Algorithms and Complexity of Constraint Satisfaction Problems (course number 5)

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We need first to determine which are the generic NP-complete constraint satisfaction problems. We already saw that

\[ \text{CSP}(or_0, or_3) = \text{CSP}(or_0, or_1, or_2, or_3) = 3\text{SAT} \]

is NP-complete, but they contain too many relations. We would need an NP-complete CSP generated by a single relation. A good candidate to generate NP-complete CSP is the relation

1-in-3 \[= \{001, 010, 100\}. \]
Theorem

\[ \text{CSP}(\text{1-in-3}) \text{ is NP-complete.} \]

Proof.

We need to construct from the relation \text{1-in-3} a set of relations \( S \) such that \( \text{CSP}(S) \) is NP-complete. We already know that \( \text{CSP}(\text{neq, or}_0) = 3\text{SAT} \), therefore it is sufficient to implement the relations

\[
\begin{align*}
\text{neq} & = \{01, 10\} \\
\text{or}_0 & = \{001, 010, 011, 100, 101, 110, 111\}
\end{align*}
\]

by the relation \text{1-in-3}. .../...
Proof (cont).

1. We will produce first the constraints 0 and 1 by variable identification. Note that

\[ [1\text{-in-}3(x_T, x_F, x_F)] = \{100\} \]

hence by the constraint $1\text{-in-}3(x_T, x_F, x_F)$ we force the variable $x_T$ to represent 1 and the variable $x_F$ to represent 0. [pinning]

2. Note also that

\[ [1\text{-in-}3(x, y, 0)] = \{01, 10\} \]

which gives us the possibility to produce the relation neq. This way we have the implementation of the constraint

\[ neq(x, y) = \exists x_T \exists x_F 1\text{-in-}3(x, y, x_F) \land 1\text{-in-}3(x_T, x_F, x_F) \]

.../...
Proof.

It is a little bit more complicated to implement $or_0$, but we can do it.

$$or_0(x_1, x_2, x_3) = \exists y_2 \exists y_3 \exists z_1 \exists z_2 \exists z_3 \exists z_4 \exists z_5$$

$$\exists 1-in-3(x_1, z_1, z_2) \land 1-in-3(y_2, z_1, z_3)$$

$$\land \exists 1-in-3(y_3, z_2, z_4) \land 1-in-3(z_2, z_3, z_5)$$

$$\land \exists neq(x_2, y_2) \land neq(x_3, y_3)$$

It is easy now to implement the relations $or_1$, $or_2$ and $or_3$ by $or_0$ and $neq$. This proves that the relation $1-in-3$ implements the necessary relations for the problem $3SAT$. 

To prove that the relation \textit{nae} generates NP-complete \textit{CSP}, we need first some lemmas.

\textbf{Lemma}

The formula $\varphi(x_1, \ldots, x_k)$ is satisfiable if and only if the disjunction $\varphi(x_1, \ldots, x_k) \lor \varphi(\neg x_1, \ldots, \neg x_k)$ is satisfiable.

\textbf{Proof.}

If $\varphi(x_1, \ldots, x_k)$ is satisfiable then $\varphi(x_1, \ldots, x_k) \lor \varphi(\neg x_1, \ldots, \neg x_k)$ is also satisfiable,

If $\varphi(x_1, \ldots, x_k) \lor \varphi(\neg x_1, \ldots, \neg x_k)$ then there exists a model $m$ satisfying the disjunction. Either $m \models \varphi(x_1, \ldots, x_k)$, and in this case the proof is finished, or $m \models \varphi(\neg x_1, \ldots, \neg x_k)$ and in this case $\neg m \models \varphi(x_1, \ldots, x_k)$.
The relations 1-in-3 and nae contain the following vectors:

\[
1\text{-in-3} = \{ \begin{array}{c} 001, 010, 100 \end{array} \}
\]
\[
nae = \{ \begin{array}{c} 001, 010, 100, 110, 101, 011 \end{array} \}
\]

Therefore we have the identity

\[
1\text{-in-3}(x, y, z) \lor 1\text{-in-3}(\neg x, \neg y, \neg z) = nae(x, y, z)
\]

Note that the relation nae implements neq by diagonalisation.

\[
[\text{neq}(x, y)] = [\text{nae}(x, y, y)] = \{01, 10\}.
\]

For each boolean vector \( m \in \{0, 1\}^k \), the function \( w(m) \) denotes its Hamming weight.
Theorem

\( \text{CSP}(\text{nae}) \) is NP-complete.

Proof.

- Let \( \varphi \) be a 1-in-3-formula, i.e., formula constructed from the relation 1-in-3. Let us construct the \( \text{nae} \)-formula \( \varphi' \) by replacement. For each clause \( 1\text{-in-3}(x, y, z) \) in \( \varphi \) we add the clause \( \text{nae}(x, y, z) \) in \( \varphi' \). If \( m \models \varphi \) then \( m \models \varphi' \) since \( 1\text{-in-3} \subseteq \text{nae} \).

- Let \( \varphi' \) be a \( \text{nae} \)-formula and \( m \) an interpretation such that \( m \models \varphi' \). For each clause \( c = \text{nae}(x_1, x_2, x_3) \) of \( \varphi' \) let \( m_c \) be the restriction of \( m \) to the variables \( \{x_1, x_2, x_3\} \) which satisfies \( c \). We construct a new \( \text{nae} \)-formula \( \varphi'' \) by replacement. If \( w(m_c) = 1 \), we add the clause \( c \) to \( \varphi'' \). If \( w(m_c) = 2 \), we replace the clause \( c \), where \( m(x) = 1 \), by the formula \( \exists y \, \text{nae}(y, x_2, x_3) \land \text{neq}(x_1, y) \). Note that we can implement the relation \( \text{neq} \) by \( \text{nae} \) as well as by 1-in-3.

\[ \ldots / \ldots \]
Proof.

- From the nae-formula $\varphi''$, we construct the 1-in-3-formula $\varphi$ by replacement. We replace in $\varphi''$ each clause $\text{nae}(x, y, z)$ by $\text{1-in-3}(x, y, z)$ and each clause $\text{neq}(x, y) = \text{nae}(x, y, y)$ by

$$\text{neq}(x, y) = \exists x_T \exists x_F \, 1\text{-in-3}(x, y, x_F) \land 1\text{-in-3}(x_T, x_F, x_F)$$

- This way, for each vector $m$, if $m \models \varphi'$ then $m \models \varphi$. 
We are finally able to prove the result announced at the beginning of this course.

**Theorem (Dichotomy Theorem (Schaefer, 1978))**

Let $S$ be a set of boolean relations. If $S$ satisfies one of the following conditions

1. $S$ is 0-valid,
2. $S$ is 1-valid,
3. $S$ is Horn,
4. $S$ is dual Horn,
5. $S$ is bijunctive,
6. $S$ is affine,

then $\text{CSP}(S)$ is decidable in polynomial time. Otherwise, $\text{CSP}(S)$ is NP-complete.
Proof.

We perform a case analysis following the 8 interesting clones $I_2, I_0, I_1, N_2, E_2, V_2, L_2$ and $D_2$, the polynomial cases first.

1. If $\text{Pol}(S) \supseteq I_0$, i.e., $S$ is 0-valid, then each relation in $S$ contains the vector $0 \cdots 0$. Hence each instance of $\text{CSP}(S)$ is satisfiable by the interpretation $0 \cdots 0$.

2. If $\text{Pol}(S) \supseteq I_1$, i.e., $S$ is 1-valid, then each relation in $S$ contains the vector $1 \cdots 1$. Hence each instance of $\text{CSP}(S)$ is satisfiable by the interpretation $1 \cdots 1$.

3. If $\text{Pol}(S) \supseteq E_2$, i.e., $S$ is closed under conjunction, then $S$ is Horn. $\text{HornSat}$ is polynomial, hence each instance of $\text{CSP}(S)$ is decidable in polynomial time.

4. If $\text{Pol}(S) \supseteq V_2$, i.e., $S$ is closed under disjunction, then $S$ is dual Horn. Hence $\text{CSP}(S)$ is decidable in polynomial time by duality with the Horn case.

.../...
Proof.

5 If $\text{Pol}(S) \supseteq L_2$, i.e., $S$ is closed under affinity then $S$ is affine. $\text{AffineSat}$ is polynomial, hence $\text{CSP}(S)$ is decidable in polynomial time.

6 If $\text{Pol}(S) \supseteq D_2$, i.e., $S$ is closed under majority, then $S$ is bijunctif. $2\text{Sat}$ is polynomial, hence $\text{CSP}(S)$ is decidable in polynomial time.

7 If $\text{Pol}(S) = N_2$, i.e., $S$ is closed under negation, then $S$ is complementive. Given that $\text{nae}$ is complementive, we have $\text{nae} \in \langle S \rangle$. Therefore $\text{CSP}(S)$ is NP-complete.

8 If $\text{Pol}(S) = I_2$, i.e., $S$ is cosed only under the identity, then $S$ is a set of all boolean relations. Given that $1\text{-in-3} \in \text{Inv}(I_2)$, we have $1\text{-in-3} \in \langle S \rangle$. Therefore $\text{CSP}(S)$ is NP-complete.
Some remarks

- Once Post's lattice as well as Galois correspondence between clones and co-clones known, the proof of the Dichotomy Theorem for boolean CSP is very easy.
- The original proof by Schaefer is totally different and much more difficult to understand.
We can now solve boolean constraint satisfaction problems very easily.

**Probleme** **Monotone 3neg-2pos Sat**

*Input:* A set of variables $V$ and a formula $\varphi$ in CNF on $V$, where each clause contains either *three* negative literals or *two* positive literals.

*Question:* Is the formula $\varphi$ satisfiable?
Monotone 3neg-2pos Sat is NP-complete.

Proof.
The problem corresponds to CSP([¬x ∨ ¬y ∨ ¬z], [x ∨ y]). The relation [¬x ∨ ¬y ∨ ¬z] is Horn, the relation [x ∨ y] is dual Horn and bijunctive. It is therefore easy to see that \( \text{Pol}([\neg x \lor \neg y \lor \neg z], [x \lor y]) = I_2 \). Therefore the problem Monotone 3neg-2pos Sat is NP-complete.
Solution of an Exercise

We can now easily answer aforementioned questions:

1. CSP($or_0$, $or_1$) is polynomial, since $Pol(or_0, or_1) \supseteq V_2$
   (exactly $Pol(or_0, or_1) = V_1$).

2. CSP($or_0$, $or_3$) also called MONOTONE 3SAT is NP-complete, since
   $Pol(or_0, or_3) = I_2$.

3. CSP($or_2$, $or_3$) is polynomial, since $Pol(or_2, or_3) \supseteq E_2$
   (exactly $Pol(or_2, or_3) = E_0$).

4. CSP($or_1$, $or_2$) is NP-complete, since $Pol(or_1, or_2) = I_2$.

Recall the relations

The relations used in the aforementioned CSP are:

\[
\begin{align*}
or_0 & = [x \lor y \lor z] \\
or_1 & = [x \lor y \lor \neg z] \\
or_2 & = [x \lor \neg y \lor \neg z] \\
or_3 & = [\neg x \lor \neg y \lor \neg z]
\end{align*}
\]
Problem \textsc{Another Sat}$(S)$

\textit{Input:} A boolean formula \( \varphi \) in CNF over \( S \) and a model \( m \) of \( \varphi \).

\textit{Question:} Is there another model \( m' \) satisfying \( \varphi \), where \( m' \neq m \)?
**Problem** **Fourth SAT(S)**

*Input:* An $S$-formula $\varphi$ and three models $M = \{m_1, m_2, m_3\}$ of $\varphi$.

*Question:* Is there another model $m \notin M$ satisfying $\varphi$?
Problem **Yet Another SAT** (Horn)

**Input:** A Horn formula $\varphi$ and a nonempty set of models $M \subseteq [\varphi]$.

**Question:** Is there another model $m \notin M$ satisfying $\varphi$?
Exercise

Determine if $\text{CSP}(S)$ are polynomial or NP-complete for the following sets of boolean relations $S$:

1. $S = \{(x \land \neg y) \equiv z\}$
2. $S = \{(x \neq y) \equiv z, (x \lor y) \equiv z\}$
3. $S = \{(x \neq y) \equiv z, (x \land y) \equiv z\}$
4. $S = \{(x \equiv y) \equiv z, (x \lor y) \land z\}$
It’s all for today.
Do you have questions?