Algorithms and Complexity of Constraint Satisfaction Problems
(seminar number 7)

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Consider the following sets of relations $S$ and show (with a proof) whether $\text{CSP}(S)$ is polynomial-time decidable or \text{NP}-complete. If $\text{CSP}(S)$ is polynomial-time decidable, determine the class of relations $S$ (Horn, dual Horn, bijunctive, affine, 0- or 1-valid, or their combination).

- $S = \{R_1, R_2, R_3\}$ where $R_1 = [(x \rightarrow y) \rightarrow \neg z]$, $R_2 = [(x \lor y) \rightarrow z]$, and $R_3 = [x \land y]$.
- $S = \{R_1, R_2\}$ where $R_1 = \{0011, 0110, 1001, 1010, 1100\}$ and $R_2 = \{0001, 0011, 0111\}$. 

Transform the relations $R_1$ and $R_2$ to clausal form, obtaining $R_1 = [(x \lor \neg z) \land (\neg y \lor \neg z)]$ and $R_2 = [(\neg x \lor z) \land (\neg y \lor z)]$. Relation $R_3$ is already in clausal form. By inspection, we see that the set of relations $S$ is Horn and bijunctive. Moreover, $R_3$ is not 0-valid since $[x \land y] = \{11\}$, $R_1$ is neither dual Horn since the clausal form of $(x \rightarrow y) \rightarrow \neg z$ contains the clause $\neg y \lor \neg z$ (this can be also checked by computing the models of $[(x \rightarrow y) \rightarrow \neg z]$ and testing the closure under disjunction), nor affine since $R_1$ contains 5 vectors which is not a power of 2. Hence $S$ is both Horn and bijunctive, and therefore $\text{CSP}(S)$ is polynomial-time decidable following Schaefer’s Dichotomy Theorem.

By inspection, we see that $R_1$ is neither 0-valid, nor 1-valid. Moreover, $R_1$ is not Horn, since $0011 \land 1100 = 0000 \notin R_1$, nor dual Horn since $0011 \lor 1100 = 1111 \notin R_1$, nor affine since $R_1$ has 5 vectors which is not a power of 2, nor bijunctive since $\text{maj}(1001, 1010, 1100) = 1000 \notin R_1$. The relation $R_2$ does not play a significant role. Hence $\text{CSP}(S)$ is NP-complete following Schaefer’s Dichotomy Theorem.
We have the relations

\[
\begin{align*}
or_0 &= [x \lor y \lor z], & or_3 &= [\neg x \lor \neg y \lor \neg z], \\
bor_0 &= [x \lor y], & bor_2 &= [\neg x \lor \neg y], \\
1\text{-in-3} &= \{100, 010, 001\}, & nae &= \{0, 1\}^3 \setminus \{000, 111\}.
\end{align*}
\]

Which of the following implementations are valid? Construct the correct implementations.

1. \(1\text{-in-3} \in \langle or_0, or_3 \rangle\)
2. \(or_0 \in \langle 1\text{-in-3} \rangle\) and \(or_3 \in \langle 1\text{-in-3} \rangle\)
3. \(or_0 \in \langle bor_0, bor_2 \rangle\) and \(or_3 \in \langle bor_0, bor_2 \rangle\)
4. \(nae \in \langle 1\text{-in-3} \rangle\)
5. \(1\text{-in-3} \in \langle nae \rangle\)
Pol 1-in-3 = I_2 and Pol⟨or_0, or_3⟩ = I_2, hence 1-in-3 ∈ ⟨or_0, or_3⟩.

For 1-in-3 ∈ ⟨or_0, or_3⟩:

\[
\begin{align*}
\text{neq}(x, y) &= or_0(x, y, y) \land or_3(x, y, y) \\
\text{1-in-3}(x, y, z) &= \exists v_1 \exists v_2 \exists v_3 \\
& \quad or_0(x, y, z) \land or_3(x, y, z) \\
& \quad \land or_3(v_1, y, z) \land \text{neq}(x, u_1) \\
& \quad \land or_3(x, u_2, z) \land \text{neq}(y, u_2) \\
& \quad \land or_3(x, y, u_3) \land \text{neq}(z, u_3)
\end{align*}
\]
Solution to Exercise 5 (2)

\[ \text{Pol 1-in-3} = I_2 \text{ hence } or_0, or_3 \in \langle 1\text{-in-3} \rangle. \]

For \( or_0 \in \langle 1\text{-in-3} \rangle \) and \( or_3 \in \langle 1\text{-in-3} \rangle \)

\[
\begin{align*}
[1\text{-in-3}(x_T, x_F, x_F)] &= \{100\} \\
[1\text{-in-3}(x, y, 0)] &= \{01, 10\} \\
neq(x, y) &= \exists x_T \exists x_F \\
& \quad 1\text{-in-3}(x, y, x_F) \land 1\text{-in-3}(x_T, x_F, x_F) \\
\text{or}_0(x_1, x_2, x_3) &= \exists y_2 \exists y_3 \exists z_1 \exists z_2 \exists z_3 \exists z_4 \exists z_5 \\
& \quad 1\text{-in-3}(x_1, z_1, z_2) \land 1\text{-in-3}(y_2, z_1, z_3) \\
& \quad \land 1\text{-in-3}(y_3, z_2, z_4) \land 1\text{-in-3}(z_2, z_3, z_5) \\
& \quad \land \neq(x_2, y_2) \land \neq(x_3, y_3) \\
\text{or}_3(x_1, x_2, x_3) &= \exists y_1 \exists y_2 \exists y_3 \\
& \quad \neq(x_1, y_1) \land \neq(x_2, y_2) \land \neq(x_3, y_3) \\
& \quad \land \text{or}_0(y_1, y_2, y_3)
\end{align*}
\]
For \( or_0 \in \langle bor_0, bor_2 \rangle \) and \( or_3 \in \langle bor_0, bor_2 \rangle \):

- **impossible** since \( \langle bor_0, bor_2 \rangle \) is bijunctive (visible from definition), but neither \( or_0 \) nor \( or_3 \) are bijunctive
- \( \text{maj}(001, 010, 100) = 000 \notin or_0 \)
- \( \text{maj}(011, 101, 110) = 111 \notin or_3 \)
For $nae \in \langle 1\text{-in-}3 \rangle$:

- $nae = or_0(x, y, z) \land or_3(x, y, z)$
- use the result of Point 2
For $1$-in-$3 \in \langle \text{nae} \rangle$:

- **impossible**, since $\text{nae}$ is complementive (closed under $\neg$), but $1$-in-$3$ is not
- $001 \in 1$-in-$3$, $\neg(001) = 110$, $110 \not\in 1$-in-$3$
Consider the relation $R(x, y, z) = \{010, 100, 101, 110, 111\}$. This relation is bijunctive, since it is closed under majority. Let us produce the projections $R_{ij}$.

### Projections $R_{ij}$

There are 6 projections $R_{ij}$ of the relation $R$ where $i \leq j$:

- $R_{11} = \{0, 1\} = R_{22} = R_{33}$
- $R_{12} = \{01, 10, 11\} = [x \lor y]$
- $R_{13} = \{00, 10, 11\} = [x \lor \neg z]$
- $R_{23} = \{10, 00, 01, 11\}$

### Solution

Hence $R(x, y, z) = R_{12}(x, y) \land R_{13}(x, z) = (x \lor y) \land (x \lor \neg z)$.
Next lesson on 13 November 2012 will take place from 16:15 to 17:45.