Outline

1. Equality
2. Ordered Resolution and Paramodulation Rules
3. Herbrand equality interpretations
4. Semantic trees
5. Generating Interpretation
6. Refutational Completeness of ORPF
7. Conclusion
Axiomes de l’égalité

\[ x = x \]  (reflexivity)

\[ x = y \lor \neg y = x \]  (symmetry)

\[ x = z \lor \neg x = y \lor \neg y = z \]  (transitivity)

\[ f(x) = f(y) \lor \neg x = y \]  (functional monotonicity)

\[ P(x) = P(y) \lor \neg x = y \]  (predicative monotonicity)

The are as many functional (resp. predicative) monotonicity axioms as the number of function (resp. predicate) symbols in the vocabulary.
Adding these axioms for each equality predicate leads to a blow up in the number of clauses generated by ordered resolution, most of which are useless.

Example (take the order rpo(f > a))

\[ \neg a = f(a) \quad f(x) = f^2(x) \]

Robinson et Wos proposed to replace all equality axioms except reflexivity by one, specific, less prolific inference rule:

\[
\frac{C \lor l = r \quad \pm D \lor A[u]}{C\sigma \lor D\sigma \lor \pm A[r\sigma]} \quad \text{si} \quad \begin{cases} u \notin \chi \\ \sigma = \text{mgu}(l = u) \end{cases}
\]
Adding these axioms for each equality predicate leads to a blow up in the number of clauses generated by ordered resolution, most of which are useless.

Example (take the order rpo(f > a))

\[ \neg a = f(a) \quad f(x) = f^2(x) \]

Robinson et Wos proposed to replace all equality axioms except reflexivity by one, specific, less prolific inference rule:

\[
\begin{align*}
C \lor l &= r \\
\pm D \lor A[u] \\
C\sigma \lor D\sigma \lor \pm A[r\sigma] \quad \text{si} \quad \begin{cases} u \notin \lambda \overbrace{\sigma = \text{mgu}(l = u)}^{
\text{mgu}} \end{cases}
\end{align*}
\]
Adding these axioms for each equality predicate leads to a blow up in the number of clauses generated by ordered resolution, most of which are useless.

Example (take the order $\text{rpo}(f > a)$)

\[
\neg a = f(a) \quad f(x) = f^2(x)
\]

Robinson et Wos proposed to replace all equality axioms except reflexivity by one, specific, less prolific inference rule:

\[
\frac{C \lor l = r \quad \pm D \lor A[u]}{C\sigma \lor D\sigma \lor \pm A[r\sigma]} \quad \text{si} \quad \begin{cases} u \notin \mathcal{X} \\ \sigma = \text{mgu}(l = u) \end{cases}
\]
Ordered Resolution and Paramodulation Rules
Herbrand equality interpretations
Semantic trees
Generating Interpretation
Refutational Completeness of \textit{ORPF}
Conclusion
Resolution

\[ +A \lor C - A' \lor C' \]
\[ \frac{C_\sigma \lor C'_\sigma}{\sigma = mgu(A = A')} \]
\[ A_\sigma \not\succ B \ \forall B \in C_\sigma \lor C'_\sigma \]

Factoring

\[ +A \lor +A' \lor C \]
\[ \frac{+A_\sigma \lor C_\sigma}{\sigma = mgu(A = A')} \]
\[ A_\sigma \not\succ B \ \forall B \in C_\sigma \]

Reflexivity

\[ -u = v \lor C \]
\[ \frac{C_\sigma}{\sigma = mg(u = v)} \]
\[ u_\sigma = v_\sigma \not\succ B \ \forall B \in C_\sigma \]
Ordered Paramodulation

\[
\begin{align*}
C \vee I &= r \\
D \vee \pm A[u] \\
\sigma = \text{mgu}(I = u) &
\end{align*}
\]

\[
\begin{align*}
I \sigma \not≺ r \sigma \\
l \sigma = r \sigma \not≺ B \\forall B \in C \sigma \\
A \sigma \not≺ B \\forall B \in D \sigma
\end{align*}
\]

Monotonic Ordered Paramodulation

\[
\begin{align*}
C \vee I &= r \\
D \vee \pm A[u] \\
\sigma = \text{mgu}(I = u) &
\end{align*}
\]

\[
\begin{align*}
A \sigma[I \sigma] \not≺ A \sigma[r \sigma] &
l \sigma = r \sigma \not≺ B \\forall B \in C \sigma \\
A \sigma \not≺ B \\forall B \in D \sigma
\end{align*}
\]
Completeness Theorem

\textit{ORPF} is refutationally complete for any partial quasi-ordering $\succeq$ satisfying the following properties:

1. $\succeq$ is stable on terms,
2. $\succeq$ restricts on ground terms to a total well-founded monotonic ordering $\succ$,
3. $\succeq$ is extended to atoms so as to satisfy:
   - \textit{monotonicity}: $s \succ t$ implies $A[s] \succ A[t]$ for any atom $A[s]$;
   - \textit{minimality of $=: s \succ t$ implies $A[s] \succ s = t$ if $A$ is not an equality atom.}
4. $\succeq$ is extended to literals and clauses in a natural way.
5. Both rules coincide under these conditions.
Subterm violated

Monotonic paramodulation is incomplete:

\[
\{ fb \neq fa, \ b = fb, \ a = fb \}
\]

\[ ff a \succ f f b \succ a \succ b \succ f a \succ f b \]

Ordered paramodulation generates:

\[
\{ f^m b \neq fa, \ f^{n+1} b \neq f^{m+1} b, \ a = f^m b, \ f^n b = f^m b, \ a = b \ \mid n \geq 0, \ m > 0 \} \cup \{ \Box \} \]

Is not a decision procedure when ground.
Subterm violated

Monotonic paramodulation is incomplete:

\[ \{ fb \neq fa, \ b = fb, \ a = fb \} \]

\[ ffa \succ ffb \succ a \succ b \succ fa \succ fb \]

\[ b = fb \quad a = fb \]

\[ \text{NO} \]

\[ ffb = fb \quad a = fb \]

\[ \text{NO} \]

\[ fb \neq fa \quad b = fb \]

\[ \text{NO} \]

\[ fb \neq fa \quad a = fb \]

\[ \text{NO} \]

Ordered paramodulation generates:

\[ \{ f^m b \neq fa, \ f^{n+1} b \neq f^{m+1} b, \ a = f^m b, \ f^n b = f^m b, \ a = b \quad | \ n \geq 0, \ m > 0 \} \cup \{ \Box \} \]

Is not a decision procedure when ground.
Subterm violated

Monotonic paramodulation is incomplete:

\[
\{ fb \neq fa, \ b = fb, \ a = fb \} \\
ffa \succ fb \succ a \succ b \succ fa \succ fb
\]

<table>
<thead>
<tr>
<th>[ b = fb ]</th>
<th>[ a = fb ]</th>
<th>[ ffb = fb ]</th>
<th>[ a = fb ]</th>
</tr>
</thead>
<tbody>
<tr>
<td>[ NO ]</td>
<td>[ NO ]</td>
<td>[ NO ]</td>
<td>[ NO ]</td>
</tr>
</tbody>
</table>

Ordered paramodulation generates:

\[
\{ f^m b \neq fa, \ f^{n+1} b \neq f^{m+1} b, \ a = f^m b, \ f^n b = f^m b, \ a = b \mid n \geq 0, \ m > 0 \} \cup \{ \square \}
\]

Is not a decision procedure when ground.
Subterm violated

Monotonic paramodulation is incomplete:

\[
\begin{align*}
\{ fb \neq fa, & \quad b = fb, \quad a = fb \} \\
ffa \succ ffb \succ a \succ b \succ fa \succ fb
\end{align*}
\]

\[
\begin{align*}
b = fb & \quad a = fb \\
\text{NO} & \quad \text{NO}
\end{align*}
\]

\[
\begin{align*}
fb \neq fa & \quad b = fb \\
\text{NO} & \quad \text{NO}
\end{align*}
\]

Ordered paramodulation generates:

\[
\begin{align*}
\{ f^m b \neq fa, & \quad f^{n+1} b \neq f^{m+1} b, \\
a = f^m b, & \quad f^n b = f^m b, \quad a = b \}
\end{align*}
\]

\[
\{ n \geq 0, \quad m > 0 \} \cup \{ \Box \}
\]

Is not a decision procedure when ground.
Monotonicity violated

Monotonic paramodulation is incomplete:

\[ \{ gb = b, \ fg^2 b \neq fb \} \]

\[ fg^3 b \succ fgb \succ fb \succ fg^2 b \succ gb \succ b \]

The set of (ground unit) clauses is already closed.

Using ordered paramodulation:

\[ \{ gb = b, \ fg^2 b \neq fb, fgb \neq fb, fb \neq fb, \square \} \]

Completeness of ordered paramodulation does not need monotonicity

[Bachmair, Ganzinger, Nieuwenhuis, Rubio, 2003]
Monotonicity violated

Monotonic paramodulation is incomplete:

\[ \{ gb = b, \ fg^2b \neq fb \} \]

\( fg^3b > fgb > fb > fg^2b > gb > b \)

The set of (ground unit) clauses is already closed.

Using ordered paramodulation:

\[ \{ gb = b, \ fg^2b \neq fb, fgb \neq fb, fb \neq fb, \square \} \]

Completeness of ordered paramodulation does not need monotonicity

[Bachmair, Ganzinger, Nieuwenhuis, Rubio, 2003]
Monotonic paramodulation is incomplete:

\[
\{ gb = b, \ fg^2 b \neq fb \}
\]

\[
fg^3 b \triangleright fgb \triangleright fb \triangleright fg^2 b \triangleright gb \triangleright b
\]

The set of (ground unit) clauses is already closed.

Using ordered paramodulation:

\[
\{ gb = b, fg^2 b \neq fb, fgb \neq fb, fb \neq fb, \square \}
\]

Completeness of ordered paramodulation does not need monotonicity

[Bachmair, Ganzinger, Nieuwenhuis, Rubio, 2003]
Compactness yields a finite unsatisfiable set of ground clauses;
- Build the tree of equality interpretations;
- Define the branch ending at an inference node;
- Reduce the tree by inference.
Completeness: the roadmap

- Compactness yields a finite unsatisfiable set of ground clauses;
- Build the tree of equality interpretations;
  - Define the branch ending at an inference node;
  - Reduce the tree by inference.
Completeness: the roadmap

- Compactness yields a finite unsatisfiable set of ground clauses;
- Build the tree of equality interpretations;
- Define the branch ending at an inference node;
- Reduce the tree by inference.
Compactness yields a finite unsatisfiable set of ground clauses;
Build the tree of equality interpretations;
Define the branch ending at an inference node;
Reduce the tree by inference.
Problem of the construction

Let \( I \) a node in the tree interpreting the atoms \( \{A_j\}_{0 \leq j < i} \) such that \( E_I \) is the set of equalities interpreted in \( T \) by \( I \) which we turn into a set of rules \( E_I \).

We want exactly three cases now:
(i) \( A_i \) is \( s = s \) that we interpret in \( T \);
(ii) \( A_i \) is reducible in \( E_I \) to some atom \( B \) and both interpretations must be the same;
(iii) \( A_i \) is irreducible and has two successors.
These three cases correspond respectively to reflexivity, paramodulation and resolution.
- $E_i$ must be confluent to ensure consistent decisions;
- The atom $B$ must belong to $\{A_j\}_{0 \leq j < i}$.
- For an arbitrary finite unsatisfiable set of ground instances of the clauses, these assumptions are usually not met.
Completion of the set of atoms

\[ \mathcal{A} : \text{finite set of atoms} \]
\[ \mathcal{E} \text{ be the set of equalities in } \mathcal{A} \]

\[ A \xrightarrow{\mathcal{E}} B \]
\[ \text{if } \]
\[ A = A[s], \quad B = A[t], \quad s = t \in \mathcal{E}, \quad s \succ t \]

Ordered completion:

\[ \frac{\mathcal{A} \cup \{A\}}{\mathcal{A} \cup \{B\}} \]

Observation: the tree of all possible completion sequences is finite.
Completion of the set of atoms

\( \mathcal{A} \) : finite set of atoms
\( \mathcal{E} \) be the set of equalities in \( \mathcal{A} \)

\[
\mathcal{A} \longrightarrow_\mathcal{E} \mathcal{B}
\]

if

\[
\mathcal{A} = \mathcal{A}[s], \quad \mathcal{B} = \mathcal{A}[t], \quad s = t \in \mathcal{E}, \quad s \succ t
\]

Ordered completion:

\[
\mathcal{A} \cup \{ \mathcal{A} \} \quad \mathcal{A} \longrightarrow_\mathcal{E} \mathcal{B}
\]

\[
\mathcal{A} \cup \{ \mathcal{B} \}
\]

Observation: the tree of all possible completion sequences is finite.
Lemma

Let $A$ be a finite set of atoms, $E$ be its subset of equality atoms, $\overline{A}$ the closure of $A$ under all possible sequences of ordered completion, and $\overline{E}$ its subset of equality atoms. Then

(i) $\overline{A}$ is finite;
(ii) $\overline{E}$ is closed under critical pair computation, hence defines a convergent set of rules again written $\overline{E}$;
(iii) $\overline{A}$ is closed under rewriting with $\overline{E}$.

Proof: easy.
We assume that $\overline{\mathcal{A}} = \mathcal{A}$, hence $\overline{\mathcal{E}} = \mathcal{E}$.

Let $E_H$ be the subset of equalities (rules) in $\mathcal{E}$ interpreted by $T$ in the Herbrand interpret. $H$.

**Lemma**

Let $\{A_i\}_{i < j \leq n}$ be an initial segment of $\mathcal{A}$, and $H$ an equality interpretation of $\{A_i\}_{i < j}$. Then,

(i) $A_i \stackrel{E_H}{\longrightarrow} B$ implies that $B = A_k$ for some $k < i$,

(ii) $E_H$ is a convergent subset of $\mathcal{E}$.

**Proof:** Since $\mathcal{A}$ is closed under ordered completion, $B \in \mathcal{A}$, and $A_i \succ B$ implies $k < i$. A critical pair between two rules of $\{A_i\}_{i < j}$ belongs to $\{A_i\}_{i < j}$ by (i), hence to $E_H$ by def. □
The tree of Herbrand equality interpretations over $A = \{A_j\}_j$ is defined inductively. Each node $I$ in the tree defines a partial equality interpretation of $\{A_j\}_{j<i}$ and a set $E_I$ of equalities interpreted by true in $I$.

1. Assume that $A_i$ has the form $s = s$. Then, $I$ has one successor $J$ s.t. $[A_i]_J = T$

2. Assume otherwise that $A_i \rightarrow_{R_i} A_j$ with $i > j$. Then $I$ has one successor $J$ s.t. $[A_i]_J = [A_j]_I$.

3. Otherwise, $I$ has a left successor $J$ and a right $K$ s.t. $[A_i]_J = T$ and $[A_i]_K = F$. 
The *tree of Herbrand equality interpretations* over $\mathcal{A} = \{A_j\}_j$ is defined inductively. Each node $I$ in the tree defines a partial equality interpretation of $\{A_j\}_{j < i}$ and a set $E_I$ of equalities interpreted by true in $I$.

1. Assume that $A_i$ has the form $s = s$. Then, $I$ has one successor $J$ s.t. $[A_i]_J = T$.

2. Assume otherwise that $A_i \rightarrow_{R_i} A_j$ with $i > j$. Then $I$ has one successor $J$ s.t. $[A_i]_J = [A_j]_I$.

3. Otherwise, $I$ has a left successor $J$ and a right $K$ s.t. $[A_i]_J = T$ and $[A_i]_K = F$. 

Herbrand equality interpretations

The *tree of Herbrand equality interpretations* over $\mathcal{A} = \{A_j\}_j$ is defined inductively. Each node $I$ in the tree defines a partial equality interpretation of $\{A_j\}_{j<i}$ and a set $E_I$ of equalities interpreted by true in $I$.

1. Assume that $A_i$ has the form $s = s$. Then, $I$ has one successor $J$ s.t. $[A_i]_J = T$

2. Assume otherwise that $A_i \rightarrow_{R_i} A_j$ with $i > j$. Then $I$ has one successor $J$ s.t. $[A_i]_J = [A_j]_I$.

3. Otherwise, $I$ has a left successor $J$ and a right $K$ s.t. $[A_i]_J = T$ and $[A_i]_K = F$. 

Lemma

The set of leaves is in bijection with the set of equality Herbrand interpretations of $\mathcal{A}$.

Proof: We show that $H$ is a partial Herbrand equality interpretation over an initial segment $\{A_i\}_{i<j\leq n}$ iff

(a) for any atom $s = s \in \{A_i\}_{i<j}$, $[s = s]_H = T$,
(b) for any two atoms $A_k, A_l$ such that $j > k > l$ and $A_k \rightarrow_{E_H} A_l$, then $[A_k]_H = [A_l]_H$.

Clearly, an Herbrand equality interpretation satisfies (i) and (ii). We show the converse.
Proof

Let \( s = t \in E_H \) and \( u[s] = u[t] \in \{A_i\}_{i < j} \) for some \( u[] \neq [] \). Then \( u[s] = u[t] \rightarrow_{E_H} u[t] = u[t] \) which belongs to \( \{A_i\}_{i < j} \) by previous lemma (i), assuming \( u[s] \succ u[t] \). By (a) and (b), \( [u[s] = u[t]]_H = [u[t] = u[t]]_H = T \), hence \( u[s] = u[t] \in E_H \).

Assuming now that \( A_k \leftrightarrow_{E_H} A_l \) with \( k > l \), we show that \( [A_k]_H = [A_l]_H \) by induction on \( k \). By previous lemma (ii), there exist atoms \( B, C \) such that \( A_k \rightarrow_{E_H} B \rightarrow_{E_H}^* C \leftarrow_{E_H}^* A_l \). By previous lemma (i), \( B = A_m \) for some \( m < k \). By induction hypothesis, \( [A_m]_H = [A_l]_H \). By assumption (b), \( [A_k]_H = [A_m]_H \). We conclude by transitivity. \( \square \)
Tree of equality interpretations for $fb \succ fb \succ a \succ b$
We assume an unsatisfiable set $\mathcal{G}$ of ground clauses built upon the atoms in $\mathcal{A}$, which is closed under the rules in $\text{ORPF}$.

**Definition**

We call *failure node* a partial equality interpretation $J$ for which there exists $C \in \mathcal{G}$ such that $[C]_J = F$ and $[C]_I$ is undefined for any $I < J$. We call semantic tree associated with $\mathcal{G}$ the tree obtained from the tree of equality interpretations by replacing each failure node $J$ by a leaf labelled with a clause in $\mathcal{G}$ refuting $J$. 
Inductive set of generating interpretations

1. If the partial interpretation $I$ is a leaf, done.
2. If $I$ has a unique successor $I'$ in the semantic tree, choose $I'$.
3. If $I$ has two successors $J$ (the left one) and $K$ (the right one) such that $K$ is a failure node, choose $J$. In case $K$ is labelled by the clause $s = t \lor D$ such that $s = t$ is maximal, we say that $s = t$ is generated.
4. If $I$ has two successors $J$ (the left one) and $K$ (the right one) such that $K$ is not a failure node and $A_{|I|}$ is an equality atom, choose $K$.
5. Otherwise, choose either $J$ or $K$.

$G$ will denote any generating interpretation.
Lemma

Assume that $G$ is a generating interpretation of a semantic tree associated with the unsatisfiable set $G = \{A_i\}_{i<n}$ of ground clauses closed under the rules in ORPF.

Let us assume that $A_i$ is reducible by $E_G$. Then, there exists a generating clause $s = t \lor C$ in $G$ such that:

(i) $A_i \xrightarrow{s=t\in E_G} A_j \in \mathcal{A}$, with $s \succ t$,
(ii) $s = t \succ A$ for every atom $A$ of $C$,
(iii) $[C]_G = F$. 


**Proof** A straightforward key property of $G$ is that the generated equations are exactly the equations $s = t$ irreducible in $E_G \setminus \{s=t\}$. Let $I$ be the father of $G$.

(i). We need proving that each reducible atom $A_i$ rewrites to atom $B \in A$ with an irreducible equation. Let $s = t$, $s \succ t$, be an equation reducing $A_i$, that is, $A_i = A_i[s]$ and $A_j = A_i[t]$ for some $j < i$, such that $(s, t)$ is minimal with respect to $\succ$. If $t$ is reducible to $t'$ by some equation $u = v$ interpreted in $T$ by $G$, then $s = t' \in E_G$, hence $A_i$ is reducible by a smaller equation. Contradiction.
If \( s \) is reducible by some equation \( u = v \) interpreted in \( T \) by \( G \), then, by monotonicity, \( w[s] \) is reducible by \( u = v \), hence \( A_i \) is reducible by an equation smaller than \( s = t \).
Contradiction, or \( s = u, t = v \) up to renaming. Therefore, \( s = t \) is irreducible for \( E_G \setminus \{s = t\} \).

(ii) and (iii). Since \( s = t \) is the last atom enumerated by \( G \), it is maximal in the clause. Since \( G \) is closed under positive factoring, we can assume that \( s = t \notin C \), hence \( [C]_G = [C]_I = F \) and \( s = t \) is strictly bigger than any atom in \( C \).
Since $I$ has two successors, by definition of the tree of Herbrand equality interpretations, $s$ and $t$ must be irreducible by $E_I$. Let now $u = v \in E_G \setminus (E_I \cup \{s \rightarrow t\})$ and assume without loss of generality that $u \succ v$. By definition of the tree of Herbrand equality interpretations, $u = v \succ s = t$. By properties of $\succ$, $u \succ s$ and $u \succ t$, hence $u$ is not a subterm of $s$ or of $t$. It follows that $s = t$ cannot be reduced by $u \rightarrow v$. \qed
**Theorem**

A set of clauses $\mathcal{C}$ is unsatisfiable iff the empty clause belongs to its closure under $\text{ORPF}$.

**Proof:** By compactness, we chose a finite unsatisfiable set $\mathcal{G}$ of ground instances of $\mathcal{C}$, built over a finite set $\mathcal{A}$ of ground atoms. By ordered completion, we complete $\mathcal{A}$ into a new finite set $\overline{\mathcal{A}}$. We can now generate a new set of ground instances of $\mathcal{C}$

$$\overline{\mathcal{G}} = \{ C_{\gamma} \mid C \in \mathcal{C}, \ C_{\gamma} \text{ ground}, \ A \in \overline{\mathcal{A}} \ \forall A \in C_{\gamma} \}$$

By construction, $\overline{\mathcal{G}}$ contains $\mathcal{G}$, hence is unsatisfiable, and is closed under $\text{ORPF}$. 
Proof continued

We construct a minimal semantic tree $\mathcal{W}$ for $\mathcal{G}$, for the ordering comparing in $(>\mathbb{N}, \succ_{mul})_{lex}$ the pair $(|\mathcal{W}|, \{\text{clauses refuting the leaves of } \mathcal{W}\})$. Assume $\mathcal{W}$ is non-empty: choose an arbitrary generating interpretation $J$ with father node $I$. By definition of $\mathcal{W}$, $J$ is refuted by a clause $\pm B \lor C$, in which $B = P(\bar{u}\gamma)$ is last enumerated atom, hence is maximal in $C$. By minimality assumption and closure of $\mathcal{G}$ under positive factoring, $B$ does not occur in $C$ when positive. By minimality assumption, definition of equality interpretations, closure of $\mathcal{A}$ under ordered paramodulation and construction of $\mathcal{G}$, $\gamma$ can be assumed in normal form for $E_J$. 
Proof continued

We now exhibit an inference between the clause refuting \( J \) and another clause in \( \mathcal{W} \). The inferred clause will belong to \( \mathcal{G} \) and refute the interpretation \( I \), therefore contradicting our minimality assumption. This is done by cases upon the definition of \( G \).

1. \( P(\overline{u} \gamma) \) is of the form \( s = s \), in which case \( I \) has \( J \) as single successor labelled by \( \neg s = s \lor C \). By closure property of \( \mathcal{G} \) under the rules in \( ORPF \), \( \mathcal{G} \) contains \( C \). There are two cases. If \( s = s \in C \), then \( C \) refutes \( J \), otherwise it refutes a node \( N < J \), contradicting our minimality assumption in both cases.
2. $P(\bar{u}\gamma)$ is irreducible by a rule in $E_l$. Then, $I$ has two successors, and by definition, $J$ must be the left successor of $I$ and the right successor must be itself a leaf. Hence $I$ has two successors labelled by clauses in both of which the atom $P(\bar{u}\gamma)$ is maximal. Let these clauses be $+P(\bar{u}\gamma) \lor C$, in which $P(\bar{u}\gamma)$ is strictly bigger than any atom occurring in $C$, and $-P(\bar{u}\gamma) \lor D$. By construction, $\mathcal{G}$ contains the clause $C \lor D$ obtained by ordered resolution from both previous clauses. By definition of $\succ$, $C \lor D$ is strictly smaller than the clause $-P(\bar{u}\gamma) \lor D$, hence refuting a node $N \leq J$, which contradicts our minimality assumption.
3. $P(\overline{u}\gamma)$ is reducible by $E_i$. Since $\gamma$ is irreducible, $P(\overline{u}\gamma)$ must be reducible at a non-variable position $p$ of $P(\overline{u})$ by an equation $s = t \in E_i$, yielding the atom $B = P(\overline{u}\gamma)[t]_p \in A$. By Lemma, $s = t$ is generated by a clause $s = t \lor D$ such that $s \triangleright t$ and $s = t$ is strictly bigger than any atom in $D$. Therefore, there is an ordered paramodulation between $l = r \lor D$ and the clause $\pm P(\overline{u}\gamma) \lor C$, yielding $B \lor C \lor D$, which belong to $\mathcal{G}$ by construction. Then, the inferred clause refutes an ancestor node, the obtained semantic tree is smaller than the starting one, a contradiction again.
Conclusion

1. Apply compactness before anything else;
2. Inductive construction of the tree of interpretations requires the subterm property;
3. Interpretation of Bachmair-Ganzinger model generation technique as a maximal branch of the semantic tree;
4. No lifting needed!