On the Interplay Between Theory and Practice in Small Characteristic DLPs

Robert Granger

Based on joint work with Faruk Göloğlu, Gary McGuire & Jens Zumbrägel, and Thorsten Kleinjung & Jens Zumbrägel

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Conclusions

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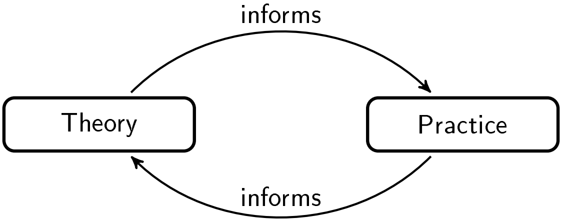

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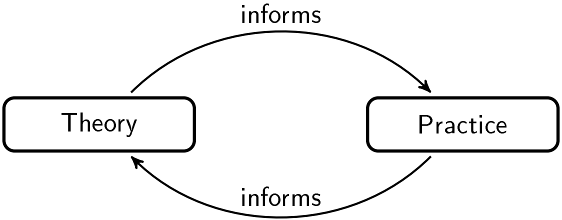

An Obvious Counterpoint

In contrast to the experimental sciences, in mathematics one can irrefutably prove things!

Overview

Background and Degree 2 Elimination

Case Study: Computing DLPs in $\mathbb{F}_{2^{4404}}$

The GKZ Quasi-Polynomial Algorithm

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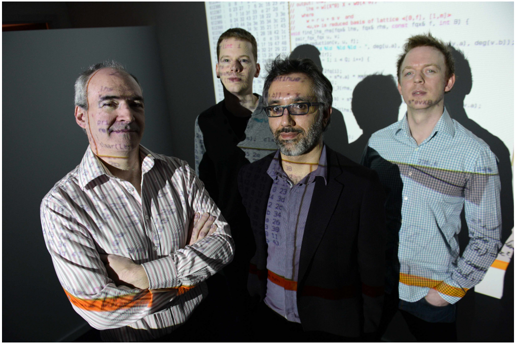
The GGMZ approach

'On the Function Field Sieve and the Impact of Higher Splitting Probabilities: Application to Discrete Logarithms in $\mathbb{F}_{2^{1971}}$ and $\mathbb{F}_{2^{3164}}$'



Faruk Göloğlu, G., Gary McGuire & Jens Zumbrägel

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CASL

The GGMZ approach

Let the target field be $\mathbb{F}_{q^{kn}}$ with $k \geq 1$ small and fixed and $n = O(q)$.

- Assume there exists $h_1, h_0 \in \mathbb{F}_{q^k}[X]$ of low degree d_h s.t.

$$h_1(X^q)X - h_0(X^q) \equiv 0 \pmod{f} \tag{1}$$

  where f is irreducible and of degree n
- Let x be a root of f so that $\mathbb{F}_{q^{kn}} = \mathbb{F}_{q^k}(x)$ and let $y = x^q$. Then by (1) we have $x = h_0(y)/h_1(y)$ and $\mathbb{F}_{q^k}(x) \cong \mathbb{F}_{q^k}(y)$
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A Basic Identity

For all $a, b, c \in \mathbb{F}_{q^k}$ we have the following equality in $\mathbb{F}_{q^{kn}}$:

$$x^{q+1} + ax^q + bx + c = \frac{1}{h_1(y)}\left(yh_0(y) + ayh_1(y) + bh_0(y) + ch_1(y)\right)$$

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- If both sides split over $\mathbb{F}_{q^k}$ then we have a relation

Bluher polynomials

Let $k \geq 3$ and consider the polynomial $X^{q+1} + aX^q + bX + c$.

If $ab \neq c$ and $a^q \neq b$, this may be transformed into

$$F_B(\overline{X}) = \overline{X}^{q+1} + B\overline{X} + B\,, \quad \text{with} \quad B = \frac{(b-a^q)^{q+1}}{(c-ab)^q}\,,$$

via $X = \frac{c-ab}{b-a^q}\,\overline{X} - a$.

Theorem (*Bluher '02*)

The number of elements $B \in \mathbb{F}_{q^k}^{\times}$ s.t. the polynomial $F_B(\overline{X}) \in \mathbb{F}_{q^k}[\overline{X}]$ splits completely over $\mathbb{F}_{q^k}$ equals

$$\frac{q^{k-1}-1}{q^2-1} \quad \textit{if } k \textit{ is odd}\,, \qquad \frac{q^{k-1}-q}{q^2-1} \quad \textit{if } k \textit{ is even}\,.$$

Degree 1 relation generation: $k \geq 3$

- Compute $\mathcal{B} = \{B \in \mathbb{F}_{q^k}^{\times} \mid X^{q+1} + BX + B \text{ splits over } \mathbb{F}_{q^k}\}$
- Since $B = (b - a^q)^{q+1}/(c - ab)^q$, for any $a, b \in \mathbb{F}_{q^k}$ s.t. $b \neq a^q$, and $B \in \mathcal{B}$, there exists a unique $c \in \mathbb{F}_{q^k}$ s.t. $x^{q+1} + ax^q + bx + c$ splits over $\mathbb{F}_{q^k}$
- For each such (a, b, c), test if $yh_0(y) + ayh_1(y) + bh_0(y) + ch_1(y)$ splits; if so then have a relation
- If $q^{3k-3} > q^k(d_h + 1)!$ then for $d_h \geq 1$ constant we expect to compute logs of degree 1 elements of $\mathbb{F}_{q^{kn}}$ in time

$$O(q^{2k+1})$$

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For the base field $\mathbb{F}_{q^2}$, relevant set of triples is

$$\{(a, a^q, c) \mid a \in \mathbb{F}_{q^2} \text{ and } c \in \mathbb{F}_q, c \neq a^{q+1}\}.$$

On the fly degree 2 elimination

For $Q(x) = x^2 + q_1 x + q_0$ let $\bar{Q}(y) = Q(x)^q = y^2 + q_1^q y + q_0^q \in \mathbb{F}_{q^{kn}}$ be an element to be eliminated, i.e., written as a product of linear elements.

- For any univariate polynomials w_0, w_1 we have

$$w_0(x^q)\,x + w_1(x^q) = \frac{1}{h_1(y)}\left(w_0(y)\,h_0(y) + w_1(y)\,h_1(y)\right)$$

- Compute a reduced basis of the lattice

$$L_{\bar{Q}} = \{(w_0(Y), w_1(Y)) \in \mathbb{F}_{q^k}[Y]^2 : w_0(Y)\,h_0(Y) + w_1(Y)\,h_1(Y) \equiv 0 \pmod{\bar{Q}(Y)}\}$$

- In general we have $(u_0, Y + u_1), (Y + v_0, v_1)$, with $u_i, v_i \in \mathbb{F}_{q^k}$, and for $s \in \mathbb{F}_{q^k}$ we have $(Y + v_0 + su_0, sY + v_1 + su_1) \in L_{\bar{Q}}$
- r.h.s. $(y + v_0 + su_0)\,h_0(y) + (sy + v_1 + su_1)\,h_1(y)$ has degree $d_h + 1$, so cofactor splits with probability $\approx 1/(d_h - 1)!$
- l.h.s. is $(x^q + v_0 + su_0)x + (sx^q + v_1 + su_1)$ which is of the form

$$x^{q+1} + ax^q + bx + c$$

On the fly degree 2 elimination

Consider the l.h.s. $x^{q+1} + sx^q + (v_0 + su_0)x + (v_1 + su_1)$.

- Recall $\mathcal{B} = \{B \in \mathbb{F}_{q^k}^{\times} \mid X^{q+1} + BX + B$ splits over $\mathbb{F}_{q^k}\}$
- For each $B \in \mathcal{B}$ we try to solve $B = (b - a^q)^{q+1}/(c - ab)^q$ for s, i.e., find $s \in \mathbb{F}_{q^k}$ that satisfies

$$B = \frac{(-s^q + u_0 s + v_0)^{q+1}}{(-u_0 s^2 + (u_1 - v_0)s + v_1)^q}$$

  by taking GCD with $s^{q^k} - s$: Cost is $O(q^2 \log q^k)$ $\mathbb{F}_{q^k}$-ops

- Expected probability of success is $\approx 1 - \left(1 - \frac{1}{(d_h-1)!}\right)^{q^{k-3}}$
- Hence need $q^{k-3} > (d_h - 1)!$ to eliminate $\bar{Q}(y)$ with good probability: Expected cost is

$$O(q^2 (d_h - 1)! \log q^k) \ \ \mathbb{F}_{q^k}\text{-ops}$$

Alternative solution finding

We need to compute $s \in \mathbb{F}_{q^k}$ that satisfy the equation:

$$B = \frac{(-s^q + u_0 s + v_0)^{q+1}}{(-u_0 s^2 + (u_1 - v_0)s + v_1)^q}$$

- Use an explicit $\mathbb{F}_{q^k}/\mathbb{F}_q$ basis $\{1, \alpha, \ldots, \alpha^{k-1}\}$, and introduce $\mathbb{F}_q$-variables $s_0, \ldots, s_{k-1}$ s.t. $s = s_0 + s_1\alpha + \cdots + s_{k-1}\alpha^{k-1}$
- Gives a quadratic system, solvable in $O((k\binom{2k}{k+1})^\omega)$ $\mathbb{F}_q$-ops
- For fixed k, d_h and $q \to \infty$ this method has cost $O(1)$ $\mathbb{F}_q$-ops, i.e., it has polylogarithmic complexity

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Computing DLPs in $\mathbb{F}_{2^{4404}}$

On 30/1/14 we (GKZ) announced the solution of a DLP in the Jacobian of $H_0/\mathbb{F}_2 : Y^2 + Y = X^5 + X^3$ over $\mathbb{F}_{2^{367}}$, which has a subgroup of prime order $r = (2^{734} + 2^{551} + 2^{367} + 2^{184} + 1)/(13 \cdot 7170258097)$ and embedding degree 12.

- $\mathbb{F}_{2^{12}} = \mathbb{F}_2[U]/(U^{12} + U^3 + 1) = \mathbb{F}_2(u)$
- $\mathbb{F}_{2^{367}} = \mathbb{F}_2[X]/(I(X)) = \mathbb{F}_2(x)$ where $I(X)$ the irreducible degree 367 divisor of $h_1(X^{64})X - h_0(X^{64})$, with

$$h_1 = X^5 + X^3 + X + 1,\ \ h_0 = X^6 + X^4 + X^2 + X + 1$$

- $\mathbb{F}_{2^{12 \cdot 367}}$ is then the compositum of $\mathbb{F}_{2^{12}}$ and $\mathbb{F}_{2^{367}}$

For small degree elimination, represent $\mathbb{F}_{2^{12}}$ as $\mathbb{F}_{q^2}$ with $q = 2^6$, $k = 2$:

- $\mathbb{F}_{2^6} = \mathbb{F}_2[U]/(T^6 + T + 1) = \mathbb{F}_2(t)$
- $\mathbb{F}_{2^{12}} = \mathbb{F}_{2^6}[V]/(V^2 + tV + 1) = \mathbb{F}_{2^6}(v)$

Factor base logs and initial descent

To have enough relations for degree one elements of $\mathbb{F}_{2^{4404}}/\mathbb{F}_{2^{12}}$ we would need $q^{2k-3} > (6+1)!$. So we used relations in $\mathbb{F}_{2^{8808}}/\mathbb{F}_{2^{24}}$:

- $\mathbb{F}_{2^{24}} = \mathbb{F}_{2^6}[W]/(W^4 + W^3 + W^2 + t^3) = \mathbb{F}_{2^6}(w)$

$\mathrm{Gal}(\mathbb{F}_{2^{24}}/\mathbb{F}_2)$ acts on the degree 1 factor base $\{x + a \mid a \in \mathbb{F}_{2^{24}}\}$:

$$(x + a)^{2^{367}} = x + a^{2^{367}} = x + a^{2^7}$$

$\Longrightarrow$ factor base has $699,252$ elements and linear system was solved in 4896 core hours on a 24 core cluster.

Initial descent: We performed a continued fraction initial split, then degree-balanced classical descent to degrees ≤ 8 in 38224 core hours.

Eliminating small degree elements over $\mathbb{F}_{2^{12}}$

We used Joux's small degree elimination, our degree 2 elimination and one other idea.

Joux's method: For $Q \in \mathbb{F}_{q^2}[X]$ of degree $D > 2$ let F, G have degree $< D$. Consider

$$G(X) \cdot \prod_{\alpha \in \mathbb{F}_q} (F(X) - \alpha G(X)) = F(X)^q G(X) - F(X) G(X)^q$$

- $F^{(q)}(y), G((h_0/h_1)(y)), F((h_0/h_1)(y)), G^{(q)}(y)$ have small degree
- Insisting r.h.s. $\equiv 0 \pmod{\bar{Q}(y)}$ results in bilinear quadratic system
- For solutions check if the cofactor is $(D-1)$-smooth

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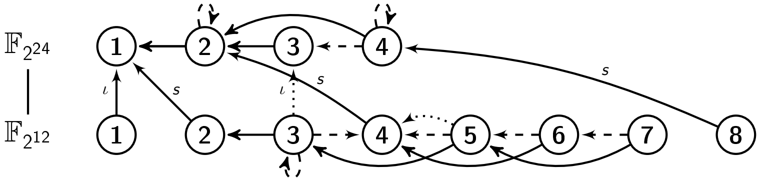

Degree 2 elimination over $\mathbb{F}_{2^{24}}$

Let $\bar{Q}(y) \in \mathbb{F}_{2^{24 \cdot 367}}$ be an element to be eliminated.

- As before we have $y = x^{64}$ and $x = h_0(y)/h_1(y)$, and for any univariate polynomials w_0, w_1 we have

$$w_0(x^q)\,x + w_1(x^q) = \frac{1}{h_1(y)}(w_0(y)\,h_0(y) + w_1(y)\,h_1(y))$$

- A reduced basis for the lattice $L_{\bar{Q}}$ is $(u_0, Y + u_1), (Y + v_0, v_1)$, with $u_i, v_i \in \mathbb{F}_{2^{24}}$. For $s \in \mathbb{F}_{2^{24}}$, $(Y + v_0 + su_0, sY + v_1 + su_1) \in L_{\bar{Q}}$
- r.h.s. $\frac{1}{h_1(y)}((y + v_0 + su_0)\,h_0(y) + (sy + v_1 + su_1)\,h_1(y))$ has degree $d_h + 1 = 7$, so cofactor splits with probability $\approx 1/5!$
- l.h.s. is $x^{q+1} + sx^q + (u_{00} + sv_{00})x + (u_{10} + sv_{10})$, which splits if

$$B = \frac{(s^{64} + u_0 s + v_0)^{65}}{(u_0 s^2 + (u_1 + v_0)s + v_1)^{64}}$$

- Probability of success is $\approx 1 - (1 - 1/5!)^{64} \approx 0.415$, but amplified to near certainty using recursive techniques

New 'traps' in the descent

During the descent, we encountered several polynomials $\bar{Q}(Y)$ that were not eliminable via Joux's method.

- All were factors of $h_1(Y) \cdot c + h_0(Y)$ for $c \in \mathbb{F}_{2^{12}}$ or $\mathbb{F}_{2^{24}}$ and hence $h_0(Y)/h_1(Y) \equiv c \pmod{\bar{Q}(Y)}$
- $\implies$ r.h.s. equals $F^{(q)}(Y)G(c) + F(c)G^{(q)}(Y) \pmod{\bar{Q}(Y)}$
- This can't be zero mod $\bar{Q}(Y)$ if the degrees of F and G are smaller than the degree of $\bar{Q}$, unless F and G are both constants
- However, writing $h_1(Y) \cdot c + h_0(Y) = \bar{Q}(Y) \cdot R(Y)$ we have $\bar{Q}(Y) = h_1(Y) \cdot ((h_0/h_1)(Y) + c)/R(Y) = h_1(Y) \cdot (X + c)/R(Y)$
- Hence $\log(\bar{Q}(y)) \equiv \log(x + c) - \log(R(y))$, since $\log(h_1(y)) \equiv 0$
- In all the cases we encountered, the log of $R(y)$ was solvable
- Note that these traps are different to those identified by Cheng, Wan and Zhuang, which are factors of $h_1(X^q)X - h_0(X^q)$ (or of $h_1(X)X^q - h_0(X)$ if using Joux's representation)

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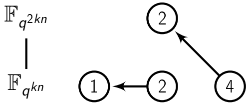

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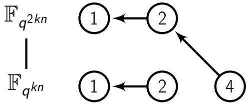

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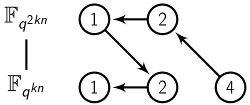

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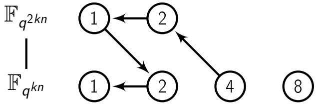

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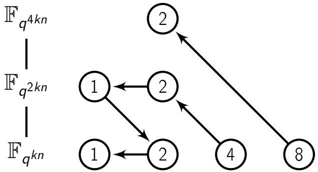

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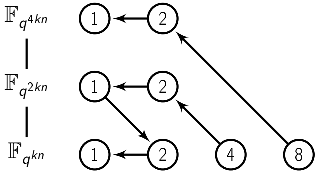

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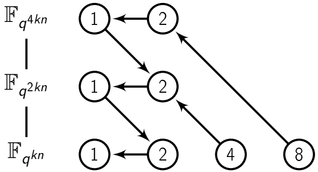

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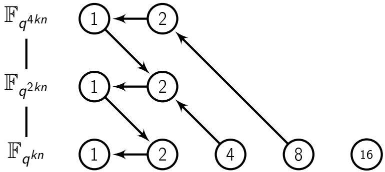

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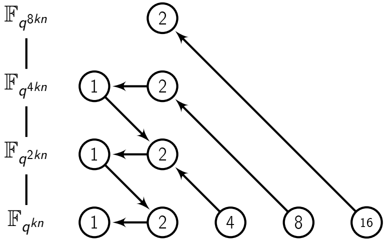

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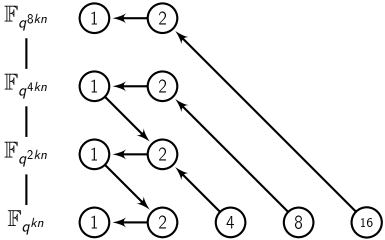
The GKZ QPA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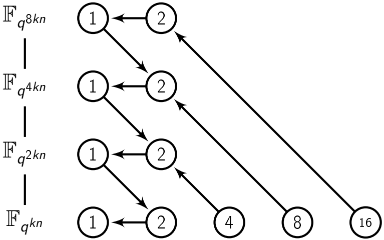

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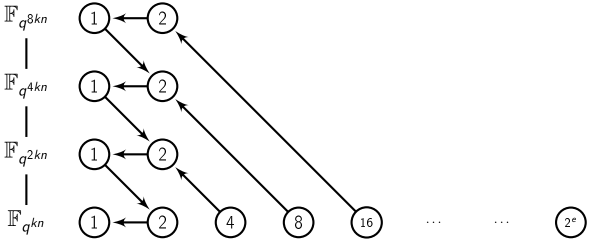
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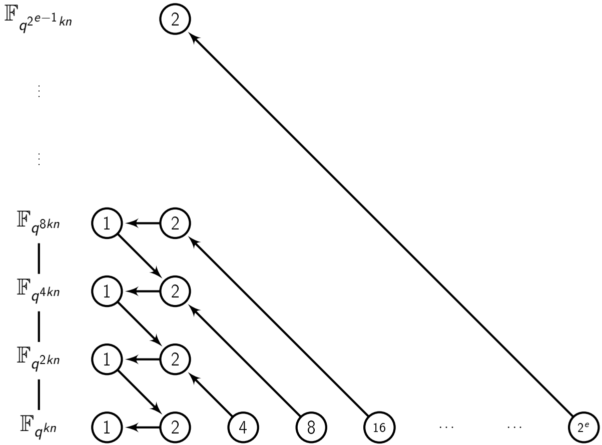
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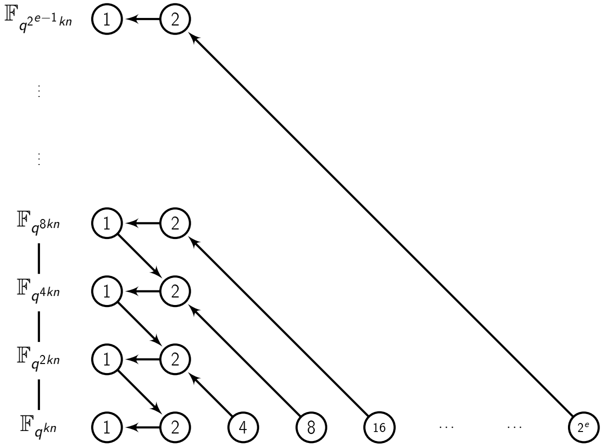

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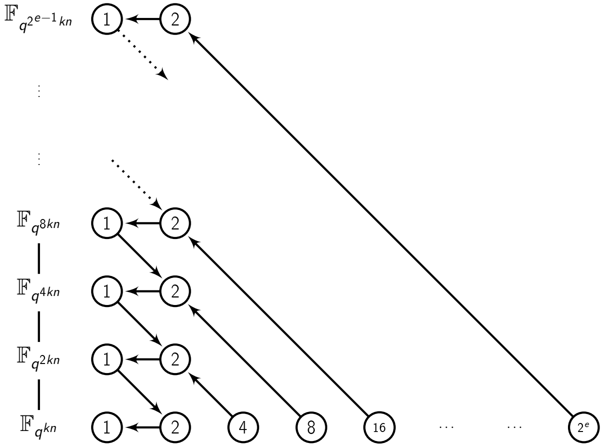
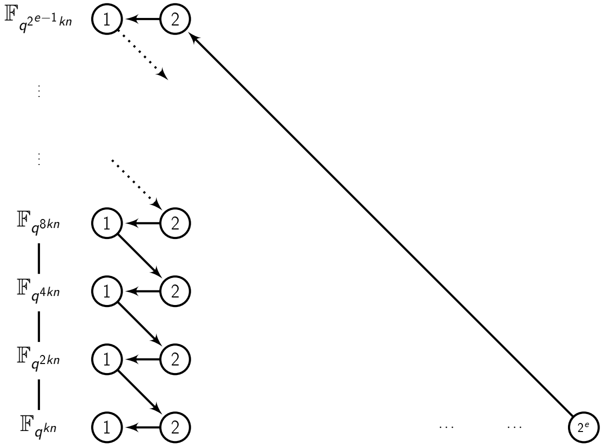
The GKZ QPA



$\mathbb{F}_{q^{2^{e-1}kn}}$
$\mathbb{F}_{q^{8kn}}$
$\mathbb{F}_{q^{4kn}}$
$\mathbb{F}_{q^{2kn}}$
$\mathbb{F}_{q^{kn}}$
1
2
4
8
16
2^e

The GKZ QPA



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1
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$\vdots$
$\vdots$
$\mathbb{F}_{q^{8kn}}$
1
2
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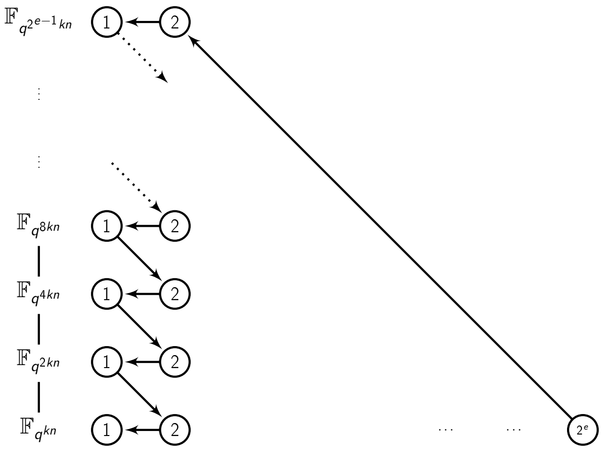
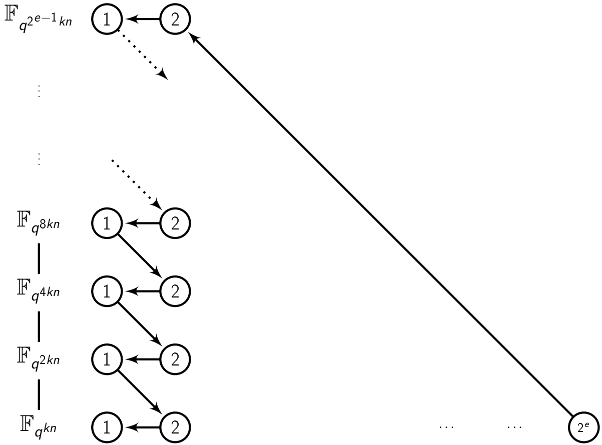
$\cdots$
$\cdots$
2^e

The GKZ QPA



- For an arbitrary element h we compute random $h' = h + r \cdot I$ s.t. $\deg h' = 2^e > 4n$ and h' is irreducible (Wan '97), then descend.

The GKZ QPA



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Hence no smoothness heuristics are needed!

Ensuring the elimination step works

To eliminate a degree 2 element $\bar{Q}(y)$ over $\mathbb{F}_{q^{kd}}$, we need to find a Bluher value B and an $s \in \mathbb{F}_{q^{kd}}$ that satisfy

$$B = \frac{(-s^q + u_0 s + v_0)^{q+1}}{(-u_0 s^2 + (u_1 - v_0)s + v_1)^q}$$

Theorem (*Helleseth-Kholosha '10*)

For $kd \geq 3$ *the set of elements* $B \in \mathbb{F}_{q^{kd}}^{\times}$ *s.t.* $X^{q+1} + BX + B$ *splits completely over* $\mathbb{F}_{q^{kd}}$ *is the image of* $\mathbb{F}_{q^{kd}} \setminus \mathbb{F}_{q^2}$ *under the map*

$$u \mapsto \frac{(u - u^{q^2})^{q+1}}{(u - u^q)^{q^2+1}}$$

Thus need lower bound for $\#\{(s, u) \in \mathbb{F}_{q^{kd}} \times (\mathbb{F}_{q^{kd}} \setminus \mathbb{F}_{q^2})\}$ on the curve

$$(u - u^{q^2})^{q+1}(-u_0 s^2 + (u_1 - v_0)s + v_1)^q - (u - u^q)^{q^2+1}(-s^q + u_0 s + v_0)^{q+1} = 0$$

Main results

Theorem

Given a prime power $q > 61$ *that is not a power of* 4, *an integer* $k \geq 18$, *coprime polynomials* $h_0, h_1 \in \mathbb{F}_{q^k}[X]$ *of degree at most two and an irreducible degree* l *factor* I *of* $h_1 X^q - h_0$, *the DLP in* $\mathbb{F}_{q^{kl}}^{\times}$ *where* $\mathbb{F}_{q^{kl}} \cong \mathbb{F}_{q^k}[X]/(I)$ *can be solved in expected time*

$$q^{\log_2 l + O(k)}$$

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Theorem

For every prime p *there exist infinitely many explicit extension fields* $\mathbb{F}_{p^n}$ *for which the DLP in* $\mathbb{F}_{p^n}^{\times}$ *can be solved in expected quasi-polynomial time*

$$\exp\left((1/\log 2 + o(1))(\log n)^2\right)$$

The GKZ QPA

'On the discrete logarithm problem in finite fields of fixed characteristic'
(previously 'On the Powers of 2')
`arxiv:1507.01495`

G., Thorsten Kleinjung & Jens Zumbrägel

(actual) Concluding remarks

- Implementing examples can be very informative
- Degree 2 elimination seems to be fundamental, sometimes complex, and theoretically very interesting (see Thorsten's talk next)
- Proving observations can be hard but worthwhile, especially due to presence of 'unknown unknowns'
- Some basic unanswered questions:
  - Can one remove the field heuristic?
  - Do faster algorithms exist for small characteristic?
  - Do faster algorithms exist for large(r) characteristic?

A comparison between the QPAs

	BGJT	GKZ
Field rep.	Heuristic	Heuristic
Elimination step	Heuristic (x 2)	Proven
Tree arity	$O(q^2)$	q
Complexity	$q^{O(\log n / \log\log q)}$	$q^{\log_2 n + o(\log q)}$
Practicality	Not yet	Yes, in $\mathbb{F}_{3^{2395}}$ and $\mathbb{F}_{2^{1279}}$