Solution of Partial Exam 2012-2013

Algorithms and Complexity of
Constraint Satisfaction Problems

MPRI 2.31.1

4 December 2012

Exercise 1 (3 points)

Can the Boolean function \(\rightarrow\) (implication) be constructed (by composition, variable identification, and projection) from the Boolean functions \(\equiv\) (equivalence) and \(\lor\) (disjunction)? If YES, explicitly write the construction. If NO, prove that the construction is impossible.

We have \((x \rightarrow y) = (x \lor y) \equiv y\) or also \((x \rightarrow y) = (x \equiv y) \lor y\). The answer is YES.

Exercise 2 (3 points)

Consider the Boolean operator of anti-implication \(\rightarrow\) defined by \(x \rightarrow y = \neg(x \rightarrow y)\). Let \(R\) be a Boolean relation closed under \(\rightarrow\). Determine if \(\text{CSP}(R)\) is polynomial-time decidable or \(\text{NP}\)-complete.

We have \(x \rightarrow x = \neg(x \rightarrow x) = 0\). Hence the relation \(R\) is 0-valid and therefore, by Schaefer’s Dichotomy Theorem, \(\text{CSP}(R)\) is polynomial-time decidable.

Exercise 3 (8 points)

We say that a Boolean vector \(b \in \{0, 1\}^n\) of arity \(n\) has no three equally spaced 0s if there are no indices \(i < j < k \leq n\), such that \(b[i] = b[j] = b[k] = 0\) and \(j - i = k - j\). Similarly, we say that a Boolean vector \(b \in \{0, 1\}^n\) of arity \(n\) has no three equally spaced 1s if there are no indices \(i < j < k \leq n\), such that \(b[i] = b[j] = b[k] = 1\) and \(j - i = k - j\).

Let \(R\) be a nonempty Boolean relation of arity 8, containing all possible vectors \(b \in \{0, 1\}^8\), which have neither three equally spaced 0s nor three equally spaced 1s. For instance, \(b = 00110011\) belongs to \(R\), but \(b' = 01001011\) does not, since \(b'[2] = b'[5] = b'[8] = 1\) and \(5 - 2 = 8 - 5 = 3\).

How many vectors does the relation \(R\) have? Write the relation \(R\). Is the relation \(R\) (1) 0-valid, (2) 1-valid, (3) Horn, (4) dual Horn, (5) bijunctive, (6) affine, or a combination thereof? What is the complexity of \(\text{CSP}(R)\)? What is the smallest relation \(Q\) (in
By inspection, we can easily determine that (2 points)
\[ R = \{00110011, 11001100, 00100110, 11011001, 01100100, 01011010\}, \]
hence \( R \) has 6 vectors (1 point). The relation \( R \) is (2 points)
1. not 0-valid by inspection,
2. not 1-valid by inspection,
3. not Horn, since \( 00110011 \lor 11001100 = 00000000 \not\in R \),
4. not dual Horn, since \( 00110011 \lor 11001100 = 11111111 \not\in R \),
5. not bijunctive, since \( \text{maj}(00110011, 00100110, 01011010) = 00110010 \not\in R \),
6. not affine, since 6 is not a integer power of 2.
7. but it is complementive (by inspection), since \( \neg R = R \).

Hence, \( \text{csp}(R) \) is NP-complete (1 point). Since \( \text{Pol} R = N_2 \) (\( R \) is complementive) and \( \text{nae} \in \text{Inv} N_2 \), the smallest relation \( Q \) implementing \( R \) is \( \text{nae} = \{001, 010, 100, 110, 101, 011\} \) (not-all-equal) (1 point). The corresponding primitive positive formula is (1 point)
\[ R(x_1, \ldots, x_8) = \bigwedge_{i=1}^{6} \text{nae}(x_i, x_{i+1}, x_{i+2}) \land \bigwedge_{i=1}^{4} \text{nae}(x_i, x_{i+2}, x_{i+4}) \land \bigwedge_{i=1}^{2} \text{nae}(x_i, x_{i+3}, x_{i+6}). \]

**Exercise 4 (6 points)**

Let \( m = (m_1, \ldots, m_k) \) and \( m' = (m'_1, \ldots, m'_k) \) be two Boolean vectors. We write \( m \leq m' \) to denote that \( m_i \leq m'_i \) holds for every \( i \in \{1, \ldots, k\} \), assuming \( 0 < 1 \), as well as \( m < m' \) for \( m \leq m' \) and \( m \neq m' \). The relation \( \leq \) is called the pointwise partial order on models. Two vectors \( a, b \) satisfying the relation \( a \leq b \) are called comparable.

We say that a relation \( R \), written as a matrix where the vectors of \( R \) are the rows, is irredundant if it does not contain two identical columns and it cannot be transformed by column permutation to a relation of the form \( Q \times \{0, 1\}^k \) for a \( k \geq 1 \), where \( Q \) is another relation.

Prove the following proposition.

**Proposition 1** Let \( R \) be an irredundant affine relation having at least two comparable vectors. Any two different comparable vectors \( a < b \) in \( R \) must differ in at least two positions \( i \) and \( j \), i.e. there exist \( i \neq j \) such that \( a[i] \neq b[i] \) and \( a[j] \neq b[j] \). Moreover, for any two comparable vectors \( a < b \) in \( R \) there must be a third vector \( c \in R \) which is not constant on the positions on which \( a \) and \( b \) differ. In other words, there must be a third vector \( c \in R \) with two positions \( i \) and \( j \), such that \( a[i] \neq b[i] \) and \( a[j] \neq b[j] \), for which we have \( c[i] \neq c[j] \).

We claim that the existence two comparable tuples \( a < b \) differing in just one position implies the relation \( R \) to be redundant. Denote by \( i \) the coordinate on which \( a \) and \( b \) differ. For any tuple \( c \) in \( R \) we construct the tuple \( c' = a + b + c \), which is identical to \( c \) except that \( c'[i] = \neg c[i] \). Since \( R \) is affine,
the tuple $c'$ must be in the relation. Hence, $R$ is redundant since it is of the form $Q \times \{0, 1\}$. Thus in any irredundant relation any two comparable tuples must differ in at least two positions. \hspace{1cm} (3 points)

Now let $a$ and $b$ differ on at least two positions, say $i$ and $j$. If all tuples $c \in R$ are constant on the positions $i$ and $j$ where $a$ and $b$ differ, i.e., $c[i] = c[j]$ for all $c \in R$, then $R$ is again redundant because in particular the columns $i$ and $j$ are identical. \hspace{1cm} (3 points)