Cubical Sets are Generalized Transition Systems

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Abstract. We show in this article that "labelled" cubical sets (or Higher-Dimensional Automata) are a natural generalization of transition systems and asynchronous transition systems. This generalizes an older result of [14] which was only holding with precubical sets and subcategories of the classical (see [29]) categories of transition systems and asynchronous transition systems. This opens up new promises on the actual use of geometric methods (such as [8]) and on comparisons with other methods for verification of concurrent programs.

keywords Models for concurrency, semantics, category theory.

1 Introduction

There is a great variety of models for concurrency, as witnessed in [29] for instance. Most of the relationships between these models are known, but the newer "geometric" models for concurrency, such as cubical sets (HDA in [23] or in [16]) or local po-spaces [9] have not been so well formally linked with older models, such as transition systems or transition systems with independence. In fact, cubical sets have a notion of generalized transition in their very definition. The idea of relating these in the style of G. Winskel et al. [29] with operational models for concurrency dates back to [14], but this was done only between fairly restricted categories. In this paper we extend this previous work to the full categories of transition systems (operational model of "interleaving" concurrency) and of transition systems with independence (operational model of "true" concurrency). The main idea is that by relating these models, we can compare the semantics of concurrent languages given in different formalisms. Moreover, it is hoped that specific methods for statically analysing concurrent programs (such as the deadlock detection algorithm of [8] in the case of cubical sets) in one model can be re-used in the other, giving some nice cross-fertilisations, some of these being hinted in Section 6.

2 Transition systems

Transition systems are one of the oldest semantic models, both for sequential and concurrent systems. There is a convenient categorical treatment of this model, that we use in the sequel, taken from [29].

Definition 1. *A transition system is a structure (S,i,L,Tran) where,*

- S *is a set of states with initial state* i
- L *is a set of labels, and*
- $Tran \subseteq S \times L \times S$ *is the transition relation*

Transition systems are made into a category by defining morphisms to be some kind of simulation (for then being able to discuss about properties modulo [weak/strong] bisimulation, see [18]). The idea is that a transition system T_1 simulates a transition system T_0 if as soon as T_0 can fire some action a in some context, then T_1 can fire a as well in some related context. A morphism $f : T_0 \rightarrow T_1$ defines the way states and transitions of T_0 are related to states and transitions of T_1 making transition systems into a category TS.

Definition 2. *Let* $T_0 = (S_0, i_0, L_0, Tran_0)$ *and* $T_1 = (S_1, i_1, L_1, Tran_1)$ *be two transition systems. A partial morphism (or morphism in [29])* $f : T_0 \rightarrow T_1$ *is a pair* $f = (\sigma, \lambda)$ *where,*

- $\sigma : S_0 \rightarrow S_1$,
- $\lambda : L_0 \rightarrow L_1$ *is a* partial *function.* (σ, λ) *are such that*
  - $\sigma(i_0) = i_1$,
  - $(s, a, s') \in Tran_0$ *and* $\lambda(a)$ *is defined implies* $(\sigma(s), \lambda(a), \sigma(s')) \in Tran_1$. *Otherwise, if* $\lambda(a)$ *is undefined then* $\sigma(s) = \sigma(s')$.

As in [29], we can restrict to "total morphisms" i.e. the ones for which λ is a total function by suitably completing transition systems. Just add "idle" transitions to transition systems, very similar in spirit to the lifting of domains in denotational semantics [17, 22], where partial functions from D to D are considered total (and strict) from $D_\perp$ to $D_\perp$ ($\perp$ is a new element such that $\forall x, \perp \leq x$). An idle (or "$\perp$-") transition is a transition $*$ such that $*$ goes from a state s to the same state s. Consider the following completion $T_* = (S_*, i_*, L_*, Tran_*)$ of a transition system $T = (S, i, L, Tran)$, by setting $S_* = S$, $i_* = i$, $L_* = L \cup \{*\}$ and $Tran_* = Tran \cup \{(s, *, s)/s \in S\}$. Now, a morphism $f = (\sigma, \lambda)$ (with λ a total function) from $(T_0)_*$ to $(T_1)_*$ such that $\lambda(*) = *$ is the same as a partial morphism f' from T_0 to T_1 by identifying $*$ with "undefined". Conversely, a partial morphism $f = (\sigma, \lambda)$ from T_0 to T_1 can be identified with $f_* = (\sigma, \lambda_*)$, $\lambda_*(x) = *$ if and only if $\lambda(x)$ is undefined.

3 Asynchronous Automata

Asynchronous Automata are a nice generalization of Mazurkiewicz traces, and have influenced a lot other models for concurrency (like transition systems with independence etc.). They have been independently introduced in [27] and [2]. The idea is to decorate transition systems with an "independence" relation (between actions) that will allow us to distinguish between true-concurrency and mutual exclusion (or non-determinism) of two actions. We actually use a slight modification for our purposes, due to [6], and called "automata with concurrency relations":

Definition 3. *An automaton with concurrency relations is a quintuple* $(S, i, E, Tran, I)$ *where,*

(1) S *and* E *are disjoint sets;* $i \in S$ *is a distinguished element (the start state);* $Tran$ *is a subset of* $S \times E \times S$,
(2) $Tran$ *is such that whenever* (s, e, s'), $(s, e, s'') \in Tran$, *then* $s = s''$; *we require that for each* $e \in E$, *there are* $s, s' \in S$ *with* $(s, e, s') \in Tran$;
(3) $I = (I_s)_{s \in S}$ *is a family of irreflexive, symmetric binary relations* I_s *on* E; *it is required that whenever* $e_1 I_s e_2$ $(e_1, e_2 \in E)$, *there exist transitions* (s, e_1, s_1), (s, e_2, s_2), (s_1, e_2, r) *and* (s_2, e_2, r) *in* $Tran$.

In the sequel, we relax condition (2). A morphism is now a morphism $f = (\sigma, \lambda)$ of the underlying transition systems such that $aI_s b$ implies $\lambda(a) I'_{\sigma(s)} \lambda(b)$. This makes automata with concurrency relations into a category, written ACR. The category of automata with concurrency relations over an alphabet E is named ACR_E.

Similarly to Section 2, we can equivalently consider ACR (and ATS) to be built using $*$ transitions and total morphisms. The condition on the independence relation is then $aI_s b \Rightarrow \lambda(a) I'_{\sigma(s)} \lambda(b)$ when $\lambda(a) \neq *$ and $\lambda(b) \neq *$.

4 Cubical sets

Cubical sets, which are classical objects in combinatorial algebraic topology, see for instance [26], have been used as an alternative "truly-concurrent" model for concurrency, in particular since the seminal paper [23]. More recently they have been used (in particular the "precubical" ones) in [8] and [9] for deriving new and interesting deadlock detection algorithms. More algorithms have been designed since then, see for instance [24], [7] and [10].

4.1 "Precubical" sets

Definition 4. *A precubical set* K *is a family of sets* $\{K_n / n \geq 0\}$ *with face maps* $\partial_i^\alpha : K_n \to K_{n-1}$ $(0 \leq i \leq n-1,\ \alpha = 0, 1)$ *satisfying the following commutation rules:*

$$\partial_i^\alpha \partial_j^\beta = \partial_{j-1}^\beta \partial_i^\alpha \ (i < j)$$

Elements of K_n *are called* n*-transitions. Let* K *and* L *be two precubical sets. Then* $f = (f_n)_{n \in \mathbb{N}}$ *is a morphism of precubical sets from* K *to* L *if for all* $n \in \mathbb{N}$, f_n *is a function from* K_n *to* L_n *such that* $f_n \circ \partial_i^\alpha = \partial_i^\alpha \circ f_{n+1}$ *(for all* i, $0 \leq i \leq n$*).*

This forms a category called Υ^S. It is a presheaf category as follows. Let $\square^S$ be the free category whose objects are $[n]$, where $n \in \mathbb{N}$, and whose morphisms are generated by $[n] \overset{\delta_i^0}{\underset{\delta_j^1}{\rightrightarrows}} [n-1]$ for all $n \in \mathbb{N}^*$ and $0 \leq i, j \leq n-1$, such that $\delta_i^k \delta_j^l = \delta_{j-1}^l \delta_i^k \ (i < j)$. Now, the category $\square^S$Set of functors from $\square^S$ to

Set (morphisms are natural transformations) is isomorphic to the category of precubical sets. This implies, by general theorems ([19] and [21]), that Υ^S is an elementary topos. Moreover it is complete and co-complete because Set is complete and co-complete. Also, we will use the general fact in the sequel that in all categories of presheaves $\mathcal{D}^{op}$Set like this one, all elements (which are contravariant functors) are direct limits of so-called representable functors $h^{\mathcal{D}}$ which to every $d \in \mathcal{D}$ associates $(x \rightarrow Hom_{\mathcal{D}}(x,d)) \in \mathcal{D}^{op}\mathrm{Set}$[1].

4.2 Cubical sets

Precubical sets are a bit like the category of transition systems with no idle transitions: paths are transformed by morphisms into paths of the same length. This is far too strict to be really useful. For instance, simulations (hence bisimulations) cannot be morphisms (respectively spans of open morphisms as in [18]) in general. Also, it is impossible to describe the restriction to some subset of transitions (projection, restriction in CCS for instance) as a morphism. This needs a generalization of idle transitions to higher-dimensions. There is in fact a close notion in cubical sets:

Definition 5. *A cubical set K is a precubical set together with degeneracy maps $\epsilon_i : K_{n-1} \rightarrow K_n$ $(0 \leq i \leq n-1)$ satisfying the extra cubical relations:*

$$\begin{aligned} \epsilon_i\epsilon_j &= \epsilon_{j+1}\epsilon_i \qquad\qquad (i \leq j) \\ \partial_i^\alpha \epsilon_j &= \begin{cases} \epsilon_{j-1}\partial_i^\alpha & (i < j) \\ \epsilon_j \partial_{i-1}^\alpha & (i > j) \\ Id & (i = j) \end{cases} \end{aligned}$$

Let K and L be two cubical sets. Then f is a morphism of cubical sets from K to L if it is a morphism of precubical sets from the underlying precubical sets, and $f_{n+1} \circ \epsilon_j = \epsilon_j \circ f_n$ (for all $n \in \mathbb{N}$, $0 \leq i \leq n$).

The corresponding category of cubical sets, Υ, is isomorphic to the category of presheaves $\square^{op}$Set over a small category $\square$. This latter can be described in a nice way, see [5]. Therefore, similarly to the case of the category of precubical sets, the category of cubical sets is an elementary topos, which is complete and co-complete. We do not talk about cubical sets with connections and compositions here [3], but they have a great interest for our purposes, see for instance [12].

4.3 Some useful functors

There again, we need two interesting (and quite classical in spirit) functors. Let Υ_n be the category of Υ, whose objects are the n-dimensional cubical sets, i.e. the "cubical sets M with $M_k = \emptyset$ for all $k > n$". This category can be seen as the presheaf category $(\square^{\leq n})^{op}$Set where $\square^{\leq n}$ is the full subcategory of $\square$ where objects are $[p]$ with $p \leq n$. Similarly, we define Υ_n^S, the category of n-dimensional precubical sets, seen as the presheaf category $\left((\square^S)^{\leq n}\right)^{op}$ Set.

[1] This is for instance classical in the categorical presentation of simplicial sets, see for instance [11].

Lemma 1. *Let T_n (respectively T_n^S) be the function from Υ (respectively Υ^S) to Υ_n (respectively Υ_n^S), which to every $M \in \Upsilon$ (respectively $M \in \Upsilon^S$) associates $N \in \Upsilon_n$ (respectively $N \in \Upsilon_n^S$) with, $N([k]) = M([k])$ if $k \leq n$, $N(e_i : [k+1] \to [k]) = M(e_i)$ for $k < n$ and $N(\delta_i^\alpha : [k-1] \to [k]) = M(\delta_i^\alpha)$ for $k < n$. It defines a functor, called the n-truncation functor.*

The second functor is one which permits to build a natural cubical set from a precubical set:

Lemma 2. *There is a functor "free cubical set from a precubical set" $F : \Upsilon^S \to \Upsilon$ which is left-adjoint to the (obvious) forgetful functor K from Υ to Υ^S. Similarly, there is a functor "free cubical set of dimension less or equal than n from a precubical set of dimension less or equal than n", $F_n : \Upsilon_n^S \to \Upsilon_n$ which is left-adjoint to the (obvious) forgetful functor K_n from Υ_n to Υ_n^S.*

The proof uses a special form of Freyd's special adjoint functor theorem (which is also some form of Kan extension in presheaf categories), which is Proposition 1.3. of [11] (see Appendix A).

4.4 Labelled Cubical Sets

One remaining problem now, is that we do not have labels on transitions. This is easily taken care of by the following trick. Consider the category Υ^L of labelled cubical sets consisting of morphisms $l : M \to E$.

The morphisms in this category are as usual $f = (g, h) : (l : M \to E) \to (l' : M' \to E')$ with $g : M \to M'$ and $h : E \to E'$ such that the diagram

$$\begin{tikzcd} M \arrow[r, "g"] \arrow[d, "l"'] & M' \arrow[d, "l'"] \\ E \arrow[r, "h"] & E' \end{tikzcd}$$

is commutative. By abuse of notation, we will sometimes identify f, g and h in the following. Of course, Υ^L is the comma category (see [20]) $(Id_\Upsilon \downarrow Id_\Upsilon)$. We will also consider in the following the category Υ_*^L of "pointed" labelled cubical sets, i.e. pairs $(l : M \to L, s)$ with $l \in \Upsilon^L$ and $s \in M_0$ (the "initial" state) and morphisms preserving initial states. We call this category, the category of Higher-Dimensional Transition Systems.

Given an alphabet ("of actions") Σ, we can construct a "labelling" cubical set $!\Sigma$ as follows. First, construct for each $\sigma \in \Sigma$ the cubical set N_σ, which is the free cubical set generated by the following precubical set (denoted also by N_σ by an abuse of notation) : $(N_\sigma)_0 = \{1_\sigma\}$, $(N_\sigma)_1 = \{\sigma\}$, $(N_\sigma)_n = \emptyset$ (for $n \geq 2$), and $\partial_0^0(\sigma) = \partial_0^1(\sigma) = 1_\sigma$. We now identify Σ with the cubical set $\coprod_{\sigma \in \Sigma} N_\sigma / \{1_\sigma = 1_\tau\}$. We therefore identify in Σ all 1_σ and we write it as 1. We now set $\Sigma^{\times^n}$ to be the cubical set $\Sigma \times \cdots \times \Sigma$ (n times) suitably symmetrised (i.e. two elements are equal iff they have the same number of each of the letters). There are a certain number of natural inclusion morphisms between all these iterated products : $\Sigma^{\times^m} \xrightarrow{i_{j_1,\cdots,j_{n-m}}} \Sigma^{\times^n}$ where, $i_{j_1,\cdots,j_{n-m}}(\sigma_1,\cdots,\sigma_m) = (\tau_1,\cdots,\tau_n)$ with, $\tau_i = \begin{cases} \sigma_{i-card\{j_k | j_k \leq i\}} & \text{if } i \neq j_k \text{ for all } 1 \leq k \leq n-m \\ 1 & \text{otherwise} \end{cases}$. Then construct $!\Sigma$ as the direct limit of the diagram whose objects are all $\Sigma^{\times^n}$ and whose morphisms are all $i_{j_1,\cdots,j_{n-m}}$.

Geometrically, $!\Sigma$ is in dimension one the wedge of a set of loops, one for each $\sigma \in \Sigma$ (giving the labels for 1-transitions). In dimension two, it is a wedge of a set of tori, one for each pair $(\sigma,\tau) \in \Sigma \times \Sigma$, now seen as a set (giving the labels for 2-transitions) etc. Notice that $!\Sigma$ is freely generated by a precubical set.

5 Some adjunctions

5.1 With transition systems

We prove that some suitable full subcategory of $(\Upsilon^L_*)_1$ is isomorphic to TS. Consider HTS to be the category whose objects are the pointed labelled cubical sets $(M, l : M \to E, i)$ such that,

- they are freely generated by precubical sets, i.e. $M = F(N)$, $l = F(l')$ with $l' : N \to F$ morphism of precubical sets,
- they are "deterministic", i.e. $\forall x, x' \in M_k$ $(k \geq 1)$,

$$\left(d^0_i(x) = d^1_i(x'), d^1_i(x) = d^1_i(x') \; (\forall 0 \leq i < k) \; , l(x) = l(x')\right) \Longrightarrow x = x'$$

and whose morphisms are all morphisms of pointed labelled cubical sets. HTS_1 is the full sub-category of HTS consisting of pointed labelled cubical sets of dimension at most one.

As a matter of fact, the categories are defined in quite similar terms. States of ordinary transition systems are of the same nature as states of labelled cubical sets and source and target representation of transitions is nothing but a functional interpretation of the relation $Tran$. This is done formally by constructing two functors $\mathcal{U} : TS \to HTS_1$ and $\mathcal{V} : HTS_1 \to TS$ inverse of each other, with,

- $(F(M), F(l) : F(M) \to F(E), i) = \mathcal{U}(S, A, Tran, j)$ with,
  - $M_0 = S$,
  - $M_1 = \{a_{s,s'} / a \in A, s \xrightarrow{a} s' \in Tran\}$,
  - $i = j$,
  - $d^0_0(a_{s,s'}) = s$, $d^1_0(a_{s,s'}) = s'$,
  - $E = K_1(T_1(!A))$,
  - $l(a_{s,s'}) = a$, $l(s) = 1$.
- $(S, A, Tran, j) = \mathcal{V}(F(M), F(l) : F(M) \to F(E), i)$ with,
  - $S = M_0$,
  - $j = i$,
  - $A = E_1$,
  - $s \xrightarrow{a} s' \in Tran$ if $\exists x \in M_1$, such that $l(x) = a$, $d^0_0(x) = s$ and $d^1_0(x) = s'$ (then this x is unique because $(F(M), F(l), i)$ is deterministic).

Action of the functors on morphisms is as follows,

– if $f = (\sigma, \lambda) : (S_0, A_0, Tran_0, j_0) \to (S_1, A_1, Tran_1, j_1)$ is a morphism of transition systems then we define $\mathcal{U}(f) = (\mathcal{U}(f)^1, \mathcal{U}(f)^2)$ where $\mathcal{U}(f)^1 : F(M_0) \to F(M_1)$ and $\mathcal{U}(f)^2 : F(E_0) \to F(E_1)$ are the two components of the morphism, where $\mathcal{U}(S_0, A_0, Tran_0, j_0) = (F(M_0), F(l_0) : F(M_0) \to F(E_0), j_0)$, $\mathcal{U}(S_1, A_1, Tran_1, j_1) = (F(M_1), F(l_1) : F(M_1) \to F(E_1), j_1)$.
  - $\mathcal{U}(f)^1(a_{s,s'}) = \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } \lambda(a) \neq * \\ \epsilon_0(\sigma(s)) & \text{otherwise} \end{cases}$,
  - $\mathcal{U}(f)^1(s) = \sigma(s)$ $(s \in M_0)$,
  - $\mathcal{U}(f)^2(a_{s,s'}) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \neq * \\ \epsilon_0(1) & \text{otherwise} \end{cases}$,
  - $\mathcal{U}(f)^2(s) = 1$ $(s \in M_0)$.

– if $f = (f^1, f^2) : (l_0 : M_0 \to E_0, i_0) \to (l_1 : M_1 \to L, i_1)$ is a morphism in HTS_1, then $\mathcal{V}(f) = (\sigma, \lambda) : \mathcal{V}(l_0 : M_0 \to E_0, i_0) \to \mathcal{V}(l_1 : M_1 \to E_1, i_1)$ with
  - $\sigma(s) = f^1(s)$ (for all s state of $\mathcal{V}(l_0 : M_0 \to E_0, i_0)$),
  - $\lambda(a) = \begin{cases} f^2(a) & \text{if } f^2(a) \notin \mathrm{Im}\ \epsilon_0 \\ * & \text{otherwise} \end{cases}$ (for all a label in $\mathcal{V}(l_0 : M_0 \to E_0, i_0)$)

In the sequel we will restrict functors and categories of models so that they have "fixed labellings". We call HTS the category of higher-dimensional transition systems labelled over a fixed cubical set $!E$ for a given (fixed once and for all in all the following arguments) set of labels E. We will no longer mention these labelling sets. Given this restriction,

Theorem 1. *$\mathcal{U}$ and $\mathcal{V}$ are inverse functors.*

Now, in order to compare the category of higher-dimensional transition systems with ordinary transition systems we only have to look at how to retract HTS onto its sub-category HTS_1. This boils down to looking at the different adjunctions we have between Υ and Υ_1 because of the few next lemmas. The first one tells us that we can lift adjunctions from unlabelled to labelled cases, and the second one tells us that we can restrict adjunctions (this is useful for dealing with the "determinism condition" of labelled cubical sets).

Lemma 3. *Let $\mathcal{C}$ and $\mathcal{D}$ be two categories and $S_{\mathcal{C},\mathcal{D}}$ be the set of for all pairs of functors (F, G) with $F : \mathcal{C} \to \mathcal{D}$ left adjoint to $G : \mathcal{D} \to \mathcal{C}$. Then all elements of $S_{\mathcal{C},\mathcal{D}}$ induce elements of $S_{(Id_{\mathcal{C}} \downarrow Id_{\mathcal{C}}),(Id_{\mathcal{D}} \downarrow Id_{\mathcal{D}})}$.*

Lemma 4. *Let $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ be a pair of adjoint functors, $\mathcal{C}'$ (respectively $\mathcal{D}'$) a full sub-category of $\mathcal{C}$ (respectively of $\mathcal{D}$). Suppose that $F(\mathcal{C}') \subseteq \mathcal{D}'$ and $G(\mathcal{D}') \subseteq \mathcal{C}'$, then $\mathcal{C}' \underset{G_{|\mathcal{D}'}}{\overset{F_{|\mathcal{C}'}}{\rightleftarrows}} \mathcal{D}'$ is a pair of adjoint functors.*

We have mainly two different adjunctions between Υ and Υ_1 using T_1 (to keep the underlying ordinary transitions unchanged in the interpretation) among all the possible ones.

Proposition 1. *There are pairs of adjoint functors as follows (for $n \geq 1$):*

- *There is a functor $\mathcal{I}_n : \Upsilon_n \to \Upsilon$ left-adjoint to the truncation functor $T_n : \Upsilon \to \Upsilon_n$. Similarly, there is a functor $\mathcal{I}_n^S : \Upsilon_n^S \to \Upsilon^S$ left-adjoint to the truncation functor $T_n : \Upsilon^S \to \Upsilon_n^S$. Moreover, $\mathcal{I}_n$ and T_n commute with the free functor.*
- *The truncation functor $T_n : \Upsilon \to \Upsilon_n$ (respectively $T_n^S : \Upsilon^S \to \Upsilon_n^S$) is left-adjoint to a functor $\mathcal{G}_n : \Upsilon_n \to \Upsilon$ (respectively $\mathcal{G}_n^S : \Upsilon_n^S \to \Upsilon^S$).*

Proof. These are direct applications of Proposition 1.3. of [11] (see Appendix A).

The intuition about these functors is as follows. $\mathcal{I}_n$ is just some kind of inclusion functor; it takes a n-dimensional cubical set and forms a cubical set with exactly the same non-degenerated elements (i.e. those elements which are not in some Im ϵ_i); in fact, exactly the same elements in dimension less or equal than n, but only degenerated elements in dimension strictly bigger than n. Seen as some kind of abstraction (in the sense of abstract interpretation [4]), it is a "minimal allocation strategy" abstraction. A n-dimensional cubical set only prescribes what can happen for degrees of concurrency less or equal than n. $\mathcal{I}_n$ interprets this as being exactly with no (interesting) actions with more than n processes busy at the same time. On the contrary $\mathcal{G}_n$ tries to interpret a n-dimensional cubical set with "maximal allocation strategy" i.e. tries to fill in all $(n+1)$-dimensional holes in a n-dimensional cubical set as imposing that this should be filled in by a $(n+1)$-transition, and up and up in all dimensions. There are "dihomotopy" properties that should be proven about this "resolution" like functor. This is left for future work.

We notice now that the adjunction $(\Upsilon_*^L)_n \underset{T_n}{\overset{\mathcal{I}_n}{\rightleftarrows}} \Upsilon_*^L$ can be restricted using Lemma 4 to the full sub-categories of free objects generated by precubical sets, in, respectively, $(\Upsilon_*^L)_n$ and Υ_*^L. This is due to the fact that (see Proposition 1) $\mathcal{I}_n$ and T_n commute with the "free functors". We can restrict this adjunction furthermore, still using Lemma 4, to see that the adjunction still holds with $n \geq 1$ when we restrict to deterministic automata. Hence we have the adjunction: $HTS_1 \underset{T_1}{\overset{\mathcal{I}_1}{\rightleftarrows}} HTS$. Given that HTS_1 and TS are isomorphic (see Theorem 1), we deduce that we have a pair of adjoint functors: $TS \underset{ht}{\overset{th}{\rightleftarrows}} HTS$. Unfortunately, we did not manage yet to "lift" the other adjunction of Proposition 1 to higher-dimensional transition systems.

5.2 With automata with concurrency relations

We first define functors $\mathcal{W}, \mathcal{Y}$, which will appear to be inverse functors:

$$ACR \underset{\mathcal{Y}}{\overset{\mathcal{W}}{\rightleftarrows}} HTS_2$$

(HTS_2 is the full subcategory of Υ_*^L consisting of higher-dimensional transition systems of dimension less than or equal to two) by,

- $(F(P), F(l : P \rightarrow L), F(j))=\mathcal{Y}(S, i, E, I, Tran)$ with,
  - $j = i$,
  - $P_0 = S$,
  - $P_1 = \{t_{s,s'} / s \xrightarrow{t} s' \in Tran\}$,
  - $L = K_2(T_2(!E))$,
  - $d_0^0(t_{s,s'}) = s$, $d_0^1(t_{s,s'}) = s'$ and $l(t_{s,s'}) = t$,
  - $P_2 = \{ab_{s,s',s'',u} / aI_sb \wedge a_{s,s'} \in P_1 \wedge b_{s,s''} \in P_1 \wedge b_{s',u} \in P_1 \wedge a_{s'',u} \in P_1\}$,
  - $d_0^0(ab_{s,s',s'',u}) = a_{s,s'}$ (or $d_1^0(ab_{s,s',s'',u}) = a_{s,s'}$, depending on the way this is coded in $!E$), $d_1^0(ab_{s,s',s'',u}) = b_{s,s''}$ (or $d_0^0(\cdots) = \cdots$), $d_1^1(ab_{s,s',s'',u})$ $= b_{s',u}$, (respectively, or $d_0^1(\cdots) = \cdots$), $d_0^1(ab_{s,s',s'',u}) = a_{s'',u}$ (respectively $\cdots$) and $l(ab_{s,s',s'',u}) = (a, b)$ (respectively $\cdots$).
- $\mathcal{W}(P, P \xrightarrow{l} L, j) = (S, i, E, I, Tran)$ with,
  - $(S, i, E, Tran) = \mathcal{V}(T_1(P), T_1(l), j)$,
  - aI_sb if there exist $x, x', y, y' \in P_1$, $C \in P_2$ with $l(x) = a$, $l(x') = a$, $l(y) = b$, $l(y') = b$ and $d_0^0(x) = d_0^0(y) = s$, $d_0^1(x) = d_0^0(y')$, $d_0^1(y) = d_0^0(x')$, $d_1^1(y') = d_1^1(x')$, $l(C) = (a, b)$, $d_0^0(C) = x$, $d_1^0(C) = y$, $d_0^1(C) = y'$ and $d_1^1(C) = x'$ (or, respectively, $d_1^0(C) = x$, $d_0^0(C) = y$, $d_1^1(C) = y'$ and $d_0^1(C) = x'$).

$\mathcal{Y}$ has the same action on the underlying ordinary transition system of an asynchronous transition system as functor $\mathcal{U}$; we will identify $\mathcal{U}(S, i, E, Tran)$ with the underlying 1-dimensional skeleton of the higher-dimensional transition system $\mathcal{Y}(S, i, E, I, Tran)$. Similarly for $\mathcal{W}$ which acts as $\mathcal{V}$ on the underlying ordinary transition systems, thus we will identify $\mathcal{V}(P, l : P \rightarrow L, j)$ as the underlying transition system of the asynchronous transition system $\mathcal{W}(P, l : P \rightarrow L, j)$. $\mathcal{Y}$ fills in all interleavings of two independent actions by 2-transitions $\mathcal{W}$ imposes two actions to be independent if and only if there exists a truly concurrent execution of them in the higher-dimensional transition system. The action on morphisms is again easy to define. Let $f = (\sigma, \lambda) : (S, i, E, I, Tran) \rightarrow (S', i', E', I', Tran')$ be a morphism of asynchronous transition systems. Then $g = \mathcal{Y}(f) : \mathcal{Y}(S, i, E, I, Tran) \rightarrow \mathcal{Y}(S', i', E', I', Tran')$ is defined by,

- $T_1(g) = \mathcal{U}(f)$ (by the identification made above),
- $$g_2(ab_{s,s',s'',u}) = \begin{cases} \lambda(a)\lambda(b)_{\sigma(s),\sigma(s'),\sigma(s''),\sigma(u)} & \text{if } \lambda(a) \neq * \text{ and } \lambda(b) \neq * \\ \epsilon_0\left(\lambda(a)_{\sigma(s),\sigma(s')}\right) & \text{if } \lambda(a) \neq * \text{ and } \lambda(b) = * \\ \epsilon_1\left(\lambda(b)_{\sigma(s),\sigma(s'')}\right) & \text{if } \lambda(b) \neq * \text{ and } \lambda(a) = * \\ \epsilon_0\epsilon_0(\sigma(s)) & \text{if } \lambda(a) = * \text{ and } \lambda(b) = * \end{cases}$$

  for $ab_{s,s',s'',u} \in \mathcal{F}(S, i, E, I, Tran)_2$.

Finally, for $g : (P, P \xrightarrow{l} L, j) \rightarrow (P', P' \xrightarrow{l'} L', j')$ a morphism of $(\Upsilon_*^L)_2$ we define $f = (\sigma, \lambda) : \mathcal{W}(P, P \xrightarrow{l} L, j) \rightarrow \mathcal{W}(P', P' \xrightarrow{l'} L, j')$ simply by (using the previous identification) $f = \mathcal{V}(T_1(g) : T_1(P), T_1(l), j) \rightarrow (T_1(P'), T_1(l'), j')$.

In the sequel we will again fix once and for all the labelling cubical set used in our higher dimensional transition systems, to be $!E$ (where E is a set of labels fixed once and for all). Then again,

Theorem 2. *$\mathcal{W}$ and $\mathcal{Y}$ are well-defined functors. Moreover, $\mathcal{Y}$ and $\mathcal{W}$ are inverse of each other.*

Proof. The only difficulty, is to show that the action of these functions on morphisms are well-defined. For $\mathcal{Y}$, the only thing to check is that the definition in dimension 2 of the underlying precubical set is coherent. We only check one of the necessary equalities: (taking the same notations as above), for $ab_{s,s',s'',u} \in \mathcal{Y}(S', i', E', I', Tran')$ with $\lambda(a) \neq *$ and $\lambda(b) = *$ (notice that we have then $\sigma(s'') = \sigma(s)$ and $\sigma(s') = \sigma(u)$),

$$
\begin{aligned}
d_l^k(g_2(ab_{s,s',s'',u})) &= d_l^k(\epsilon_0(\lambda(a)_{\sigma(s),\sigma(s')})) \\
&= \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \epsilon_0(d_0^0(\lambda(b)_{\sigma(s),\sigma(s'')})) = \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=1,\, l=0 \\ \epsilon_0(d_0^1(\lambda(b)_{\sigma(s'),\sigma(u)})) = \epsilon_0(\sigma(u)) & \text{if } k=1,\, l=1 \end{cases} \\
g_1(d_l^k(ab_{s,s',s'',u})) &= \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s''),\sigma(u)} = \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=1,\, l=0 \\ \epsilon_0(\sigma(u)) & \text{if } k=1,\, l=1 \end{cases}
\end{aligned}
$$

which are equal. The rest of the proof goes along the same lines (see the rest in Appendix A).

For $\mathcal{W}$ we have to check that, for $f = (\sigma, \lambda) = \mathcal{Y}(g : (P, l : P \rightarrow L, i) \rightarrow (P', l' : P' \rightarrow L', i'))$, $aI_s b$ and $\lambda(a) \neq *$, $\lambda(b) \neq *$ implies $\lambda(a) I'_{\sigma(s)} \lambda(b)$. Suppose $aI_s b$ in $\mathcal{Y}(P, l : P \rightarrow L, i)$. Then there exist $x, x', y, y' \in P_1$ with $l(x) = a$, $l(x') = a$, $l(y) = b$, $l(y') = b$ and $d_0^0(x) = d_0^0(y) = s$, $d_0^1(x) = d_0^0(y')$, $d_0^1(y) = d_0^0(x')$, $d_1^1(y') = d_1^1(x')$, and we have a $C \in P_2$ with $l(C) = (a, b)$, $d_0^0(C) = x$, $d_1^0(C) = y$, $d_0^1(C) = y'$ and $d_1^1(C) = x'$. We know that $g(C) \in P'_2$ and that $l' \circ g(C) = (f(a), f(b))$ since $f(a) \neq *$ and $f(b) \neq *$. Similarly, $l'(g(x)) = f(a)$, $l'(g(x')) = f(a)$, $l'(g(y)) = f(b)$, $l'(g(y')) = f(b)$. Furthermore, because g is a morphism of cubical sets, $d_0^0(g(x)) = d_0^0(g(y)) = \sigma(s)$, $d_0^1(g(x)) = d_0^0(g(y'))$, $d_0^1(g(y)) = d_0^0(g(x'))$, $d_1^1(g(y')) = d_1^1(g(x'))$, so $\lambda(a) I'_{\sigma(s)} \lambda(b)$.

The adjunctions of Proposition 1, in the particular case $n = 2$, together with the result of Theorem 2 imply that we have a pair of adjoint functors: $ACR \underset{ha}{\overset{ah}{\rightleftarrows}} HTS$.

6 Conclusion and further work

6.1 Other adjunctions

In [29] some adjunctions are described between a variety of models for concurrency. We hope be able to lift some of these functors to the case of labelled cubical sets. In particular, we believe that the equivalence between traces defined in the category TL of Mazurkiewicz traces should be mapped onto homotopy classes of traces in HTS, therefore the partially commutative monoid defined in Mazurkiewicz trace theory should be some analog of the fundamental category in cubical sets (defined for instance in [15]). This is left for future work. The domain of configurations of an event structure is a dI-domain (stable domain, à la Berry, see for instance [29]) and we believe that through adjunctions with HTS (and through the adjunctions between cubical sets and local po-spaces [9], using the geometrical realization functor), this is linked to the fact that partially ordered topological spaces are related to some particular forms of Scott domains (see again [15]). Finally, we believe that there is an equivalence of categories between some form of higher-dimensional transition systems and general Petri nets. One of the difficulties is in finding the right notion of independence between any number of transitions in Petri nets. One possible start is to use the adjunction between ACR and Petri nets in [6].

We have seen that cubical sets are complete and co-complete. This means that the category of labelled cubical sets (with a fixed alphabet of the form $!E$) is complete and co-complete. Because it is related through left and right adjoints to transition systems (and asynchronous transition systems), there are some correspondances between limits and co-limits in these categories. For instance, products in higher-dimensional transition systems correspond to the parallel combination (with no interference) of the two higher-dimensional transition systems (as does the cartesian product of two partially ordered topological spaces); co-products correspond to non-deterministic choice. Fibred products, i.e. synchronized products as in the category of ordinary transition systems [1], allow for nice semantical definitions. This allows also for nice comparison of semantics through adjunctions.

Stubborn sets [28], sleep sets and persistent sets [13] are methods used for diminishing the complexity of model-checking using transition systems. They are based on semantic observations using Petri nets in the first case and Mazurkiewicz trace theory in the other one. We believe that these are special forms of "homotopy retracts" when cast (using the adjunctions we have hinted) in the category of higher-dimensional transition systems. We hope to make this statement more formal, through these adjunctions, and use this to design new state-space reduction methods.

Last but not least, in [18] is defined an abstract notion of bisimulation. Given a model for concurrency, i.e. a category of models $\mathbf{M}$ and a "path category", i.e. a subcategory of $\mathbf{M}$ which somehow represents what should be thought of as being paths in the models, then we can define two elements of $\mathbf{M}$ to be bisimilar if there exists a span of special morphisms linking them. These special morphisms have a path-lifting property that we believe would be in higher-dimensional transition systems a (geometric) fibration property. We thus hope that homotopy invariants could be useful for the study of a variety of bisimulation equivalences. Some work has been done in that direction in [25] (and in some sense also in [16]).

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A Proofs

Most proofs are based on a particular case of the existence of Kan extensions, taken here from [11] (Proposition 1. 3. Page 22):

Proposition 2. *Let $\mathcal{C}$ be a category with direct limits and $G : \mathcal{D}^{op}\mathrm{Set} \to \mathcal{C}$ a functor. Then the following statements are equivalent :*

(i) G commutes with direct limits.
(ii) G is left adjoint to a functor $D : \mathcal{C} \to \mathcal{D}^{op}\mathrm{Set}$. Moreover, the functor $G \to G \circ h^{\mathcal{D}}$ is an equivalence of the full subcategory of $Hom(\mathcal{D}^{op}\mathrm{Set}, \mathcal{C})$ formed by the functors G which commute with direct limits on $Hom(\mathcal{D}, \mathcal{C})$.

In fact, D is the functor which associates $h^{\mathcal{C}}(c) \circ G \circ h^{\mathcal{D}}$ with $c \in \mathcal{C}$.

<u>Lemma 1</u> is obvious.
<u>Lemma 2</u>:

Proof. It suffices to use Proposition 2 with $\mathcal{D} = \Box^S$, $\mathcal{C} = \Box^{op}\mathrm{Set}$ and functor $w \in Hom(\mathcal{D}, \mathcal{C})$ with $w([p]) = h^{\Box}([p])$. This defines F and its right-adjoint K. It is easy to see that the unit η of the adjunction is in fact the identity natural transformation $\eta : Id \to K \circ F$. This means that K induces an equivalence of categories between $F(\Upsilon^S)$ and Υ^S.

The case of cubical sets of dimension less or equal than n is treated in exactly the same manner.

<u>Theorem 1</u>:

Proof. We now forget about the given labelling set E, even in the definition of transition systems and labelled cubical sets. Thus, given a transition system $T = (S, i, Tran) \in TS$ we have, $\mathcal{U}(T) = N$ where $N = F(M, l, j)$ with,

- $M_0 = S$,
- $M_1 = \{a_{s,s'} / a \in E, s \xrightarrow{a} s' \in Tran\}$,
- $d_0^0(a_{s,s'}) = s$, $d_0^1(a_{s,s'}) = s'$,
- $l(a_{s,s'}) = a$, $l(s) = 1$.

Therefore, $\mathcal{V}(\mathcal{U}(T)) = (S', i', Tran')$ with,

- $S' = N_0 = M_0 = S$,
- $i' = j = i$,
- $s \xrightarrow{a} s' \in Tran'$ if $\exists x \in N_1$, such that $l(x) = a$, $d_0^0(x) = s$ and $d_0^1(x) = s'$. The only possible $x \in N_1$ such that $l(x) = a \in E$ is actually $x \in M_1$, and the only possible x satisfying all the conditions above is $a_{s,s'}$. Therefore, $s \xrightarrow{a} s' \in Tran'$ if and only if $s \xrightarrow{a} s' \in Tran$, hence $Tran' = Tran$.

Now, take $(M, l, j) \in HTS_1$, then $(S, i, Tran) = \mathcal{V}(M, l, j)$ with,

- $S = M_0$,
- $j = i$,

- $A = E_1 \backslash \mathrm{Im}\ \epsilon_0$,
- $s \xrightarrow{a} s' \in Tran$ if $\exists x \in M_1$, such that $l(x) = a$, $d_0^0(x) = s$ and $d_0^1(x) = s'$.

And then, $F(M', l', j') = \mathcal{U}(S, i, Tran)$ with,

- $M'_0 = S = M_0$,
- $M'_1 = \{a_{s,s'} / a \in A, s \xrightarrow{a} s' \in Tran\} = M_1 \backslash \mathrm{Im}\ \epsilon_0$ (because l is free),
- $j' = i = j$,
- $d_0^0(a_{s,s'}) = s$, $d_0^1(a_{s,s'}) = s'$,
- $l(a_{s,s'}) = a$, $l(s) = 1$.

Therefore, $F(M', l', j') = (M, l, j)$ because M and l are free.

This proof extends readily on morphisms: Let first $f : (l_0 : M_0 \to E_0, i_0) \to (l_1 : M_1 \to E_1, i_1)$ be a morphism of HTS, $f = (f_1, f_2)$. Then let $(\sigma, \lambda) = \mathcal{V}(f) : (S_0, i_0, Tran_0) \to (S_1, i_1, Tran_1)$. We have:

- $\sigma(s) = f^1(s)$ (for all s state of $\mathcal{V}(l_0 : M_0 \to E_0, i_0)$),
- $\lambda(a) = \begin{cases} f^2(a) & \text{if } f^2(a) \notin \mathrm{Im}\ \epsilon_0 \\ * & \text{otherwise} \end{cases}$ (for all a label in $\mathcal{V}(l_0 : M_0 \to E_0, i_0)$)

Let now $(g_1, g_2) = \mathcal{U}(\sigma, \lambda)$. We have,

- $\mathcal{U}(\sigma, \lambda)^1(a_{s,s'}) = \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } \lambda(a) \neq * \\ \epsilon_0(\sigma(s)) & \text{otherwise} \end{cases}$,
- $\mathcal{U}(\sigma, \lambda)^1(s) = \sigma(s)$ $(s \in M_0)$,
- $\mathcal{U}(\sigma, \lambda)^2(a_{s,s'}) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \neq * \\ \epsilon_0(1) & \text{otherwise} \end{cases}$,
- $\mathcal{U}(\sigma, \lambda)^2(s) = 1$ $(s \in M_0)$.

But,

- $\lambda(a)_{\sigma(s),\sigma(s')}$ is the unique x (because of the determinism condition in HTS_1) going from $\sigma(s) = f^1(s)$ to $\sigma(s') = f^1(s')$, with label $\lambda(a) = f^2(a)$, hence is equal to $f^1(a_{s,s'})$,
- when $\lambda(a) = *$, i.e. when $f^2(a) \in \mathrm{Im}\ \epsilon_0$, $f^1(s)$ is necessarily in $\mathrm{Im}\ \epsilon_0$: (f^1, f^2) being a morphism between l_0 and l_1, we have $l_1(f^1(a_{s,s'})) = f^2(l_0(a_{s,s'})) = f^2(a) \in \mathrm{Im}\ \epsilon_0$; In order to have this, it is necessary that $f^1(a_{s,s'}) \in \mathrm{Im}\ \epsilon_0$. Furthermore, $d_0^0(f^1(a_{s,s'})) = f^1(s) = \sigma(s)$ and $d_0^1(f^1(a_{s,s'})) = \sigma(s') = \sigma(s)$ therefore $f^1(a_{s,s'}) = \epsilon_0(\sigma(s)) = \mathcal{U}(\sigma, \lambda)^1(a_{s,s'})$.

Now let $f = (\sigma, \lambda) : (S_0, i_0, Tran_0) \to (S_1, i_1, Tran_1)$ be a morphism of labelled transition system and $g = \mathcal{U}(f)$. We have,

- $\mathcal{U}(\sigma, \lambda)^1(a_{s,s'}) = \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } \lambda(a) \neq * \\ \epsilon_0(\sigma(s)) & \text{otherwise} \end{cases}$,
- $\mathcal{U}(\sigma, \lambda)^1(s) = \sigma(s)$ $(s \in M_0)$,
- $\mathcal{U}(\sigma, \lambda)^2(a_{s,s'}) = \begin{cases} \lambda(a) & \text{if } \lambda(a) \neq * \\ \epsilon_0(1) & \text{otherwise} \end{cases}$,
- $\mathcal{U}(\sigma, \lambda)^2(s) = 1$ $(s \in M_0)$.

Then consider $f' = (\sigma', \lambda') = \mathcal{V}(g)$. We have,

- $\sigma'(s) = g_1(s)$ (for all s state),
- $\lambda'(a) = \begin{cases} g_2(a) & \text{if } g_2(a) \notin \text{Im } \epsilon_0 \\ * & \text{otherwise} \end{cases}$ (for all a label)

Therefore,

- if $g_2(a) \notin \text{Im } \epsilon_0$, i.e. if $\lambda(a) \neq *$, then $\lambda'(a) = g_2(a) = \lambda(a)$. If not, $\lambda'(a) = *$ and $\lambda(a) = *$ at the same time.
- $\sigma'(s) = g_1(s) = \sigma(s)$.

Lemma 3:

Proof. Let $(F, G) \in S_{\mathcal{C},\mathcal{D}}$, $l \in Hom_{\mathcal{C}}(M, N)$ and $l' \in Hom_{\mathcal{D}}(M', N')$. Let now $f \in Hom_{(Id_{\mathcal{D}} \downarrow Id_{\mathcal{D}})}(F(l), l')$; this means that $f = (f_1, f_2)$ where f_1 and f_2 are morphisms in $\mathcal{D}$ which make the following diagram commutative:

$$\begin{tikzcd} F(M) \arrow[r, "f_1"] \arrow[d, "F(l)"'] & M' \arrow[d, "l'"] \\ F(N) \arrow[r, "f_2"'] & N' \end{tikzcd}$$

So the following diagram is also commutative by functoriality of G:

$$\begin{tikzcd} G \circ F(M) \arrow[r, "G(f_1)"] \arrow[d, "G \circ F(l)"'] & G(M') \arrow[d, "G(l')"] \\ G \circ F(N) \arrow[r, "G(f_2)"'] & G(N') \end{tikzcd}$$

But the unit η of the adjunction between F and G is a natural transformation, thus the first square of the following diagram also commutes, entailing that the outer square itself is a commutative one:

$$\begin{tikzcd} M \arrow[r, "\eta_M"] \arrow[d, "l"'] & G \circ F(M) \arrow[r, "G(f_1)"] \arrow[d, "G \circ F(l)"'] & G(M') \arrow[d, "G(l')"] \\ N \arrow[r, "\eta_N"'] & G \circ F(N) \arrow[r, "G(f_2)"'] & G(N') \end{tikzcd}$$

Hence we get naturally, a morphism in $Hom_{(Id_{\mathcal{C}} \downarrow Id_{\mathcal{C}})}(l, G(l'))$:

$$A_{l,l'}(f_1, f_2) = (G(f_1) \circ \eta_M, G(f_2) \circ \eta_N)$$

Similarly in the other direction, we get a morphism in $Hom_{(Id_{\mathcal{D}} \downarrow Id_{\mathcal{D}})}(F(l), l')$,

$$B_{l,l'}(g_1, g_2) = (\epsilon_{M'} \circ F(g_1), \epsilon_{N'} \circ F(g_2))$$

where ϵ is the co-unit of the adjunction(F, G).

We now prove that this is a natural bijection between $Hom_{(Id_{\mathcal{D}} \downarrow Id_{\mathcal{D}})}(F(l), l')$ and $Hom_{(Id_{\mathcal{C}} \downarrow Id_{\mathcal{C}})}(l, G(l'))$. The composite of $A_{l,l'}$ with $B_{l,l'}$ being the identity is a direct consequence of the (right) identity 8 page 80 of [20]:

$$F(M) \xrightarrow{F(\eta_M)} FGF(M) \xrightarrow{\epsilon_{F(M)}} F(M)$$

is the identity natural transformation on F. This means that the following diagram is commutative:

$$\begin{tikzcd}
F(M) \arrow[r, "F(\eta_M)"] \arrow[d, "f_1"'] & FGF(M) \arrow[r, "\epsilon_{F(M)}"] \arrow[d, "FG(f_1)"'] & F(M) \arrow[d, "f_1"'] \\
M' \arrow[r, "\eta_{M'}"] & FG(M') \arrow[r, "\epsilon_{M'}"] & M'
\end{tikzcd}$$

Hence,

$$F(M) \xrightarrow{F\eta_M} FGF(M) \xrightarrow{FG(f_1)} FG(M') \xrightarrow{\epsilon_{M'}} M' \quad = f_1$$

Similarly, the composite $B_{l,l'} \circ A_{l,l'} = Id$ because of (left) identity 8 page 80 of [20], so we have:

$$G(M') \xrightarrow{\eta_{M'}G} GFG(M') \xrightarrow{GF(f_2)} GF(M) \xrightarrow{\eta_M} M \quad = f_2$$

Thus (F, G) induces a pair of adjoint functors between $(Id_{\mathcal{C}} \downarrow Id_{\mathcal{C}})$ and $(Id_{\mathcal{D}} \downarrow Id_{\mathcal{D}})$.

<u>Lemma 4</u>:

Proof. The natural bijection between $Hom_{\mathcal{D}}(F(X), Y)$ and $Hom_{\mathcal{C}}(X, G(Y))$ naturally restricts to a bijection between $Hom_{\mathcal{D}}(F(X), Y) = Hom_{\mathcal{D}'}(F(X), Y)$ ($\mathcal{D}'$ is full in $\mathcal{D}$) and $Hom_{\mathcal{C}}(X, G(Y)) = Hom_{\mathcal{C}'}(X, G(Y))$ ($\mathcal{C}'$ is full in $\mathcal{C}$) for $X \in \mathcal{D}'$ and $Y \in \mathcal{C}'$.

<u>Proposition 1</u>:

Proof. Take as a first instance of Proposition 2 $\mathcal{D} = \square^{\leq^n}$ and $\mathcal{C} = \square^{op}\mathrm{Set}$. We define functor $u \in Hom(\mathcal{D}, \mathcal{C})$ as follows :

$$u([p]) = h^{\square}([p])$$

Then functor G of Proposition 2 is the functor which commutes with direct limits and which is such that,

$$G(h^{\Box^{\leq n}}([p])) = h^{\Box}([p])$$

$\mathcal{I}_n$ of the proposition is therefore this functor G. Its right adjoint D given by the same proposition is such that (see [11]),

$$D(c) : a \to Hom_{\mathcal{C}}(G(h^{\mathcal{D}}(a)), c)$$

i.e. in our case, for $p \leq n$,

$$\begin{aligned} D(c)([p]) &= Hom_{\Upsilon}(h^{\Box}([p]), c) \\ &= c([p]) \end{aligned}$$

the last equality holding because of Yoneda's lemma [20]. We recognize D as being the truncation functor.

Restricting the adjunction to the categories of cubical sets with morphisms respecting the initial states is obvious. The adjunction $(\Upsilon_*^L)_n \underset{T_n}{\overset{\mathcal{I}_n}{\rightleftarrows}} \Upsilon_*^L$ is a direct consequence of Lemma 3.

We proceed in a similar manner for the adjunction $\mathcal{I}_n^S$, T_n^S. We define again by Proposition 2 $\mathcal{I}_n^S(h^{\Box^{S\leq n}}[p]) = h^{\Box^S}[p]$. Notice that $h^{\Box^{\leq n}}[p] = F_n(h^{S\leq n}[p])$ and $h^{\Box}[p] = F(h^{\Box^S}[p])$, therefore $\mathcal{I}_n(F(h^{\Box^{S\leq n}}[p])) = F(\mathcal{I}_n^S(h^{\Box^{S\leq n}}[p]))$, hence the commuting diagram, by taking the direct limit. The proof for the commutation of the diagram involving T_n is similar.

The last part of the proposition is by taking $\mathcal{D} = \Box$, $\mathcal{C} = (\Box^{\leq n})^{op}\mathbf{Set}$ and functor $v \in Hom(\mathcal{D}, \mathcal{C})$ as follows,

$$v([p])([q]) = Hom_{\Box}([q], [p])$$

which gives as G functor T_n. Now, its right adjoint is functor D with (for $N \in (\Box^{\leq n})^{op}\mathrm{Set}$ and $[p] \in \Box$),

$$D(N)([p]) = Hom_{\Upsilon_n}(T_n(h^{\Box}([p])), N)$$

<u>Theorem 2</u>:

Proof. The only difficulty in the first part, is to show that the action of these functions on morphisms are well-defined. For $\mathcal{Y}$, the only thing to check is that the definition in dimension 2 of the underlying precubical set is coherent. We compute first (taking the same notations as above), for $ab_{s,s',s'',u} \in \mathcal{Y}(S', i', E', I', Tran')$:

- if $\lambda(a) \neq *$ and $\lambda(b) \neq *$,

$$\begin{aligned} d_l^k(g_2(ab_{s,s',s'',u})) &= d_l^k\left(\lambda(a)\lambda(b)_{\sigma(s),\sigma(s'),\sigma(s''),\sigma(u)}\right) \\ &= \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s''),\sigma(u)} & \text{if } k=1,\, l=0 \\ \lambda(b)_{\sigma(s'),\sigma(u)} & \text{if } k=1,\, l=1 \end{cases} \end{aligned}$$

We also have,

$$g_1(d_l^k(ab_{s,s',s'',u})) = \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s''),\sigma(u)} & \text{if } k=1,\, l=0 \\ \lambda(b)_{\sigma(s'),\sigma(u)} & \text{if } k=1,\, l=1 \end{cases}$$

which are equal.

- if $\lambda(a) \neq *$ and $\lambda(b) = *$ (notice that we have then $\sigma(s'') = \sigma(s)$ and $\sigma(s') = \sigma(u)$),

$$\begin{aligned} d_l^k(g_2(ab_{s,s',s'',u})) &= d_l^k(\epsilon_0(\lambda(a)_{\sigma(s),\sigma(s')})) \\ &= \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \epsilon_0(d_0^0(\lambda(b)_{\sigma(s),\sigma(s'')})) = \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=1,\, l=0 \\ \epsilon_0(d_0^1(\lambda(b)_{\sigma(s'),\sigma(u)})) = \epsilon_0(\sigma(u)) & \text{if } k=1,\, l=1 \end{cases} \end{aligned}$$

We also have,

$$g_1(d_l^k(ab_{s,s',s'',u})) = \begin{cases} \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=0,\, l=0 \\ \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=1 \\ \lambda(a)_{\sigma(s''),\sigma(u)} = \lambda(a)_{\sigma(s),\sigma(s')} & \text{if } k=1,\, l=0 \\ \epsilon_0(\sigma(u)) & \text{if } k=1,\, l=1 \end{cases}$$

which are equal.

- if $\lambda(b) \neq *$ and $\lambda(a) = *$ (notice then that we have $\sigma(s') = \sigma(s)$),

$$\begin{aligned} d_l^k(g_2(ab_{s,s',s'',u})) &= d_l^k(\epsilon_1(\lambda(b)_{\sigma(s),\sigma(s'')})) \\ &= \begin{cases} \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=0 \\ \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=0,\, l=1 \\ \epsilon_0(\sigma(s'')) & \text{if } k=1,\, l=0 \\ \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=1,\, l=1 \end{cases} \end{aligned}$$

We also have,

$$g_1(d_l^k(ab_{s,s',s'',u})) = \begin{cases} g_1(a_{s,s'}) = \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=0 \\ g_1(b_{s,s''}) = \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=0,\, l=1 \\ g_1(a_{s'',u}) = \epsilon_0(\sigma(s'')) & \text{if } k=1,\, l=0 \\ g_1(b_{s',u}) = \lambda(b)_{\sigma(s'),\sigma(u)} = \lambda(b)_{\sigma(s),\sigma(s'')} & \text{if } k=1,\, l=1 \end{cases}$$

which are equal.

- if $\lambda(a) = *$ and $\lambda(b) = *$ (notice that then we have $\sigma(s) = \sigma(s') = \sigma(s'') = \sigma(u)$),

$$\begin{aligned} d_l^k(g_2(ab_{s,s',s'',u})) &= d_l^k(\epsilon_0\epsilon_0(\sigma(s))) \\ &= \begin{cases} \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=0 \\ \epsilon_0(d_0^0(\epsilon_0(\sigma(s)))) = \epsilon_0(\sigma(s)) & \text{if } k=0,\, l=1 \\ \epsilon_0(\sigma(s)) & \text{if } k=1,\, l=0 \\ \epsilon_0(d_0^1(\epsilon_0(\sigma(s)))) = \epsilon_0(\sigma(s)) & \text{if } k=1,\, l=1 \end{cases} \end{aligned}$$

We also have,

$$g_1(d_l^k(ab_{s,s',s'',u})) = \begin{cases} g_1(a_{s,s'}) = \epsilon_0(\sigma(s)) & \text{if } k=0,\ l=0 \\ g_1(b_{s,s''}) = \epsilon_0(\sigma(s)) & \text{if } k=0,\ l=1 \\ g_1(a_{s'',u}) = \epsilon_0(\sigma(s'')) = \epsilon_0(\sigma(s)) & \text{if } k=1,\ l=0 \\ g_1(b_{s',u}) = \epsilon_0(\sigma(s')) = \epsilon_0(\sigma(s)) & \text{if } k=1,\ l=1 \end{cases}$$

which are equal.

For $\mathcal{W}$ we have to check that, for $f = (\sigma, \lambda) = \mathcal{Y}(g : (P, l : P \to L, i) \to (P', l' : P' \to L', i'))$,

$$aI_s b \text{ and } \lambda(a) \neq *,\ \lambda(b) \neq * \text{ implies } \lambda(a) I'_{\sigma(s)} \lambda(b)$$

Suppose aI_sb in $\mathcal{Y}(P, l : P \to L, i)$. Then there exist $x, x', y, y' \in P_1$ with $l(x) = a$, $l(x') = a$, $l(y) = b$, $l(y') = b$ and $d_0^0(x) = d_0^0(y) = s$, $d_0^1(x) = d_0^0(y')$, $d_0^1(y) = d_0^0(x')$, $d_1^1(y') = d_1^1(x')$, and we have a $C \in P_2$ with $l(C) = (a, b)$, $d_0^0(C) = x$, $d_1^0(C) = y$, $d_0^1(C) = y'$ and $d_1^1(C) = x'$. We know that $g(C) \in P'_2$ and that $l' \circ g(C) = (f(a), f(b))$ since $f(a) \neq *$ and $f(b) \neq *$. Similarly, $l'(g(x)) = f(a)$, $l'(g(x')) = f(a)$, $l'(g(y)) = f(b)$, $l'(g(y')) = f(b)$. Furthermore, because g is a morphism of cubical sets, $d_0^0(g(x)) = d_0^0(g(y)) = \sigma(s)$, $d_0^1(g(x)) = d_0^0(g(y'))$, $d_0^1(g(y)) = d_0^0(g(x'))$, $d_1^1(g(y')) = d_1^1(g(x'))$, so $\lambda(a) I'_{\sigma(s)} \lambda(b)$.

It is easy to see that these functors restricted to the 1-skeleton are inverse of each other (this is the consequence of Theorem 1). Now more generally, it is easy to check that $\mathcal{W} \circ \mathcal{Y} = Id$.

Finally, for all free 2-dimensional cubical sets (from precubical sets) (P, l, j), $\mathcal{Y} \circ \mathcal{W}(P, l, j)$ is naturally equal to (P, l, j).