# Complexity of Default Logic on Generalized Conjunctive Queries*


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**Abstract.** Reiter's default logic formalizes nonmonotonic reasoning using default assumptions. The semantics of a given instance of default logic is based on a fixpoint equation defining an extension. Three different reasoning problems arise in the context of default logic, namely the existence of an extension, the presence of a given formula in an extension, and the occurrence of a formula in all extensions. Since the end of 1980s, several complexity results have been published concerning these default reasoning problems for different syntactic classes of formulas. We derive in this paper a complete classification of default logic reasoning problems by means of universal algebra tools using Post's clone lattice. In particular we prove a trichotomy theorem for the existence of an extension, classifying this problem to be either polynomial, $\mathrm{NP}$-complete, or $\Sigma_2\mathrm{P}$-complete, depending on the set of underlying Boolean connectives. We also prove similar trichotomy theorems for the two other algorithmic problems in connection with default logic reasoning.


## 1 Introduction

Nonmonotonic reasoning is one of the most important topics in computational logic and artificial intelligence. Different logics formalizing nonmonotonic reasoning have been developed and studied since the late 1970s. One of the most known is Reiter's *default logic* [21], which formalizes nonmonotonic reasoning using default assumptions. Default logic can express facts like "by default, a formula $\varphi$ is true", in contrast with standard classical logic, which can only express that a formula $\varphi$ is true or false.

Default logic is based on the principle of defining the semantics of a given set of formulas $W$ (also called *premises* or *axioms*) through a fixpoint equation by means of a finite set of defaults $D$. The possible extensions of a given set $W$ of axioms are the sets $E$, stable under a specific transformation, i.e., satisfying the identity $\Gamma(E) = E$. These fixpoint sets $E$ represent the different possible sets of knowledge that can be adopted on the base of the premises $W$. Three important decision problems arise in the context of reasoning in default logic. The first is to decide whether for a given set of axioms $W$ and defaults $D$ there exists a fixpoint. The second, called *credulous reasoning*, is the task to determine whether a formula $\varphi$ occurs in at least one extension


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of the set $W$. The third one, called *skeptical reasoning* asks to determine whether a given formula $\varphi$ belongs to *all* extensions of $W$.

At the end of 1980s and the beginning of 1990s, several complexity results were proved for default logic reasoning. Several authors have investigated the complexity of the three aforementioned problems for syntactically restricted versions of propositional default logic. Kautz and Selman [13] proved the $\mathrm{NP}$-completeness of propositional default reasoning restricted to disjunction-free formulas, i.e., all propositional formulas occurring in the axioms $W$ and the defaults $D$ are conjunctions of literals. Furthermore, they show that for very particular restrictions default reasoning is feasible in polynomial time. Stillman [23,24] extends the work of Kautz and Selman by analyzing further subclasses of disjunction-free default theories, as well as some other classes that allow a limited use of disjunction. The work of Kautz and Selman [13], as well as of Stillman [23, 24] provided a good understanding of the tractability frontier of propositional default reasoning. The complexity of the general case was finally settled by Gottlob in [11], where he proved that propositional default reasoning is complete for the second level of the polynomial hierarchy. All these complexity results indicate that default logic reasoning is more complicated than that of the standard propositional logic.

In the scope of the aforementioned results a natural question arises whether the previous analysis covers all possible cases. We embark on this challenge by making two generalizations. First, the usual clauses have been generalized to constraints based on Boolean relations. Second, we allow in the axioms $W$ and the defaults $D$ not only formulas built as conjunctions of constraints, but also conjunctive queries, i.e., existential positive conjunctive formulas built upon constraints. This approach using a restricted existential quantification can be seen as a half way between the usual propositional formulas and the default query language DQL defined in [6]. Moreover, this approach is natural in the scope of relation-based constraints, since it allows us to use the universal algebra tools to reason about complexity. We take advantage of the closed classes of Boolean functions and relations, called clones and co-clones, which allow us to prove a complexity result for a single representant of this class, that extends by means of closure properties to all Boolean functions or queries, respectively, in the same class. Using these algebraic tools we deduce a complete classification of the three default reasoning problems parametrized by sets of Boolean constraints. Similar classification, using universal algebra tools and Post lattice, had been already done for other nonmonotonic reasoning formalisms, namely circumscription [16] and abduction [8,17]. Finally, a complexity classification of propositional default logic along other lines, studying the structural aspects of the underlying formulas, had been done in [1]. Our approach to the complexity classification differs from Ben-Eliyahu's [1] in the following points: (1) the class of formulas in the the axioms, prerequisite, justification, and consequence of defaults is always the same; (2) the classification is performed on the set of underlying Boolean relations $S$, taking the role of a parameter, from which the formulas are built and not on the input formulas itself; (3) the studied classes of formulas are closed under conjunction and existential quantification. The aforementioned requirements for uniformity of the formulas in all three parts of defaults and in the axioms, plus the closure under conjunction exclude prerequisite-free, justification-free, normal, semi-normal, or any other syntactically restricted default theories from this classification.

## 2 Preliminaries

Throughout the paper we use the standard correspondence between predicates and relations. We use the same symbol for a predicate and its corresponding relation, since the meaning will always be clear from the context, and we say that the predicate *represents* the relation.

An $n$-ary *logical relation* $R$ is a Boolean relation of arity $n$. Each element of a logical relation $R$ is an $n$-ary Boolean vector $m = (m_1, \dots, m_n) \in \{0,1\}^n$. Let $V$ be a set of variables. A *constraint* is an application of $R$ to an $n$-tuple of variables from $V$, i.e., $R(x_1, \dots, x_n)$. An assignment $I \colon V \to \{0,1\}$ satisfies the constraint $R(x_1, \dots, x_n)$ if $(I(x_1), \dots, I(x_n)) \in R$ holds.

*Example 1.* Equivalence is the binary relation defined by $eq = \{00, 11\}$. Given the ternary relations $nae = \{0,1\}^3 \smallsetminus \{000, 111\}$ and 1-in-3 $= \{100, 010, 001\}$, the constraint $nae(x,y,z)$ is satisfied if not all variables are assigned the same value and 1-in-3$(x,y,z)$ is satisfied if exactly one of the variables $x$, $y$, and $z$ is assigned to 1.

Throughout the text we refer to different types of Boolean constraint relations following Schaefer's terminology [22]. We say that a Boolean relation $R$ is *1-valid* if $1 \cdots 1 \in R$ and it is *0-valid* if $0 \cdots 0 \in R$; *Horn* (*dual Horn*) if $R$ can be represented by a conjunctive normal form (CNF) formula having at most one unnegated (negated) variable in each clause; *bijunctive* if it can be represented by a CNF formula having at most two variables in each clause; *affine* if it can be represented by a conjunction of linear functions, i.e., a CNF formula with $\oplus$-clauses (XOR-CNF); *complementive* if for each $(\alpha_1, \dots, \alpha_n) \in R$, also $(\neg\alpha_1, \dots, \neg\alpha_n) \in R$. A set $S$ of Boolean relations is called 0-valid (1-valid, Horn, dual Horn, affine, bijunctive, complementive) if every relation in $S$ is 0-valid (1-valid, Horn, dual Horn, affine, bijunctive, complementive).

Let $R$ be a Boolean relation of arity $n$. The *dual relation* to $R$ is the set of vectors $\mathrm{dual}(R) = \{(\neg\alpha_1, \dots, \neg\alpha_n) \mid (\alpha_1, \dots, \alpha_n) \in R\}$. Note that $R^{\neg} = R \cup \mathrm{dual}(R)$ is a complementive relation called the complementive closure of $R$. The set $\mathrm{dual}(S) = \{\mathrm{dual}(R) \mid R \in S\}$ denotes the corresponding dual relations to the set of relations $S$.

Let $S$ be a non-empty finite set of Boolean relations. An *$S$-formula* is a finite conjunction of *$S$-clauses*, $\varphi = c_1 \wedge \cdots \wedge c_k$, where each $S$-clause $c_i$ is a constraint application of a logical relation $R \in S$. An assignment $I$ satisfies the formula $\varphi$ if it satisfies all clauses $c_i$. We denote by $\mathrm{sol}(\varphi)$ the set of satisfying assignments of a formula $\varphi$.

Schaefer in his seminal paper [22] developed a complexity classification of the satisfiability problem of $S$-formulas. *Conjunctive queries* turn out to be useful in order to obtain this result. Given a set $S$ of Boolean relations, we denote by $\mathrm{COQ}(S)$ the set of all formulas of the form

$$F(x_1, \dots, x_k) \quad = \quad \exists y_1 \exists y_2 \cdots \exists y_l \; \varphi(x_1, \dots, x_k, y_1, \dots, y_l),$$

where $\varphi$ is an $S$-formula. We call these existentially quantified formulas *conjunctive queries over* $S$, with $\boldsymbol{x} = (x_1, \dots, x_k)$ being the vector of *distinguished variables*.

As usually in computational complexity, we denote by $A \leq_m B$ a polynomial-time many-one reduction from the problem $A$ to problem $B$. If there exist reductions $A \leq_m B$ and $B \leq_m A$, we say that the problems $A$ and $B$ are *polynomially equivalent*, denoted by $A \equiv_m B$.

| | | | |
|---|---|---|---|
| $\mathrm{Pol}(R) \supseteq \mathrm{E}_2 \Leftrightarrow R$ is Horn | | $\mathrm{Pol}(R) \supseteq \mathrm{V}_2 \Leftrightarrow R$ is dual Horn | |
| $\mathrm{Pol}(R) \supseteq \mathrm{D}_2 \Leftrightarrow R$ is bijunctive | | $\mathrm{Pol}(R) \supseteq \mathrm{L}_2 \Leftrightarrow R$ is affine | |
| $\mathrm{Pol}(R) \supseteq \mathrm{N}_2 \Leftrightarrow R$ is complementive | | $\mathrm{Pol}(R) \supseteq \mathrm{R}_2 \Leftrightarrow R$ is disjunction-free | |
| $\mathrm{Pol}(R) \supseteq \mathrm{I}_0 \Leftrightarrow R$ is 0-valid | | $\mathrm{Pol}(R) \supseteq \mathrm{I}_1 \Leftrightarrow R$ is 1-valid | |
| $\mathrm{Pol}(R) \supseteq \mathrm{I} \Leftrightarrow R$ is 0- and 1-valid | | $\mathrm{Pol}(R) \supseteq \mathrm{I}_2 \Leftrightarrow R$ is Boolean | |

**Fig. 1.** Polymorphism correspondences

## 3 Closure Properties of Constraints

There exist easy criteria to determine if a given relation is Horn, dual Horn, bijunctive, or affine. We recall these properties here briefly for completeness. An interested reader can find a more detailed description with proofs in the paper [5] or in the monograph [7]. Given a logical relation $R$, the following *closure properties* fully determine the structure of $R$, where $\oplus$ is the exclusive or and $\mathrm{maj}$ is the majority operation:

- $R$ is Horn if and only if $m, m' \in R$ implies $(m \wedge m') \in R$.
- $R$ is dual Horn if and only if $m, m' \in R$ implies $(m \vee m') \in R$.
- $R$ is affine if and only if $m, m', m'' \in R$ implies $(m \oplus m' \oplus m'') \in R$.
- $R$ is bijunctive if and only if $m, m', m'' \in R$ implies $\mathrm{maj}(m, m', m'') \in R$.

The notion of closure property of a Boolean relation has been defined more generally, see for instance [12,18]. Let $f\colon \{0,1\}^k \to \{0,1\}$ be a Boolean function of arity $k$. We say that $R$ is *closed under* $f$, or that $f$ is a *polymorphism* of $R$, if for any choice of $k$ vectors $m_1, \ldots, m_k \in R$, not necessarily distinct, we have that

$$\Big(f\big(m_1[1], \ldots, m_k[1]\big),\ \ldots,\ f\big(m_1[n], \ldots, m_k[n]\big)\Big) \in R, \tag{1}$$

i.e., that the new vector constructed coordinate-wise from $m_1$, ..., $m_k$ by means of $f$ belongs to $R$. We denote by $\mathrm{Pol}(R)$ the set of all polymorphisms of $R$ and by $\mathrm{Pol}(S)$ the set of Boolean functions that are polymorphisms of every relation in $S$. It turns out that $\mathrm{Pol}(S)$ is a *closed set of Boolean functions*, also called a *clone*, for every set of relations $S$. In fact, a clone is a set of functions containing all projections and closed under composition. A clone generated by a set of functions $F$, i.e., a set containing $F$, all projections, and closed under composition, is denoted by $[F]$. All closed classes of Boolean functions were identified by Post [20]. Post also detected the inclusion structure of these classes, which is now referred to as *Post's lattice*, presented in Fig. 2 with the notation from [2]. We did not use the previously accepted notation for the clones, as in [18,19], since we think that the new one used in [2] is better suited mnemotechnically and also scientifically than the old one. The correspondence of the most studied classes with respect to the polymorphisms of a relation $R$ is presented in Fig. 1. The class $\mathrm{I}_2$ is the closed class of Boolean functions generated by the identity function, thus for every Boolean relation $R$ we have $\mathrm{Pol}(R) \supseteq \mathrm{I}_2$. If the condition $\mathrm{Pol}(S) \supseteq C$ holds for $C \in \{\mathrm{E}_2, \mathrm{V}_2, \mathrm{D}_2, \mathrm{L}_2\}$, i.e., $S$ being Horn, dual Horn, bijunctive, or affine, respectively, then we say that the set of relations $S$ belongs to the *Schaefer's class*.

A Galois correspondence has been exhibited between the sets of Boolean functions $\mathrm{Pol}(S)$ and the sets of Boolean relations $S$. A basic introduction to this correspondence can be found in [18] and a comprehensive study in [19]. See also [5]. This theory helps us to get elegant and short proofs for results concerning the complexity of conjunctive queries. Indeed, it shows that the smaller the set of polymorphisms is, the more expressive the corresponding conjunctive queries are, which is the cornerstone for applying the algebraic method to complexity (see [2] and [5] for surveys). The following proposition can be found, e.g., in [5, 18, 19].

**Proposition 2.** *Let $S_1$, $S_2$ be two sets of Boolean relations. The inclusion* $\mathrm{Pol}(S_1) \subseteq \mathrm{Pol}(S_2)$ *implies* $\mathrm{COQ}(S_1 \cup \{eq\}) \supseteq \mathrm{COQ}(S_2 \cup \{eq\})$.

Given a $k$-ary Boolean function $f\colon \{0,1\}^k \longrightarrow \{0,1\}$, the set of *invariants* $\mathrm{Inv}(f)$ of $f$ is the set of Boolean relations closed under $f$. More precisely, a relation $R$ belongs to $\mathrm{Inv}(f)$ if the membership condition (1) holds for any collection of not necessarily distinct vectors $m_i \in R$ for $i = 1, \ldots, k$. If $F$ is a set of Boolean functions then $\mathrm{Inv}(F)$ is the set of invariants for each function $f \in F$. It turns out that $\mathrm{Inv}(F)$ is a *closed set of Boolean relations*, also called a *co-clone*, for every set of functions $F$. In fact, a co-clone is a set of relations (identified by their predicates) closed under conjunction, variable identification, and existential quantification. A co-clone generated by a set of relations $S$ is denoted by $\langle S \rangle$. Polymorphisms and invariants relate clones and co-clones by a Galois correspondence. This means that $F_1 \subseteq F_2$ implies $\mathrm{Inv}(F_1) \supseteq \mathrm{Inv}(F_2)$ and $S_1 \subseteq S_2$ implies $\mathrm{Pol}(S_1) \supseteq \mathrm{Pol}(S_2)$. Geiger [10] proved the identities $\mathrm{Pol}(\mathrm{Inv}(F)) = [F]$ and $\mathrm{Inv}(\mathrm{Pol}(S)) = \langle S \rangle$ for all sets of Boolean functions $F$ and relations $S$.

## 4 Default Logic

A *default* [21] is an expression of the form

$$\frac{\alpha : \mathsf{M}\beta_1, \ldots, \mathsf{M}\beta_m}{\gamma} \tag{2}$$

where $\alpha, \beta_1, \ldots \beta_m, \gamma$ are propositional formulas. The formula $\alpha$ is called the *prerequisite*, $\beta_1, \ldots, \beta_m$ the *justification* and $\gamma$ the *consequence* of the default. The notation with $\mathsf{M}$ serves only to syntactically and optically distinguish the justification from the prerequisite. A *default theory* is a pair $T = (W, D)$, where $D$ is a set of defaults and $W$ a set of propositional formulas also called the *axioms*. For a default theory $T = (W, D)$ and a set $E$ of propositional formulas let $\Gamma(E)$ be the minimal set such that the following properties are satisfied:

**(D1)** $W \subseteq \Gamma(E)$
**(D2)** $\Gamma(E)$ is deductively closed
**(D3)** If

$$\frac{\alpha : \mathsf{M}\beta_1, \ldots, \mathsf{M}\beta_m}{\gamma} \in D, \quad \alpha \in \Gamma(E), \quad \text{and} \quad \neg\beta_1, \ldots, \neg\beta_m \notin E$$

then $\gamma \in \Gamma(E)$

Any fixed point of $\Gamma$, i.e., a set $E$ of formulas satisfying the identity $\Gamma(E) = E$, is an *extension* for $T$. Each extension $E$ of a default theory $T = (W, D)$ is identified by a subset $gd(E, T)$ of $D$, called the *generating defaults* of $E$, defined as

$$gd(E,T) = \left\{ \frac{\alpha : \mathsf{M}\beta_1, \ldots, \mathsf{M}\beta_m}{\gamma} \in D \;\middle|\; \alpha \in E, \neg\beta_1 \notin E, \ldots, \neg\beta_m \notin E \right\}.$$

There exists an equivalent constructive definition of the extension. It has been proved equivalent to the previous definition by Reiter in [21], whereas some authors, like Kautz and Selman [13], take it for the initial definition of the extension. Define $E_0 = W$ and

$$E_{i+1} = \mathrm{Th}(E_i) \cup \left\{ \gamma \;\middle|\; \frac{\alpha : \mathsf{M}\beta_1, \ldots, \mathsf{M}\beta_m}{\gamma} \in D,\ \alpha \in E_i,\ \text{and}\ \neg\beta_1, \ldots, \neg\beta_m \notin E \right\},$$

where $\mathrm{Th}(E)$ is the deductive closure of the set of formulas $E$. Then the *extension* of the default theory $T = (W, D)$ is the union $E = \bigcup_{i=0}^{\infty} E_i$. Notice the presence of the final union $E$ in the conditions $\neg\beta_i \notin E$.

We generalize the default theories in the same way as propositional formulas are generalized to $S$-formulas. For a non-empty finite set of Boolean relations $S$, an $S$-*default* is an expression of the form (2), where $\alpha, \beta_1, \ldots \beta_m, \gamma$ are formulas from $\textsc{coq}(S)$. An $S$-*default theory* is a pair $T(S) = (D, W)$, where $D$ is a set of $S$-defaults and $W$ a set of formulas from $\textsc{coq}(S)$. An $S$-*extension* is a minimal set of $\textsc{coq}(S)$-formulas including $W$ and closed under the fixpoint operator $\Gamma$.

Three algorithmic problems are investigated in connection with default logic, namely the existence of an extension for a given default theory $T$, the question whether a given formula $\varphi$ belongs to some extension of a default theory (called credulous or brave reasoning), and the question whether $\varphi$ belongs to every extension of a theory (called skeptical or cautious reasoning). We express them as constraint satisfaction problems.

**Problem:** $\textsc{extension}(S)$
*Input:* An $S$-default theory $T(S) = (W, D)$.
*Question:* Does $T(S)$ have an $S$-extension?

**Problem:** $\textsc{credulous}(S)$
*Input:* An $S$-default theory $T(S) = (W, D)$ and an $S$-formula $\varphi$.
*Question:* Does $\varphi$ belong to *some* $S$-extension of $T(S)$?

**Problem:** $\textsc{skeptical}(S)$
*Input:* An $S$-default theory $T(S) = (W, D)$ and an $S$-formula $\varphi$.
*Question:* Does $\varphi$ belong to *every* $S$-extension of $T$.

To be able to use the algebraic tools for exploration of complexity results by means of clones and co-clones, and to exploit Post's lattice, we need to establish a Galois connection for the aforementioned algorithmic problems.

**Theorem 3.** *Let $S_1$ and $S_2$ be two sets of relations such that the inclusion $\mathrm{Pol}(S_1) \subseteq \mathrm{Pol}(S_2)$ holds. Then we have the following reductions among problems:*

$$\textsc{extension}(S_2) \leq_m \textsc{extension}(S_1) \qquad \textsc{credulous}(S_2) \leq_m \textsc{credulous}(S_1)$$
$$\textsc{skeptical}(S_2) \leq_m \textsc{skeptical}(S_1)$$

*Proof.* Since $\mathrm{Pol}(S_1) \subseteq \mathrm{Pol}(S_2)$ holds, then any conjunctive query on $S_2$ can be expressed by a logically equivalent conjunctive query using only relations from $S_1$, according to Proposition 2. Let $T(S_2) = (W_2, D_2)$ be an $S_2$-default theory. Perform the aforementioned transformation for every conjunctive query in $W_2$ and $D_2$ to get corresponding sets of preliminaries $W_1$ and defaults $D_1$, equivalent to $W_2$ and $D_2$, respectively. Therefore the default theory $T(S_2)$ has an $S$-extension if and only if $T(S_1) = (W_1, D_1)$ has one. An analogous result holds for credulous and skeptical reasoning. □

Post's lattice is symmetric according to the main vertical line $\mathrm{BF} \longleftrightarrow \mathrm{I}_2$ (see Figure 2), expressing graphically the duality between various clones and implying the duality between the corresponding co-clones. This symmetry extends to all three algorithmic problems observed in connection with default logic, as we see in the following lemma. It will allow us to considerably shorten several proofs.

**Lemma 4.** *Let $S$ be a set of relations. Then the following equivalences hold:*

$$\begin{aligned} \text{EXTENSION}(S) &\equiv_m \text{EXTENSION}(\mathrm{dual}(S)) \\ \text{CREDULOUS}(S) &\equiv_m \text{CREDULOUS}(\mathrm{dual}(S)) \\ \text{SKEPTICAL}(S) &\equiv_m \text{SKEPTICAL}(\mathrm{dual}(S)) \end{aligned}$$

*Proof.* It is clear that $\varphi(\boldsymbol{x}) = R_1(\boldsymbol{x}) \wedge \cdots \wedge R_k(\boldsymbol{x})$ belongs to an $S$-extension $E$ of the default theory $T(S)$ if and only if the $\mathrm{dual}(S)$-formula $\varphi'(\boldsymbol{x}) = \mathrm{dual}(R_1)(\boldsymbol{x}) \wedge \cdots \wedge \mathrm{dual}(R_k)(\boldsymbol{x})$ belongs to a $\mathrm{dual}(S)$-extension $E'$ of the default theory $T(\mathrm{dual}(S))$. □

## 5 Complexity Results

Complexity results for reasoning in default logic started to be published in early 1990s. Gottlob [11] proved that deciding the existence of an extension for a propositional default theory is $\Sigma_2\mathrm{P}$-complete. Kautz and Selman [13] investigated the complexity of propositional default logic reasoning with unit clauses. They proved that deciding the existence of an extension for this special case is $\mathrm{NP}$-complete. Zhao and Ding [26] also investigated the complexity of several special cases of default logic, when the formulas are restricted to special cases of bijunctive formulas. We complete here the complexity classification for default logic by the algebraic method.

**Proposition 5.** *If $S$ is 0-valid or 1-valid, i.e., if $\mathrm{Pol}(S) \supseteq \mathrm{I}_0$ or $\mathrm{Pol}(S) \supseteq \mathrm{I}_1$, then every $S$-default theory always has a unique $S$-extension.*

*Proof.* Consider Reiter's constructive definition of the extension of an $S$-default theory $T(S) = (W, D)$. Since every formula in $W$ and $D$ is 0-valid (respectively 1-valid), every justification $\beta$ of any default is also 0-valid (1-valid). Then $\neg\beta$ is *not* 0-valid (1-valid) and therefore it cannot appear in any $S$-extension $E$. Therefore any default from $D$ is satisfied if and only if its prerequisite $\alpha$ is in the set $E_i$ for some $i$. Since every formula in $D$ is 0-valid (1-valid), whatever consequence $\gamma$ is added to $E_i$, there cannot be a contradiction with the formulas previously included into $E_i$. Hence we just have to add to $E$ every consequence $\gamma$ recursively derived from the prerequisites until we reach a fixpoint $E$. Since we start with a finite set of axioms $W$ and there is only a finite set of defaults $D$, an $S$-extension $E$ always exists and it is unique. $\square$

We need to distinguish the $\Sigma_2\mathrm{P}$-complete cases from the cases included in $\mathrm{NP}$. The following proposition identifies the largest classes of relations for which the existence of an extension is a member of $\mathrm{NP}$. According to the Galois connection, we need to identify the smallest clones that contain the corresponding polymorphisms. The reader is invited to consult Figure 1 to identify the clones of polymorphisms corresponding to the mentioned relational classes.

**Proposition 6.** *If $S$ is Horn, dual Horn, bijunctive, or affine, i.e., if the inclusions $\mathrm{Pol}(S) \supseteq \mathrm{E}_2$, $\mathrm{Pol}(S) \supseteq \mathrm{V}_2$, $\mathrm{Pol}(S) \supseteq \mathrm{D}_2$, or $\mathrm{Pol}(S) \supseteq \mathrm{L}_2$ hold, then the problem* EXTENSION$(S)$ *is in* $\mathrm{NP}$.

*Proof.* We present a nondeterministic polynomial algorithm which finds an extension for an $S$-default theory $T(S) = (W, D)$.

1. Guess a set $D' \subseteq D$ of generating defaults.
2. For every COQ$(S)$-formula $\varphi \in W \cup \{\gamma \mid \gamma$ consequence of $d \in D'\}$ verify that $\varphi \nvDash \neg\beta$ holds for every justification $\beta$ in $D'$, i.e., check that $\varphi \wedge \beta$ is satisfiable.
3. Check that $D'$ is minimal, i.e., for every $S$-default $\frac{\alpha:\mathsf{M}\beta_1,\ldots,\mathsf{M}\beta_m}{\gamma} \in D \smallsetminus D'$ and every COQ$(S)$-formula $\varphi \in W \cup \{\gamma \mid \gamma$ consequence of $d \in D'\}$ verify that $\varphi \nvDash \alpha$ or $\varphi \nvDash \beta_i$ holds for an $i$.

Step 1 ensures $\mathit{\Gamma}(E) \subseteq E$. Instead of $\varphi \nvDash \alpha$ and $\varphi \nvDash \beta_i$ for an $i$ we check whether $\varphi \Rightarrow \alpha$ and $\varphi \Rightarrow \beta_i$ hold, respectively. Note that $\theta \Rightarrow \rho$ holds if and only if $\theta \equiv \rho \wedge \theta$. Equivalence is decidable in polynomial time for $S$-formulas from Schaefer's class [3], which extends to conjunctive queries. Therefore we can decide if $\varphi \wedge \beta$, $\varphi \Rightarrow \alpha$, $\varphi \Rightarrow \beta_i$ hold, and also if $\varphi \nvDash \alpha$, $\varphi \nvDash \beta_i$ for an $i$, in polynomial time. Hence, Steps 2 and 3 can be performed in polynomial time. $\square$

Now we need to determine the simplest relational classes for which the extension problem is $\mathrm{NP}$-hard. The first one has been implicitly identified by Kautz and Selman [13] as the class of formulas consisting only of literals.

**Proposition 7.** *If $\mathrm{Pol}(S) \subseteq \mathrm{R}_2$ holds then* EXTENSION$(S)$ *is* $\mathrm{NP}$*-hard.*

*Proof.* Kautz and Selman proved in [13] using a reduction from 3SAT, that the extension problem is $\mathrm{NP}$-hard for default theories $T = (W, D)$, where all formulas in the axioms $W$ and the defaults $D$ are literals. Böhler *et al.* identified in [4] that the relational class generated by the sets of satisfying assignments to a literal is the co-clone $\mathrm{Inv}(\mathrm{R}_2)$. Therefore from the Galois connection and Theorem 3 follows that the inclusion $\mathrm{Pol}(S) \subseteq \mathrm{R}_2$ implies that the extension problem for $T(S)$ is $\mathrm{NP}$-hard. $\square$

The second simplest class with an $\mathrm{NP}$-hard extension problem contains all relations which are at the same time bijunctive, affine, and complementive.

**Proposition 8.** *If $\mathrm{Pol}(S) \subseteq \mathrm{D}$ holds, then* EXTENSION$(S)$ *is* $\mathrm{NP}$*-hard.*

*Proof.* Recall first that $\mathrm{Inv}(D)$ is generated by the relation $\{01, 10\}$ (see [4]), which is the set of satisfying assignments of the clause $x \oplus y$, or equivalently of the affine clause $x \oplus y = 1$. Note that the affine clause $x \oplus y = 0$ represents the equivalence relation $x \equiv y$ belonging to every co-clone. Hence, the co-clone $\mathrm{Inv}(D)$ contains both relations generated by $x \oplus y = 1$ and $x \oplus y = 0$.

We present a polynomial reduction from the NP-complete problem NAE-3SAT (Not-All-Equal 3SAT [9, page 259]) to EXTENSION$(S)$. Consider the following instance of NAE-3SAT represented by the formula $\varphi(x_1, \ldots, x_n) = \bigwedge_{i=1}^{k} nae(u_i, v_i, t_i)$ built upon the variables $x_1, \ldots, x_n$, where $nae(x, y, z)$ ensures that the variables $x$, $y$, $z$ do not take the same Boolean value. We first build the following $2(n-1)$ defaults

$$d_i^0 = \frac{\top : \mathsf{M}(x_i \oplus x_{i+1} = 0)}{x_i \oplus x_{i+1} = 0} \quad \text{and} \quad d_i^1 = \frac{\top : \mathsf{M}(x_i \oplus x_{i+1} = 1)}{x_i \oplus x_{i+1} = 1}$$

for each $i = 1, \ldots, n-1$. For each clause $nae(u, v, t)$ in the formula $\varphi$ we build the corresponding default

$$d(u, v, t) = \frac{\top : \mathsf{M}(u \oplus z = 1), \mathsf{M}(v \oplus z = 1), \mathsf{M}(t \oplus z = 1)}{\bot}$$

where $z$ is a new variable. From each pair $(d_i^0, d_i^1)$ exactly one default will apply. It will assign two possible pairs of truth values $(b_i, b_{i+1})$ to the variables $x_i$ and $x_{i+1}$. This way the first set of default pairs separates the variables $x_1, \ldots, x_n$ into two equivalence classes. All variables in one equivalence class take the same truth value.

Note that the formula $(u \oplus z = 1) \wedge (v \oplus z = 1) \wedge (t \oplus z = 1)$ is satisfied only if the identity $u = v = t$ holds. Therefore the default $d(u, v, t)$ applies if and only if the clause $nae(u, v, t)$ is not satisfied. Let $D$ be the set of all constructed defaults $d_i^0$, $d_i^1$, and $d(u, v, t)$ for each clause $nae(u, v, t)$ from $\varphi$. This implies that the formula $\varphi(x_1, \ldots, x_n)$ has a solution if and only if the default theory $(\emptyset, D)$ has an extension. The proposition then follows from Theorem 3. $\square$

Finally, we deal with the most complicated case of default theories. The following proposition presents a generalization of Gottlob's proof from [11] that the existence of an extension is $\Sigma_2\mathrm{P}$-complete.

**Proposition 9.** *If* $\mathrm{Pol}(S) \subseteq \mathrm{N}_2$ *holds then* EXTENSION$(S)$ *is* $\Sigma_2\mathrm{P}$*-hard.*

*Proof.* Let $\psi = \exists \boldsymbol{x}\, \forall \boldsymbol{y}\; \varphi(\boldsymbol{x}, \boldsymbol{y})$ be a quantified Boolean formula, with the variable vectors $\boldsymbol{x} = (x_1, \ldots, x_n)$ and $\boldsymbol{y} = (y_1, \ldots, y_m)$, such that the relation $R = \mathrm{sol}(\varphi(\boldsymbol{x}, \boldsymbol{y}))$ satisfies the condition $\mathrm{Pol}(R) = \mathrm{I}_2$. Let $R^{\neg}$ be the dual closure of the relation $R$. It is clear that $R(\boldsymbol{x}, \boldsymbol{y})$ is satisfiable if and only if $R^{\neg}(\boldsymbol{x}, \boldsymbol{y})$ is. Suppose that $\mathrm{Pol}(S) = \mathrm{N}_2$ holds, meaning that $S$ is a set of complementive relations. Since $R^{\neg}$ is complementive, the relation $\bar{R} = \{0,1\}^{n+m} \smallsetminus R^{\neg}$ must be complementive as well. Therefore both relations $R^{\neg}$ and $\bar{R}$ must be in the co-clone $\langle S \rangle = \mathrm{Inv}(\mathrm{Pol}(S)) = \mathrm{Inv}(\mathrm{N}_2)$ generated by the relations $S$. Moreover, we have that $\bar{R}(\boldsymbol{x}, \boldsymbol{y}) = \neg R^{\neg}(\boldsymbol{x}, \boldsymbol{y})$.

The identity relation is included in every co-clone, therefore we can use the identity predicate $(x = y)$. Since $S$ is complementive, the co-clone $\langle S \rangle$ contains the relation

$nae$, according to [4]. By identification of variables we can construct the predicate $nae(x, y, y)$ which is identical to the inequality predicate $(x \neq y)$.

Construct the $S$-default theory $T(S) = (W, D)$ with the empty set of axioms $W = \emptyset$ and the defaults $D = D_1 \cup D_2$, where

$$D_1 = \left\{ \frac{\top : \mathsf{M}(x_i = x_{i+1})}{x_i = x_{i+1}},\ \frac{\top : \mathsf{M}(x_i \neq x_{i+1})}{x_i \neq x_{i+1}} \;\middle|\; i = 1, \ldots, n-1 \right\},$$

$$D_2 = \left\{ \frac{\top : \mathsf{M}\bar{R}(\boldsymbol{x}, \boldsymbol{y})}{\perp} \right\}.$$

The satisfiability of $\psi$ is the generic $\Sigma_2\mathrm{P}$-complete problem [25]. To prove $\Sigma_2\mathrm{P}$-hardness for $S$-EXTENSION where $\mathrm{Pol}(S) = \mathrm{N}_2$, it is sufficient to show that $\psi$ is valid if and only if $T(S)$ has an extension by same reasoning as in the proof of Theorem 5.1 in [11]. Since $\mathrm{I}_2 \subseteq \mathrm{N}_2$ and $\mathrm{Pol}(S) = \mathrm{N}_2$ hold, the proof of our proposition follows. $\square$

Gottlob [11] proved the $\Sigma_2\mathrm{P}$ membership of the extension problem using a constructive equivalence between default logic and autoepistemic logic, previously exhibited by Marek and Truszczyński [14], followed by a $\Sigma_2\mathrm{P}$-membership proof of the latter, which itself follows from a previous result of Niemelä [15]. A straightforward generalization of these results to $S$-default theories and the aforementioned propositions allow us to prove the following trichotomy theorem.

**Theorem 10.** *Let $S$ be a set of Boolean relations. If $S$ is $0$-valid or $1$-valid then the problem* EXTENSION$(S)$ *is decidable in polynomial time. Else if $S$ is Horn, dual Horn, bijunctive, or affine, then* EXTENSION$(S)$ *is* $\mathrm{NP}$*-complete. Otherwise* EXTENSION$(S)$ *is* $\Sigma_2\mathrm{P}$*-complete.*

Gottlob exhibited in [11] an intriguing relationship between the EXTENSION problem and the two other algorithmic problems observed in connection with default logic reasoning. In fact, the constructions used in the proofs for the EXTENSION problem can be reused for the CREDULOUS and SKEPTICAL problems, provided we make some minor changes. These changes can be carried over to our approach as well, as we see in the following theorems.

**Theorem 11.** *Let $S$ be a set of Boolean relations. If $S$ is $0$-valid or $1$-valid then the problem* CREDULOUS$(S)$ *is decidable in polynomial time. Else if $S$ is Horn, dual Horn, bijunctive, or affine, then* CREDULOUS$(S)$ *is* $\mathrm{NP}$*-complete. Otherwise the problem* CREDULOUS$(S)$ *is* $\Sigma_2\mathrm{P}$*-complete.*

*Proof.* The extension $E$ constructed in the proof of Proposition 5 is unique and testing whether a given $S$-formula $\varphi$ belongs to $E$ takes polynomial time. The nondeterministic polynomial-time algorithm from the proof of Proposition 6 can be extended by the additional polynomial-time step

4. Check whether $\varphi \in \mathrm{Th}(W \cup \{\gamma \mid \gamma \text{ consequence of } d \in D'\})$ holds.

to test whether a given $S$-formula $\varphi$ belongs to $E$. If $\mathrm{Pol}(S) \subseteq \mathrm{R}_2$ holds, it is sufficient to take the default theory $T = (W, D)$ with the axiom $W = \{\varphi(x_1, \ldots, x_n)\}$ and the defaults

$$D = \left\{ \frac{\top : \mathsf{M} x_i}{x_i},\ \frac{\top : \mathsf{M} \neg x_i}{\neg x_i} \;\middle|\; i = 1, \ldots, n \right\}.$$

Note that the possible truth value assignments correspond to different extensions of the default theory $T$. Hence $\varphi$ belongs to an extension of $T$ if and only if there exists an extension of $T$. The same construction also works for $\mathrm{Pol}(S) \subseteq \mathrm{D}$ and $\mathrm{Pol}(S) \subseteq \mathrm{N}_2$, provided that we take the set of defaults $D = \{d_i^0, d_i^1 \mid i = 1, \ldots, n-1\}$ in the former and $D_1$ in the latter case. □

**Theorem 12.** *Let $S$ be a set of Boolean relations. If $S$ is $0$-valid or $1$-valid then the problem* SKEPTICAL$(S)$ *is decidable in polynomial time. Else if $S$ is Horn, dual Horn, bijunctive, or affine, then* SKEPTICAL$(S)$ *is* $\mathrm{coNP}$*-complete. Otherwise the problem* SKEPTICAL$(S)$ *is* $\Pi_2\mathrm{P}$*-complete.*

*Proof.* Skeptical reasoning is dual to the credulous one. For each credulous reasoning question whether an $S$-formula $\varphi(\boldsymbol{x}) = R_1(\boldsymbol{x}) \wedge \cdots \wedge R_k(\boldsymbol{x})$ belongs to an extension of a default theory $T(S) = (W, D)$, we associate the (dual) skeptical reasoning question whether the $\mathrm{dual}(S)$-formula $\varphi'(\boldsymbol{x}) = \mathrm{dual}(R_1)(\boldsymbol{x}) \wedge \cdots \wedge \mathrm{dual}(R_k)(\boldsymbol{x})$ belongs to *no* extension of the corresponding dual default theory $T(\mathrm{dual}(S)) = (W', D')$. Every $S$-formula in $W$ and $D$ is replaced by its corresponding $\mathrm{dual}(S)$-formula in $W'$ and $D'$. Note that the co-clones $\mathrm{Inv}(\mathrm{N}_2)$, $\mathrm{Inv}(\mathrm{L}_2)$, $\mathrm{Inv}(\mathrm{D}_2)$, $\mathrm{Inv}(\mathrm{D})$, and $\mathrm{Inv}(\mathrm{R}_2)$ are closed under duality, i.e., for each $X \in \{\mathrm{Inv}(\mathrm{N}_2), \mathrm{Inv}(\mathrm{L}_2), \mathrm{Inv}(\mathrm{D}_2), \mathrm{Inv}(\mathrm{D}), \mathrm{Inv}(\mathrm{R}_2)\}$ we have $X = \mathrm{dual}(X)$. Moreover we have the identities $\mathrm{dual}(\mathrm{Inv}(\mathrm{E}_2)) = \mathrm{Inv}(\mathrm{V}_2)$ and $\mathrm{dual}(\mathrm{Inv}(\mathrm{V}_2)) = \mathrm{Inv}(\mathrm{E}_2)$, what relates the co-clones of Horn and dual Horn relations. Using now Lemma 4, the result follows from Theorem 11. □

## 6 Concluding Remarks

We found a complete classification for reasoning in propositional default logic, observed for the three corresponding algorithmic problems, namely of the existence of an extension, the presence of a given formula in an extension, and the membership of a given formula in all extensions. To be able to take advantage of the algebraic proof methods, we generalized the propositional default logic formulas to conjunctive queries. This generalization is in the same spirit and it is done along the same guidelines as the one going from the satisfiability problem SAT for Boolean formulas in conjunctive normal form to the constraint satisfaction problem CSP on the Boolean domain. Gottlob [11], Kautz and Selman [13], Stillman [23,24], and Zhao with Ding [26] explored a large part of the complexity results for default logic reasoning. We completed the aforementioned results and found that only a trivial subclass of default theories have the three algorithmic problems decidable in polynomial time. The corresponding polymorphism clones are colored white in Figure 2. Another part of default theories (composed

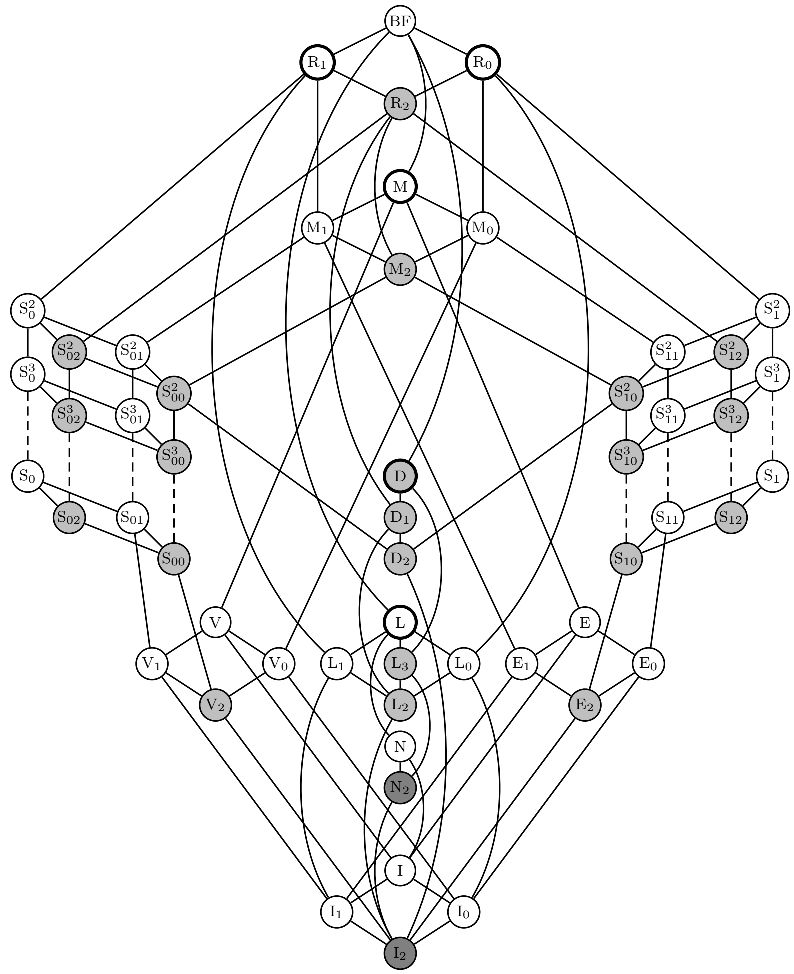


**Fig. 2.** Graph of all closed classes of Boolean functions

of Horn, dual Horn, bijunctive, or affine relations) have $\mathrm{NP}$-complete (resp. $\mathrm{coNP}$-complete) algorithmic problems, with the corresponding polymorphism clones colored light gray in Figure 2. Finally, for the default theories, based on complementive or on all relations, the algorithmic problems are $\Sigma_2\mathrm{P}$-complete (resp. $\Pi_2\mathrm{P}$-complete), with the corresponding polymorphism clones colored dark gray in Figure 2. This implies the existence of a trichotomy theorem for each of the studied algorithmic problems.

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