2(1):59–83, 1993.
[15] T. Tammet. Resolution, inverse method and the sequent calculus. In *Proceedings of KGC'97*, pages 65–83. Springer-Verlag LNCS 1289, 1997.
[16] K. Watkins, I. Cervesato, F. Pfenning, and D. Walker. A concurrent logical framework: The propositional fragment. In *Proceedings of TYPES-2003*, pages 355–377. Springer-Verlag LNCS 3085, January 2004.