# Individual Discrete Logarithm in GF($p^k$)
## *(last step of the Number Field Sieve algorithm)*


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# Link with Logjam attack (N. Heninger's talk)

Solving actual practical problem:
Given a **fixed** finite field GF($q$),

Huge massive precomputation (weeks, months, years)

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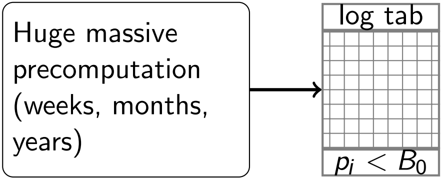

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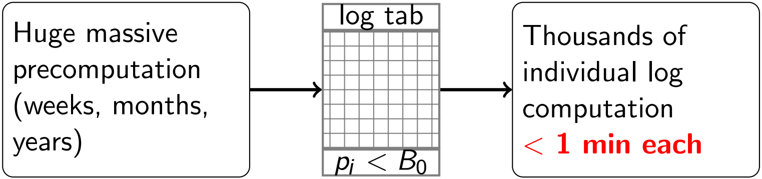

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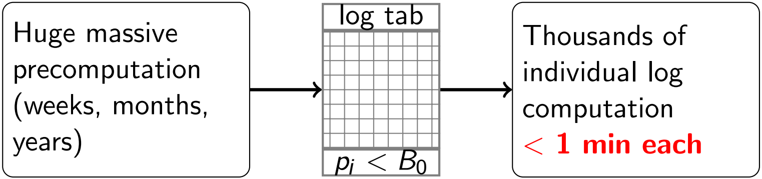


- Logjam: GF($q$) = GF($p$) (standardized) prime field

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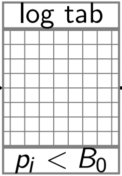


Thousands of individual log computation
**$<$ 1 min each**

- Logjam: GF($q$) $=$ GF($p$) (standardized) prime field
- Pairing-based cryptosystems: GF($q$) $=$ GF($p^2$), GF($p^6$), GF($p^{12}$)

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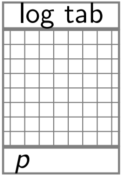

# DLP in the target group of pairing-friendly curves

# Why DLP in finite fields $\mathbb{F}_{p^2}$, $\mathbb{F}_{p^3}$;:::?

In a subgroup $\mathbb{G} = \langle g \rangle$ of order $\ell$,

- $(g, x) \mapsto g^x$ is easy (polynomial time)
- $(g, g^x) \mapsto x$ is (in well-chosen subgroup) hard: DLP.

**pairing:**

$$\begin{array}{ccccc} \mathbb{G}_1 & \times & \mathbb{G}_2 & \to & \mathbb{G}_T \\ \cap & & \cap & & \cap \\ E(\mathbb{F}_p) & & E(\mathbb{F}_{p^k}) & & \mathbb{F}_{p^k}^* \end{array}$$

- where $E/\mathbb{F}_p$ is a *pairing-friendly* curve
- $\mathbb{G}_1, \mathbb{G}_2, \mathbb{G}_T$ of large prime order $\ell$ (generic attacks in $O(\sqrt{\ell})$: take e.g. 256-bit $\ell$)
- $1 \leq k \leq 12$ embedding degree: very specific property (specific attacks (NFS): take 3072-bit $p^k$)

# DL records in small characteristic

✗ Small characteristic:

- supersingular curves $E/\mathbb{F}_{2^n}$: $\mathbb{G}_T \subset \mathbb{F}_{2^{4n}}$, $E/\mathbb{F}_{3^m}$: $\mathbb{G}_T \subset \mathbb{F}_{3^{6m}}$

Practical attacks (first one and most recent):

- Hayashi, Shimoyama, Shinohara, Takagi: GF($3^{6\cdot 97}$) ( 923 bit field) (2012)
- Granger, Kleinjung, Zumbragel: GF($2^{9234}$), GF($2^{4404}$) (2014)
- Adj, Menezes, Oliveira, Rodríguez-Henríquez: GF($3^{822}$), GF($3^{978}$) (2014)
- Joux: GF($3^{2395}$) (with Pierrot, 2014), GF($2^{6168}$) (2013)

Theoretical attacks:

- [Barbulescu Gaudry Joux Thomé 14] Quasi-Polynomial-time Algorithm (QPA)
- $\vdots$

# Common used pairing-friendly curves

✓ Curves over prime fields $E/\mathbb{F}_p$ where QPA does NOT apply
(with $\log p \geq \log \ell \approx 256$ bits, s.t. $k \log p \geq 3072$)

- supersingular: $\mathbb{G}_T \subset \mathbb{F}_{p^2}$ ($\log p = 1536$)
- [Miyaji Nakabayashi Takano 01] (MNT): $\mathbb{G}_T \subset \mathbb{F}_{p^3}$ ($\log p = 1024$), $\mathbb{F}_{p^4}$ ($\log p = 768$), $\mathbb{F}_{p^6}$ ($\log p = 512$)
- [Barreto Naehrig 05] (BN): $\mathbb{G}_T \subset \mathbb{F}_{p^{12}}$ ($\log p = 256$, optimal)
- [Kachisa Schaefer Scott 08] (KSS): $\mathbb{G}_T \subset \mathbb{F}_{p^{18}}$ (used for 192-bit security level: 384-bit $\ell$, $\log p = 512$, $k \log p = 9216$)

# Theoretical attacks in non-small characteristic fields

Variants of NFS, generic fields

- MNFS [Coppersmith 89]: $\mathbb{F}_p$, [Barbulescu Pierrot 14], [Pierrot 15]: $\mathbb{F}_{p^k}$

Specific to pairing target groups, when $p = P(x_0)$, with $\deg P \geq 2$

- [Joux Pierrot 13]
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These attacks were not taken into account in the 3072-bit target field recommendation.

# Last DL records, with the NFS-DL algorithm

| GF($p$) | GF($p'^2$), $p'^2 = q$ [BGGM15] | |
|---|---|---|
| **Massive precomputation** (d=core-day, y=core-year) | | |
| [Logjam] 512-bit $p$: 10y | 530-bit $q$: 0.2y + 1.25 GPU d | |
| [BGIJT14] 596-bit $p$: 131y | 598-bit $q$: 0.75y + 18 GPU-d | **175× faster** |
| **Individual Discrete Log** | | |
| 512-bit $p$: 70s median ✓ | 530-bit $q$ : few d | **slow** |
| 768-bit $p$: 2d | 600-bit $q$ : few d | **slow** |

[Logjam]: see weakdh.org
[BGGM15]: Barbulescu, Gaudry, G., Morain
[BGIJT14]: Bouvier, Gaudry, Imbert, Jeljeli, Thomé

This talk:

- Faster **individual** discrete logarithm in $\mathbb{F}_{p^k}$, especially $k = 2, 3, 4, 6$
- Apply to pairing target group $\mathbb{G}_T$
- source code: part of `http://cado-nfs.gforge.inria.fr/`

# NFS – Number Field Sieve algorithm

# Number Field Sieve algorithm for DL in $\mathbb{F}_{p^k}$

1. *Polynomial selection:*
   compute $f(x)$, $g(x)$ with
   $\varphi = \gcd(f, g) \pmod p$ and
   $\mathbb{F}_{p^k} = \mathbb{F}_p[x]/(\varphi(x))$

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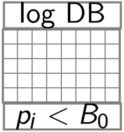
2. *Relation collection*
3. *Linear algebra modulo $\ell \mid p^k - 1$.*

➙ here we know the discrete log of a subset of elements.



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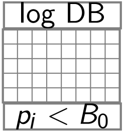

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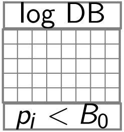


1. *Individual target discrete logarithm*

# Number Field Sieve algorithm for DL in $\mathbb{F}_{p^k}$

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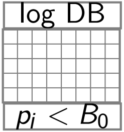


1. **Individual target discrete logarithm for each given DLP instance**

- not so trivial
- this talk: pratical improvements very efficient for small $k$

# Example: [MNT01] parameters (explicitly advised to NOT use them)

Polynomial selection: Conjugation method [BGGM15]

- $k = 3,\ p = 12y_0^2 + 1,\ t = -6y_0 - 1,\ \ell \mid p + 1 - t = 12y_0^2 + 6y_0 + 2$, with $y_0 = -870230335309004989831690 2$
- $f = 12x^6 - 24x^5 - 85x^4 + 70x^3 + 215x^2 + 96x + 12$
- $\varphi_y = g = x^3 - yx^2 - (y+3)x - 1$, where $y = y_0 + 1$ ($\varphi_{y_0}$ not irr.) $= x^3 + 8702303353090049898316901x^2 + 8702303353090049898316898x - 1$
- $f \pmod{p} = 12\varphi_y\varphi_{-y} = \mathrm{Res}_y(\varphi_y, 12y^2 + 1)$

$$G = X + 6 \in \mathbb{F}_{p^3}^* = \mathbb{F}_p[X]/(\varphi(X))$$

randomized target $T = t_0 + t_1X + t_2X^2 \in \mathbb{F}_{p^3}^*$

# Preimage in $\mathbb{Z}[x]$=$(f(x))$ and map

randomized target $T = t_0 + t_1X + t_2X^2 \in \mathbb{F}_{p^3}^* = \mathbb{F}_p[X]/(\varphi(X))$
Most simple preimage $\mathbf{T}$ choice:
$\mathbf{T} = \mathbf{t_0} + \mathbf{t_1}x + \mathbf{t_2}x^2 \in \mathbb{Z}[x]/(f(x))$, with $\mathbf{t_i} \equiv t_i \pmod p$.
We can always choose $\mathbf{T}$ s.t.

- $|\mathbf{t_i}| < p$
- $\deg \mathbf{T} < \deg f$

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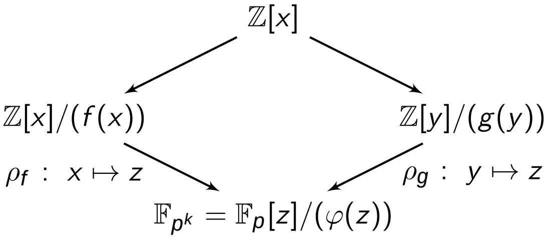
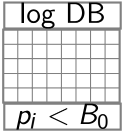
We need $\rho(\mathbf{T}) = T$ (where $\rho$ is simply a reduction modulo $(\varphi, p)$) when $f$ (resp. $g$) is monic



# Individual DL of random target $T_0 \in \mathbb{F}^*_{p^k}$

Given $G$ and a log database s.t. for all $p_i < B$, $\log p_i \in$



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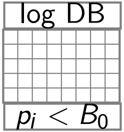


Given $G$ and a log database s.t. for all $p_i < B$, $\log p_i \in$

1. booting step (a.k.a. smoothing step): **DO**
   1.1 take $t$ at random in $\{1, \ldots, \ell - 1\}$ and set $T = G^t T_0$ (hence $\log_G(T_0) = \log_G(T) - t$)
   1.2 factorize $\mathrm{Norm}(\mathbf{T}) = \underbrace{q_1 \cdots q_i}_{\text{too large: } B_0 < q_i \leq B_1} \times$(elements in DL database),

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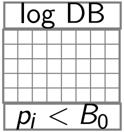


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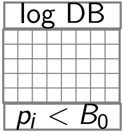
Given $G$ and a log database s.t. for all $p_i < B$, $\log p_i \in$ 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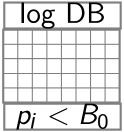
Given $G$ and a log database s.t. for all $p_i < B$, $\log p_i \in$ 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   - 1.2 factorize $\underbrace{\text{Norm}(\mathbf{T})}_{\text{reduce this}} = \underbrace{q_1 \cdots q_i}_{\text{too large: } B_0 < q_i \leq B_1} \times$(elements in DL database),

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# Booting Step

# Norm computation

$f$ monic,

$\mathbf{T} = t_0 + t_1 x + \ldots + t_d x^d \in \mathbb{Z}[x]/(f(x))$, $d < \deg f$:

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A||\mathbf{T}||_\infty^{\deg f} ||f||_\infty^d$$

with $||f||_\infty = \max_{1 \leq i \leq \deg f} |f_i|$

Example: [MNT01], $k = 3$, $\deg g = 3$, $||g||_\infty = O(p^{1/2})$

$$\begin{aligned} p &= 908761003790427908077548955758380356675829026531247 \\ \mathbf{T} &= 314159265358979323846264338327950288419716939937510+ \\ & 582097494459230781640628620899862803482534211706798x+ \\ & 214808651328230664709384460955058223172535940812829x^2 \\ f &= 12x^6 - 24x^5 - 85x^4 + 70x^3 + 215x^2 + 96x + 12 \\ g &= x^3 + 8702303353090049898316901x^2 + 8702303353090049898316898x - 1 \end{aligned}$$

$\mathrm{Norm}_f(\mathbf{T})(\approx ||\mathbf{T}||_\infty^6 ||f||_\infty^2) =$ **1017**bits $\sim p^6$

$\mathrm{Norm}_g(\mathbf{T})(\approx ||\mathbf{T}||_\infty^3 ||g||_\infty^2) =$ **665**bits $\sim p^4$

# Booting step complexity

Given random target $T_0 \in \mathbb{F}_{p^k}^*$, and $G$ a generator of $\mathbb{F}_{p^k}^*$
**repeat**

1. take $t$ at random in $\{1, \ldots, \ell - 1\}$ and set $T = g^t T_0$
2. factorize $\mathrm{Norm}(\mathbf{T})$

**until** it is $B_1$-smooth: $\mathrm{Norm}(\mathbf{T}) = \prod_{q_i \leq B_1} q_i \prod_{p_i \leq B_0} p_i$

$L$-notation: $Q = p^k$, $L_Q[1/3, \mathbf{c}] = e^{(\mathbf{c}+o(1))(\log Q)^{1/3} (\log\log Q)^{2/3}}$ for $\mathbf{c} > 0$.
Norm factorization done with ECM method, in time $L_{B_1}[1/2, \sqrt{2}]$

**Lemma (Booting step running-time)**

*if* $\mathrm{Norm}(\mathbf{T}) \leq Q^e$, *take* $B_1 = L_Q[2/3, (e^2/3)^{1/3}]$, *then the running-time is* $L_Q[1/3, (3e)^{1/3}]$ *(and this is optimal).*

# Booting step complexity

- $\mathbb{F}_p$: Norm(preimage) $\leq p = Q$, running-time: $L_Q[1/3, \mathbf{1.44}]$ with $B_1 = L_Q[\mathbf{2/3}, 0.69]$ [Commeine Semaev 06, Barbulescu 13]
- med. char. $\mathbb{F}_{p^k}$, JLSV1 poly. select.: $\deg f = \deg g = k$, $||f||_\infty = ||g||_\infty = O(p^{1/2})$, Norm(preimage) $\leq Q^{3/2}$, running-time: $L_Q[1/3, \mathbf{1.65}]$, with $B_1 = L_Q[\mathbf{2/3}, 0.91]$ [Joux Lercier Naccache Thomé 09, Barbulescu Pierrot 14]

| field | $\mathbb{F}_p$ | $\mathbb{F}_{p^k}$ | | |
|---|---|---|---|---|
| polynomial selec. | | gJL | JLSV$_1$ | Conj |
| NFS dominating, $c$ | 1.92 | 1.92 | 2.42 | 2.20 |
| $L_Q[\frac{1}{3}, c]$, 512-bit $Q$ | $2^{64}$ | $2^{64}$ | $2^{81}$ | $2^{73}$ |
| Norm($\mathbf{T}$) $< Q^e =$ | $Q$ | $Q$ | $Q^{3/2}$ | $Q$ |
| time $L_Q[1/3, c]$, c | 1.44 | 1.44 | 1.65 | 1.44 |
| nb of operations, 512-bit $Q$ | $2^{48}$ | $2^{48}$ | $2^{55}$ | $2^{48}$ |
| $q_i$ bound $B_1$ | $2^{90}$ | $2^{90}$ | $2^{118}$ | $2^{90}$ |

# Optimizing the Preimage Computation

# Preimage optimization

$f$, $\deg f$, $||f||_\infty$, $g$, $\deg g$, $||g||_\infty$ are given by the polynomial selection step (NFS-DL step 1)

To reduce the norm,

- reduce $||\mathbf{T}||_\infty$
- and/or reduce $d = \deg \mathbf{T}$

# Previous work

- $\mathbb{F}_p$: Rational Reconstruction. $T \in \mathbb{Z}/p\mathbb{Z}$, $\mathbf{T}$ is an integer $< p$. Rational Reconstruction gives $\mathbf{T} = u/v \pmod p$ with $u, v < \sqrt{p}$
  - booting step: we want $u$,$v$ to be $B_1$-smooth at the same time, instead of $\mathbf{T}$ to be $B_1$-smooth. $\mathbf{T}$ is already split in two integers of half size each.
- [Blake Mullin Vanstone 84] Waterloo algorithm in $\mathbb{F}_2[x]$: $\mathbf{T} = U/V = \frac{u_0 + \ldots + u_{bd/2c} x^{bd/2c}}{v_0 + \ldots + v_{bd/2c} x^{bd/2c}}$ **reduce degree**
- [Joux Lercier Smart Vercauteren 06] in $\mathbb{F}_{p^k}$: $\mathbf{T} = U/V = \frac{u_0 + \ldots + u_d x^d}{v_0 + \ldots + v_d x^d}$, where $|u_i|, |v_i| \sim p^{1/2}$ **reduce coefficient size**

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*How much is the booting step improved?*

# Booting step: First experiments

Commonly assumed: launch at morning coffee ... finished for afternoon tea.

- $\mathbb{F}_{p^2}$ 600 bits was easy (BGGM15 record), as fast as for $\mathbb{F}_{p^0}$ ($<$ one day)
- $\mathbb{F}_{p^3}$ 400 bits and MNT 508 bits were much slower (days, week)
- $\mathbb{F}_{p^4}$ 400 bits was even worse ($>$ one week)

What happened?

- $\mathbb{F}_{p^3}$: $||\mathbf{T}||_\infty = p$, $\deg f = 6$, [JLSV06] method: $\mathrm{Norm}(\mathbf{T}) \leq Q \rightarrow c = 1.44$ (but still much slower)
- $\mathbb{F}_{p^4}$: $||f||_\infty = O(p^{1/2})$, $\mathrm{Norm}(\mathbf{T}) \leq Q^{3/2} \rightarrow c = 1.65$

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Because of the constant hidden in the $O()$?

# Our solution

**Lemma**

*Let* $T \in \mathbb{F}_{p^k}$.
$\log(T) = \log(u \cdot T) \pmod{\ell}$ *for any* $u$ *in a proper subfield of* $\mathbb{F}_{p^k}$.

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- $\mathbb{F}_p$ is a proper subfield of $\mathbb{F}_{p^k}$
- target $T = t_0 + t_1 x + \ldots + t_d x^d$
- we divide the target by its leading term:

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Similar technique in pairing computation: Miller loop denominator elimination [Boneh Kim Lynn Scott 02]

# Subfield Simplification + LLL

We want to reduce $||\mathbf{T}||_\infty$. Example with $\mathbb{F}_{p^3}$:

- $f = x^6 + 19x^5 + 90x^4 + 95x^3 + 10x^2 - 13x + 1$
- $\varphi = x^3 - yx^2 - (y+3)x - 1 \; y \in \mathbb{Z}$
- $\mathbf{T} = t_0 + t_1 x + x^2$
- define $L = \begin{bmatrix} p & 0 & 0 & 0 & 0 & 0 \\ 0 & p & 0 & 0 & 0 & 0 \\ t_0 & t_1 & 1 & 0 & 0 & 0 \\ \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 & 0 \\ 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 & 0 \\ 0 & 0 & \varphi_0 & \varphi_1 & \varphi_2 & 1 \end{bmatrix}$
- LLL$(L)$ outputs a short vector $r$, linear combination of $L$'s rows.
  $r = \lambda_0 p + \lambda_1 px + \lambda_2 T + \lambda_3 \varphi + \lambda_4 x\varphi + \lambda_5 x^2 \varphi$.
  $r = r_0 + \ldots + r_5 x^5, \; ||r_i||_\infty \leq C \det(L)^{1/6} = O(p^{1/3})$

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- $\log \rho(r) = \log(T) \pmod{\ell}$ because $\rho(r) = \lambda_2 T$ with $\lambda_2 \in \mathbb{F}_p$

# Subfield Simplification + LLL

$$\mathrm{Norm}_f(\mathbf{T}) = Res(f, \mathbf{T}) \leq A||\mathbf{T}||_\infty^{\deg f} ||f||_\infty^d$$

- $\mathrm{Norm}_f(r) \leq ||r||_\infty^6 ||f||_\infty^5 = O(p^2) = O(Q^{2/3}) < O(Q)$

MNT example: $\log Q = 508$ bits

| | $\mathrm{Norm}_f(\mathbf{T})$ | | $\mathrm{Norm}_g(\mathbf{T})$ | | $L_Q[1/3, c]$ | | $q_i \leq B_1 =$ |
|---|---|---|---|---|---|---|---|
| | $Q^e$ | bits | $Q^e$ | bits | $c$ | time | $L_Q[\frac{2}{3}, c]$ |
| Nothing | $Q^2$ | 1010 | $Q^{4/3}$ | 667 | 1.58 | $2^{53}$ | $2^{109}$ |
| [JLSV06] | $Q$ | 508 | $Q^{5/3}$ | 847 | 1.44 | $2^{48}$ | $2^{90}$ |
| **This work** | $\mathbf{Q^{2/3}}$ | **340** | **Q** | **508** | **1.26** | $\mathbf{2^{42}}$ | $\mathbf{2^{69}}$ |

# $\mathbb{F}_{p^4}$: JLSV$_1$ polynomial selection and booting step improvement

# $\mathbb{F}_{p^4}$ of 400 bits

[JLSV06] first method: choose $f$ of degree 4 and very small coefficients, and set $g = f + p$. Booting step on $f$ side, with the $\mathbf{T} = U/V$ method.

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Relation collection and Linear algebra do not scale well for large $p$

We use JLSV06 other method: $\deg f = \deg g = k$, $||f||_\infty = ||g||_\infty = p^{1/2}$

$$
\begin{aligned}
p &= 314159265358979323846270891033 \text{ of 98 bits (30 dd)} \\
\ell &= 986960440108935861883490271847705742814406423277877598070 9 \text{ of 192 bits} \\
f &= x^4 - 560499121640472x^3 - 6x^2 + 560499121640472x + 1 \\
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&\quad -4898685125033473x + 560499121639105 \\
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\end{aligned}
$$

Terribly slow booting step (more than one week)

# Terribly slow booting step

- $T = t_0 + t_1 x + t_2 x^2 + x^3$
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$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ t_0 & t_1 & t_2 & 1 \end{bmatrix}$$
- dim 4 because $\max(\deg f, \deg g) = 4$
- compute LLL($L$), get $r$, $||r||_\infty \approx p^{3/4}$,
  $\mathrm{Norm}_f(r) \approx ||r||_\infty^4 ||f||_\infty^3 \approx p^{9/2} = Q^{9/8}$ of 450 bits!
- Booting step, nb of operations: $2^{44}$
- Large prime bound $B_1$ of 82 bits

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$$L = \begin{bmatrix} p & 0 & 0 & 0 \\ 0 & p & 0 & 0 \\ 0 & 0 & p & 0 \\ t_0 & t_1 & t_2 & 1 \end{bmatrix} \leftarrow \text{could we find something else, } \textit{monic}?$$
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# Our solution: quadratic subfield simplification

**Lemma**

*Let $T \in \mathbb{F}_{p^k}$, $k$ even. We can always find $u \in \mathbb{F}_{p^{k/2}}$ and $T' \in \mathbb{F}_{p^k}$, such that $T' = u \cdot T$ and $T'$ is of degree $k-2$ instead of $k-1$.*

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- $\log \rho(r) = \log(T) \pmod{\ell}$

$\mathrm{Norm}_f(r) = ||r||_\infty^4 ||f||_\infty^3 = p^{7/2} = Q^{7/8} < Q$

# Summary of results

| $\mathbb{G}_T \subset$ | $\mathbb{F}_{p^2}$ | $\mathbb{F}_{p^3}$ | $\mathbb{F}_{p^4}$ | $\mathbb{F}_{p^6}$ |
|---|---|---|---|---|
| Norm bound | | | | |
| prev. | $Q$ [JLSV06] | | $Q^{3/2}$ (nothing) | |
| **this work** | $Q^{1/2}$ | $Q^{2/3}$ | $Q^{7/8}$ | $Q^{11/12}$ |
| Booting step running time in $L_Q[1/3, c]$ | | | | |
| prev. $c$ (*) | 1.44 | | 1.65 | |
| new $c$ | **1.14** | **1.26** | **1.38** | **1.40****** |
| numerical values for a 512-bit $Q$ | | | | |
| prev. nb of operations | $2^{48}$ | | $2^{55}$ | |
| **new nb of operations** | $\mathbf{2^{38}}$ | $\mathbf{2^{42}}$ | $\mathbf{2^{46}}$ | $\mathbf{2^{47}}$ |
| $q_i$ bound $B_1 = L_Q[2/3, c']$ | | | | |
| previous $B_1$ | $2^{90}$ | | $2^{118}$ | |
| **new $\mathbf{B_1}$** | $\mathbf{2^{57}}$ | $\mathbf{2^{69}}$ | $\mathbf{2^{83}}$ | $\mathbf{2^{85}}$ |

* [CommeineSemaev06, JouxLercierNaccacheThomé09, Barbulescu13, Bar.Pierrot14]

** with cubic subfield simplification

# Summary of results

- Accepted paper at Asiacrypt 2015, Auckland, New Zealand
- online version HAL 01157378
- `guillevic@lix.polytechnique.fr`

# DL record computation in $\mathbb{F}_{p^4}$ of 392 bits (120dd)

**Joint work with R. Barbulescu, P. Gaudry, F. Morain**

$$
\begin{aligned}
p &= 314159265358979323846270891033 \text{ of 98 bits (30 dd)} \\
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&\quad -4898685125033473x + 560499121639105 \\
\varphi &= g \\
G &= x + 3 \in \mathbb{F}_{p^4} \\
T_0 &= 31415926535897x^3 + 93238462643383x^2 + 27950288419716x + 93993751058209
\end{aligned}
$$

$$\mathbf{\log_G(T_0) =}$$

$$\mathbf{1364394725868398385294409072195832018219505919841942570 22 \pmod{\ell}}$$