Components of the Fundamental Category II

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Abstract. In this article we carry on the study of the fundamental category (Goubault and Raussen, 2002; Goubault, 2003) of a partially ordered topological space (Nachbin, 1965; Johnstone, 1982), as arising in e.g. concurrency theory (Fajstrup et al., 2006), initiated in (Fajstrup et al., 2004). The "algebra" of dipaths modulo dihomotopy (the fundamental category) of such a po-space is essentially finite in a number of situations. We give new definitions of the component category that are more tractable than the one of (Fajstrup et al., 2004), as well as give definitions of future and past component categories, related to the past and future models of (Grandis, 2005). The component category is defined as a category of fractions, but it can be shown to be equivalent to a quotient category, much easier to portray. A van Kampen theorem is known to be available on fundamental categories (Grandis, 2003; Goubault, 2003), we show in this paper a similar theorem for component categories (conjectured in (Fajstrup et al., 2004)). This proves useful for inductively computing the component category in some circumstances, for instance, in the case of simple PV mutual exclusion models (Goubault and Haucourt, 2005), corresponding to partially ordered subspaces of $\mathbb{R}^n$ minus isothetic hyperrectangles. In this last case again, we conjecture (and give some hints) that component categories enjoy some nice adjunction relations directly with the fundamental category.

Keywords: partially ordered space, po-space, dihomotopy, fundamental category, category of fractions, component, *Yoneda* morphism, *Yoneda* system, pure system

1. Introduction

Partially ordered spaces or *po-spaces* appeared in (Eilenberg, 1941). Motivated by functional analysis, several results from general topology have been extended to po-spaces by *L.Nachbin* (Nachbin, 1965). In the meantime, *E.W. Dijkstra* has introduced the notion of *progress graphs*, a particular case of po-spaces, as a natural model for concurrency (Dijkstra, 1968).

The main motivation of this paper is to apply methods from algebraic topology, after several suitable modifications, to classify parallel programs via their geometric representation. Concurrent processes naturally define po-spaces, whose points are states of the parallel machine and the partial-order is the causal ordering. Let us recap the now classical example of (Fajstrup et al., 2004), where two processes share two resources a and b:

$$T1 = Pa.Pb.Vb.Va$$

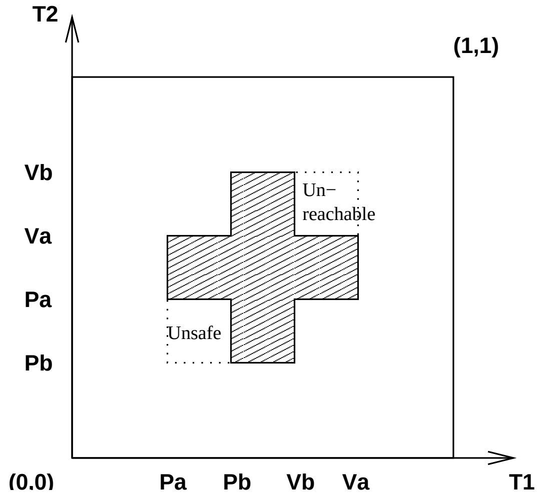


Figure 1. The Swiss Flag example - two processes sharing two resources

$$T2 = Pb.Pa.Va.Vb$$

the geometric model is the "Swiss flag", Figure 1, regarded as a subset of $\mathbb{R}^2$ with the componentwise partial order $(x_1, y_1) \leq (x_2, y_2)$ if $x_1 \leq y_1$ and $x_2 \leq y_2$. The (interior of the) horizontal dashed rectangle comprises global states that are such that T_1 and T_2 both hold a lock on a: this is impossible by the very definition of a binary semaphore. Similarly, the (interior of the) vertical rectangle consists of states violating the mutual exclusion property on b. Therefore both dashed rectangles form the *forbidden region*, which is the complement of the space X of (legal) states.

This space with the inherited partial order provides us with a particular po-space X, as defined in Section 2. Moreover, legal execution paths, called *dipaths*, are increasing maps from the po-space $\overrightarrow{I}$ (the unit segment with its natural order) to X. The partial order on X thus reflects (at least) the time ordering on all possible execution paths.

Po-spaces also appear in several other contexts, all having their own mathematical interest:

- The positive cone $\mathcal{P}$ of any C^*-algebra $\mathcal{A}$ is naturally provided with an order relation $\sqsubseteq$ and thus becomes a po-space (Takesaki, 2002). Because $\mathcal{P}$ is a convex subset of $\mathcal{A}$, its fundamental category is isomorphic to the poset $(P, \sqsubseteq)$, where P is the underlying set of $\mathcal{P}$. Moreover, *Sherman*'s theorem claims the order $\sqsubseteq$ is a lattice if and only if the C^*-algebra $\mathcal{A}$ is abelian. Since a poset is a lattice if and only if its category of components (as introduced in (Fajstrup et al.,

2004) and fully worked out in this paper) is trivial (Haucourt, 2007), the commutativity of a C^*-algebra is characterized by the triviality of the category of components of its positive cone.

- Any physically reasonable spacetime (i.e. time-oriented connected *Lorentz* manifold or globally hyperbolic spacetime) ordered by causality (Dodson and Poston, 1997; Hawkins and Ellis, 1973) is a po-space whose category of components can be thus seen as an abstraction of the metric and differential structure preserving causal information. In the same stream of ideas, *K. Martin* and *P. Panangaden* have given an abstraction of any physically reasonable spacetime M, based on causality, which is rich enough to recover the topology of M from it (Martin and Panangaden, 2005).

- Given a closed Riemannian manifold M and a Morse-Smale function $f : M \to \mathbb{R}$, Smale (Smale, 1961) defines a partial order $<$ on the set of critical points of f as follows: $a < b$ if and only if there is a flow line from a to b (which is a sub-partial order of the po-space with the same order, but for all points of M). Note that, as noticed by an anonymous referee, there should be some connections, yet to formalize, between our notion of component category, and the generalization of the $<$-partial order above, introduced in (Cohen et al., 1995): the category C_f whose objects are critical points of f, and whose morphisms between two critical points a and b are in some sense "piecewise flow lines" of the gradient flow of f which connect a to b.

To study progress graphs and po-spaces, we easily turn the notion of fundamental groupoid of a topological space (Higgins, 1971) into the notion of fundamental (loop-free) category of a po-space (Grandis, 2003; Goubault and Raussen, 2002; Goubault, 2003). The next step consists in computing these invariants automatically (Goubault and Haucourt, 2005). In order to do so, we need to "discretize" the representation of fundamental categories which often have uncountably many objects. In classical algebraic topology, the fundamental groupoid of a space X is entirely determined by the set of its arcwise connected components and their fundamental groups which are, for spaces in our scope of interest, finitely generated. The problem becomes highly more intricate in the case of the fundamental category of a po-space. Though the construction described in (Fajstrup et al., 2004) provides the expected discretization in all "concrete" examples, it has severe theoretical drawbacks, one of which is that we do not know whether the construction makes sense in general cases (including all progress graphs for instance). By slightly reformulating the definition of component

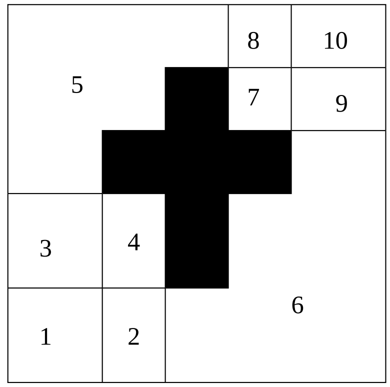
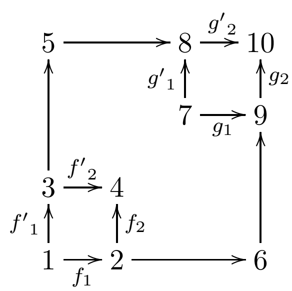


Figure 2. The components of the Swiss flag

categories, we obtain much stronger results, although ending up with the same component categories, at least in the case of progress graphs[1].

For instance, the resulting components of the po-space of Figure 1 are shown on Figure 2 while its category of components is depicted by the following diagram, is obtained with our new definition (Definition 5), and is the same as the one given as an example in (Fajstrup et al., 2004):



together with relations $g'_2 \circ g'_1 = g_2 \circ g_1$ and $f'_2 \circ f'_1 = f_2 \circ f_1$.

Organization and contributions of the paper: We summarize the main (classical) definitions of po-space, dihomotopy and fundamental category in Section 2. We then introduce the first contribution of the paper: Definition 5, which gives a set of axioms for "*Yoneda*-systems". A particular *Yoneda*-system of the fundamental category will be used to define the component category, as the category of fractions of the fundamental category with respect to this *Yoneda*-system, Definition 6.

We show that the family of *Yoneda*-systems of a particular type of small categories (including fundamental categories of po-spaces),

[1] or PV models which are special instances of progress graphs

admits a maximal element under the subset ordering (of sets of morphisms), consequence of Theorem 1: this will be the *Yoneda*-system of interest for Definition 6. This theorem asserts even more in that it shows that the set of *Yoneda*-systems, together with subset inclusion, forms a locale: this is the second main contribution of this paper.

The third major contribution of the paper is Corollary 2 in which we show that the component category just defined could instead have been defined as a (much more "practical") quotient construction, defined in Section 4.1 and Section 4.2, after the work of (Bednarczyk et al., 1999).

The fourth contribution of the paper is a van Kampen theorem for component categories, Proposition 4, which allow for practical, inductive computations.

In Section 6 we refine our understanding of component categories by splitting the axioms defining *Yoneda*-systems in two parts: one for "future components" (*Yoneda*-f-systems), the other for "past components" (*Yoneda*-p-systems). They still enjoy interesting lifting properties, like for component categories, Proposition 6 (this is the fifth major contribution of the paper).

Last but not least, and sixth contribution of this article, we show in Theorem 5 that a category very similar to the component category (that we conjecture to be equivalent again), the orthogonal subcategory of a category $\mathcal{C}$ with respect to its biggest *Yoneda*-f-systems, is reflective in $\mathcal{C}$.

Note: In order to avoid any confusion with the notion of weak equivalences in algebraic topology, the terminology "weak(ly) invertible morphism", "system of weak equivalences", that was used in (Fajstrup et al., 2004), is replaced by "*Yoneda* morphism" and "*Yoneda* system". We also have slightly modified the notion of directed homotopy of (Fajstrup et al., 2004), to take the one of (Grandis, 2003).

For a small category $\mathcal{C}$, $Ob(\mathcal{C})$ (respectively $Mo(\mathcal{C})$) denotes the set of objects (respectively, morphisms) of $\mathcal{C}$. For a morphism $f \in Mo(\mathcal{C})$ $src(f)$ (respectively $tgt(f)$) denotes the source of f (respectively the target of f).

2. Basic definitions

The framework for the applications we have in mind is mostly based on the simple notion of a po-space:

DEFINITION 1.

1. *A po-space is a topological space X with a (global) closed partial order $\leq$ (i.e. $\leq$ is a closed subset of $X \times X$). The unit segment* $[0,1]$

with the usual topology and the usual order, is a po-space denoted $\overrightarrow{I}$ and called the directed interval.

2. *A dimap $f : X \to Y$ between po-spaces X and Y is a continuous map that respects the partial orders (is non-decreasing).*

3. *A dipath $f : \overrightarrow{I} \to X$ is a dimap whose source is the interval $\overrightarrow{I}$.*

Po-spaces and dimaps form a complete and co-complete category denoted PoTop, see (Haucourt, 2005). To a certain degree, our methods apply to the more general categories of lpo-spaces (Fajstrup et al., 2006) (with a local partial order), of flows (Gaucher, 2002) and of d-spaces (Grandis, 2003), but for the sake of simplicity, we stick to po-spaces in the present paper.

Dihomotopies between dipaths f and g (with fixed extremities α and β in X) are dimaps $H : \overrightarrow{I} \times \overrightarrow{I} \to X$ such that for all $x \in \overrightarrow{I}$, $t \in \overrightarrow{I}$,

$$H(x, 0) = f(x),\ H(x, 1) = g(x),\ H(0, t) = \alpha,\ H(1, t) = \beta.$$

Given two directed paths f and g, we write $f \sqsubseteq g$ when there exists a dihomotopy from f to g, the relation $\sqsubseteq$ thus define a partial order on the collection of directed paths of a given po-space $\overrightarrow{X}$. Then we define $\sim_{dih}$ as the equivalence relation induced by $\sqsubseteq$, its equivalence classes are called dihomotopy classes. Let us insist on the important fact that, given two directed paths f and g and their underlying continuous path f' and g', we might have a classical homotopy between f' and g' though $f \not\sim_{dih} g$. Furthermore, this situation is rather common, and is not by any means an exception.

Now, we can define the main object of study of this paper:

DEFINITION 2. *The fundamental category is the category $\overrightarrow{\pi_1}(\overrightarrow{X})$ with:*

– *as objects: the points of X,*

– *as morphisms, the dihomotopy classes of dipaths: a morphism from x to y is a dihomotopy class $[f]$ of a dipath f from x to y.*

Concatenation of dipaths factors over dihomotopy and yields the composition of morphisms in the fundamental category. A dimap $f : X \to Y$ between po-spaces induces a functor $f_\# : \overrightarrow{\pi_1}(\overrightarrow{X}) \to \overrightarrow{\pi_1}(\overrightarrow{Y})$, and we obtain thus a functor $\overrightarrow{\pi_1}$ from the category of po-spaces to the category of categories. The fundamental category of a po-space generalizes the fundamental groupoid $\pi_1(X)$ of a topological space X (same set of objects as $\overrightarrow{\pi_1}(\overrightarrow{X})$; morphisms from x to y are homotopy classes of paths from x to y). As indicated by its name, the fundamental groupoid of a topological space is always a small category in which any morphism is an isomorphism. It follows that π_1 is a functor from Top (the category of topological spaces and continuous maps between them) to Grpd (the category of groupoids and functors between them). Given a po-space $\overrightarrow{X}$, one can remark that $\overrightarrow{\pi_1}(\overrightarrow{X})$ satisfies:

– $\forall x \in \overrightarrow{\pi_1}(\overrightarrow{X}) \;\; \overrightarrow{\pi_1}(\overrightarrow{X})[x, x] = \{id_x\}$

– $\forall x, y \in \overrightarrow{\pi_1}(\overrightarrow{X}) \;\; x \neq y \;\Rightarrow\; (\overrightarrow{\pi_1}(\overrightarrow{X})[x, y] = \emptyset \text{ or } \overrightarrow{\pi_1}(\overrightarrow{X})[y, x] = \emptyset)$

Any category $\mathcal{L}$ satisfying these properties are called **loop-free**[2]. The small loop-free categories and functor between them form an epi-reflective subcategory of Cat denoted LfCat. Hence $\overrightarrow{\pi}_1$ is a functor from PoTop to LfCat. In other words, the fundamental category of a po-space is loop-free, this property will be helpful in the sequel.

The fundamental category is often an enormous gadget (with uncountably many objects and morphisms) and possesses less structure than a group. It is the aim of this paper to "shrink" the essential information in the fundamental category to an associated component category, that in many cases is finite and possesses a comprehensible structure.

3. A convenient framework for components

The definition of components given in (Fajstrup et al., 2004) is strengthened in this section, and, from a theoretical point of view, improved. We first define "*Yoneda* systems" in Section 3.1 and then consider the category of fractions based on the fundamental category, where we invert the morphisms of the maximal (see Section 3.2) *Yoneda*-system, Definition 6.

"Pureness", that was required as an axiom in (Fajstrup et al., 2004) to have in particular the lifting property, Proposition 7 of (Fajstrup et al., 2004) (and recapped as Proposition 5 in this article), becomes a fairly easy consequence of the new set of axioms for *Yoneda* systems, Definition 3. Recall that a subcategory $\Sigma \subseteq \mathcal{C}$ is pure if for all morphisms $f \in \Sigma$, whenever $f = g \circ h$ with $g, h \in \mathcal{C}$, g and h necessarily belong to Σ.

[2] Appeared in (Haefliger, 1992) as "small categories without loops" or "scwols". We also refer the reader to (Bridson and Haefliger, 1999) for details.

3.1. Definition of *Yoneda* systems

First we recall the definition of *Yoneda* (invertible) morphisms (Fajstrup et al., 2004): it expresses basic requirements for morphisms to "bring no information", leading to the lifting property, Proposition 7 of (Fajstrup et al., 2004) and Proposition 5 of this article.

DEFINITION 3. *Given a (small) category* $\mathcal{C}$*, a morphism* $x \xrightarrow{\sigma} y \in \mathcal{C}$ *is* Yoneda (invertible) morphism *when for each object* z *of* $\mathcal{C}$ *such that* $\mathcal{C}[y,z] \neq \emptyset$*, the following map*[3]*:*

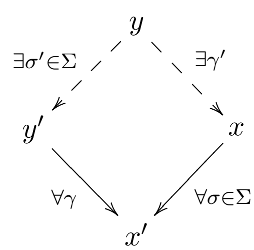
$$\mathcal{C}[y,z] \xrightarrow{-\circ\sigma} \mathcal{C}[x,z] \text{ is a } \underline{\text{bijection}},$$

and for each object z *of* $\mathcal{C}$ *such that* $\mathcal{C}[z,x] \neq \emptyset$*, the following map*[4]*:*

$$\mathcal{C}[z,x] \xrightarrow{\sigma\circ-} \mathcal{C}[z,y] \text{ is a } \underline{\text{bijection}}.$$

As showed in (Fajstrup et al., 2004), *Yoneda* morphisms do not necessarily make good calculi of fractions, i.e. calculi having right and/or left extension properties as defined below:

DEFINITION 4. **R***ight* **E***xtension* **P***roperty*
Σ *has the right extension property with respect to* $\mathcal{C}$ *iff for all* $\gamma : y' \longrightarrow x'$*, for all* $\sigma : x \longrightarrow x' \in \Sigma$*, there exists* $\sigma' : y \longrightarrow y' \in \Sigma$*, there exists* $\gamma' : y \longrightarrow x$ *such that* $\sigma \circ \gamma' = \gamma \circ \sigma'$*, i.e. the following diagram is commutative:*



Left **E**xtension **P**roperty is obtained by "dualizing" Definition 4

This can be fixed, see Lemma 5 of (Fajstrup et al., 2004), by restricting ourselves to the maximal subset of *Yoneda* morphisms which has REP and LEP. Unfortunately, this does not provide us with a pure calculus in general. To circumvent these problems, we strengthen LEP and REP so that to have canonical extensions, by pushouts and pullbacks:

[3] This is a form of preservation of the <u>future</u> cone.
[4] This is a form of preservation of the <u>past</u> cone.

DEFINITION 5. *Let $\mathcal{C}$ be a small category, $\Sigma \subseteq Mo(\mathcal{C})$ is a* Yoneda-*system if and only if:*

(A1) Σ *is stable under composition (of* $\mathcal{C}$*)*

(A2) $Iso(\mathcal{C}) \subseteq \Sigma \subseteq Yoneda(\mathcal{C})$[5]

(A3) Σ *is stable under pushouts (with any morphism in* $\mathcal{C}$*).*

(A4) Σ *is stable under pullbacks (with any morphism in* $\mathcal{C}$*).*

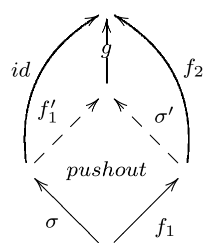
The last two points mean that Σ has both **REP** *and* **LEP** *with respect to $\mathcal{C}$ and further the commutative squares provided by* **REP** *and* **LEP** *can be chosen in order to be respectively* **pullback** *and* **pushout** *squares in $\mathcal{C}$.*

Let us mention here that (private communication of Lisbeth Fajstrup, Aalborg University), most of the following results would still hold with weaker axioms, in particular, one can ask for only having *weak* pushouts (no unicity required, only finiteness).

Any *Yoneda*-system of any small category $\mathcal{C}$ is pure and has left and right extension properties:

LEMMA 1. *Let $\mathcal{C}$ be a small category such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. Then any* Yoneda-*system of $\mathcal{C}$ is pure in $\mathcal{C}$.*

PROOF. Take $\sigma \in \Sigma$ and $f_1, f_2 \in Mo(\mathcal{C})$ such that $\sigma = f_2 \circ f_1$. By (A3) of Definition 5, we have a $\sigma' \in \Sigma$ and f_1' which form a pushout square and a unique $g \in Mo(\mathcal{C})$ making the following diagram commutative.



By pureness of $Iso(\mathcal{C})$ in $\mathcal{C}$, f_1' and g are isomorphisms, hence by (A2) of definition 5, belong to Σ. So by (A1) of Definition 5, $f_2 = g \circ \sigma' \in \Sigma$. In the same way, using the pullback (instead of pushout) extension property, one proves that $f_1 \in \Sigma$. Thus Σ is pure in $\mathcal{C}$. ■

[5] $Iso(\mathcal{C})$ and $Yoneda(\mathcal{C})$ are subcategories of $\mathcal{C}$ respectively generated by isomorphisms and *Yoneda* invertible morphisms of $\mathcal{C}$.

3.2. Locale of *Yoneda*-systems

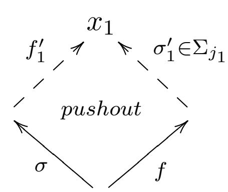
In this section, we give several results which will be combined to prove that the collection of *Yoneda*-systems of a small category $\mathcal{C}$ such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ forms a locale (Theorem 1). We recall that a **locale** is a poset $(L, \leq_L)$ such that for all $U \subseteq L$, U has a least upper bound and a greatest lower bound (it is a complete lattice) and $\forall (b_j)_{j \in J} \in L^J\ \forall a \in L,\ a \wedge \left(\bigvee_{j \in J} b_j\right) = \bigvee_{j \in J}(a \wedge b_j)$ (see (Borceux, 1994b) or (Johnstone, 1982)).

LEMMA 2. *Let $\mathcal{C}$ be a small category, the collection $Iso(\mathcal{C})$ is a* Yoneda-*system of $\mathcal{C}$.*

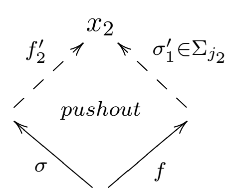
Proof. It is routine verification which does not involve more category theory than the fact that the pushout (respectively the pullback) of an isomorphism along any morphism is an identity. ■

LEMMA 3. *If $(\Sigma_j)_{j \in J}$ is a non empty family of* Yoneda *systems of a small category $\mathcal{C}$ then $\bigcap_{j \in J} \Sigma_j$ is a* Yoneda-*system of $\mathcal{C}$.*

Proof. $\bigcap_{j \in J} \Sigma_j$ obviously enjoys (A1) and (A2) of Definition 5. Suppose $\sigma \in \bigcap_{j \in J} \Sigma_j$ and $f \in Mo(\mathcal{C})$ with $src(f) = src(\sigma)$. Take $j_1, j_2 \in J$, since $\sigma \in \Sigma_{j_1}$ we have a pushout square

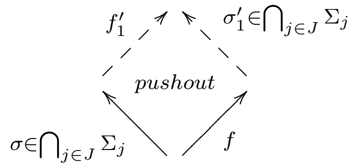


and also



because $\sigma \in \Sigma_{j_2}$. By uniqueness (up to isomorphism) of the pushout, we have an isomorphism τ from x_2 to x_1 such that $\sigma'_1 = \tau \circ \sigma'_2$. By (A2) of Definition 5, $\tau \in \Sigma_{j_2}$ which is stable under composition by (A1), thus $\sigma'_1 = \tau \circ \sigma'_2 \in \Sigma_{j_2}$. By the same argument, for all $j \in J, \sigma'_1 \in \Sigma_j$

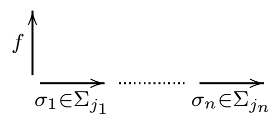
i.e. $\sigma'_1 \in \bigcap_{j\in J} \Sigma_j$ and we have



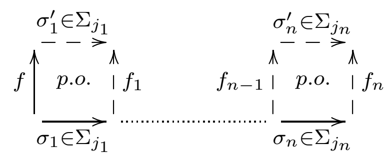
The same proof holds for pullback squares. ■

LEMMA 4. *If* $(\Sigma_j)_{j\in J}$ *is a* *non empty* *family of* Yoneda *systems of a small category* $\mathcal{C}$ *then* $\biguplus_{j\in J} \Sigma_j$ *is a* Yoneda*-system of* $\mathcal{C}$*, where* $\biguplus_{j\in J} \Sigma_j$ *is the least sub-category of* $\mathcal{C}$ *including all the* Σ_j*'s.*

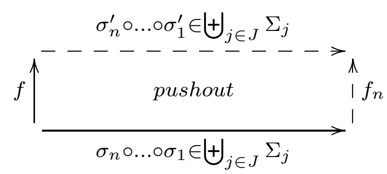
PROOF. By definition, $\biguplus_{j\in J} \Sigma_j = \{\sigma_n \circ \ldots \circ \sigma_1 \mid n \in \mathbb{N}^* , \{j_1, \ldots, j_n\} \subseteq J$ *and* for all $k \in \{1, \ldots, n\}$, $\sigma_k \in \Sigma_{j_k}\}$, property (A1) of Definition 5 immediately follows. The second one is obvious for the family is non empty and because a composition of *Yoneda* invertible morphisms is *Yoneda* invertible. Take $\sigma_n \circ \ldots \circ \sigma_1 \in \biguplus_{j\in J} \Sigma_j$ with $n \in \mathbb{N}^*$, $\{j_1, \ldots, j_n\} \subseteq J$, for all $k \in \{1, \ldots, n\}$, $\sigma_k \in \Sigma_{j_k}$ and $f \in Mo(\mathcal{C})$ with $src(\sigma_1) = src(f)$. We have



By a finite induction (apply consecutively (A3) of Definition 5 for $\Sigma_{j_1}, \ldots, \Sigma_{j_n}$), we have



Now, it is a general fact that a "composition" of push-out squares is a push-out square (see (Mac Lane, 1971; Borceux, 1994a)) hence



This works analogously for pullback squares, thus property (A4) of Definition 5 is satisfied. ■

LEMMA 5. *Let $\mathcal{C}$ be a (small) category. If $\mathcal{A}$ is a <u>pure</u> subcategory $\mathcal{C}$ then for all families $(\mathcal{C}_j)_{j\in J}$ of subcategories of $\mathcal{C}$, $\mathcal{A} \cap \left(\biguplus_{j\in J} \mathcal{C}_j\right) = \biguplus_{j\in J}(\mathcal{A} \cap \mathcal{C}_j)$*

PROOF. The inclusion $\mathcal{A} \cap \left(\biguplus_{j\in J} \mathcal{C}_j\right) \supseteq \biguplus_{j\in J}(\mathcal{A} \cap \mathcal{C}_j)$ is always satisfied. Indeed, if f is an element of the right member, then one has $n \in \mathbb{N}^*$, $\{j_1, \dots, j_n\} \subseteq J$, for all $k \in \{1, \dots, n\}$, $\sigma_k \in \mathcal{A} \cap \Sigma_{j_k}$ and $f = \sigma_n \circ \dots \circ \sigma_1$. Now $\mathcal{A}$ is a subcategory of $\mathcal{C}$ and in particular, for all $k \in \{1, \dots, n\}$, $\sigma_k \in \mathcal{A}$, hence $f \in Mo(\mathcal{A})$. Conversely, suppose that we have $n \in \mathbb{N}^*$, $\{j_1, \dots, j_n\} \subseteq J$, for all $k \in \{1, \dots, n\}$, $\sigma_k \in \Sigma_{j_k}$ and $f = \sigma_n \circ \dots \circ \sigma_1 \in Mo(\mathcal{A})$, by pureness of $\mathcal{A}$, $\sigma_n, \dots, \sigma_1 \in Mo(\mathcal{A})$, then for all $k \in \{1, \dots, n\}$, $\sigma_k \in \mathcal{A} \cap \Sigma_{j_k}$ and f is an element of the left member. ■

In fact, having

$$\mathcal{A} \cap \left(\biguplus_{j\in J} \mathcal{C}_j\right) = \biguplus_{j\in J}(\mathcal{A} \cap \mathcal{C}_j)$$

is equivalent to the existence of the right adjoint of the functor $\mathcal{A}\cap\_$: $(\{\text{subcategories of } \mathcal{C}\}, \subseteq) \longrightarrow (\{\text{subcategories of } \mathcal{C}\}, \subseteq)$, where the continuous lattice $(\{\text{subcategories of } \mathcal{C}\}, \subseteq)$ is seen as a complete and co-complete small category. The equivalence directly comes from the special adjoint functor theorem, see (Freyd and Scedrov, 1990). This equivalence is related to the link between locales and complete **Heyting algebras**, see (Borceux, 1994b) for further details.

COROLLARY 1. *Let $(\Sigma_j)_{j\in J}$ be a family of* Yoneda-*systems of a small category $\mathcal{C}$ such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$ and Σ a* Yoneda-*system of $\mathcal{C}$. Then $\Sigma \cap \left(\biguplus_{j\in J} \Sigma_j\right) = \biguplus_{j\in J}(\Sigma \cap \Sigma_j)$.*

PROOF. By Lemma 1, Σ is pure in $\mathcal{C}$, the result follows by Lemma 5. ■

REMARK 1. *$\bigcap$ and $\biguplus$ are associative over the family of subcategories of a small category $\mathcal{C}$.*

We can now state:

THEOREM 1. *Let $\mathcal{C}$ be a small category such that $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. Then, the family of* Yoneda-*systems of $\mathcal{C}$ is not empty and, together with $\subseteq$ it forms a <u>locale</u> whose l.u.b. operator is $\biguplus$ and g.l.b operator is $\bigcap$. Moreover, the least element of this locale ("bottom") is $Iso(\mathcal{C})$.*

Proof. The collection of *Yoneda*-systems of $\mathcal{C}$ has a least element by Lemma 2 and the other axioms of a locale are given by Lemmas 3, 4 and Corollary 1. ■

Remark that Lemmas 2, 3 and 4 are satisfied for any small category $\mathcal{C}$, thus proving that the collection of *Yoneda*-systems of a small category is not empty and, ordered by inclusion, forms a complete lattice; the pureness of $Iso(\mathcal{C})$ is only involved in Lemma 5 and its Corollary 1.
As explained in (Borceux, 1994b) and (Johnstone, 1982), the notion of locale generalizes the notion of family of open subsets of a topological space, thus, Theorem 1 gives us a kind of topology over $\mathcal{C}$ as soon as $Iso(\mathcal{C})$ is pure in $\mathcal{C}$. This pureness hypothesis is actually very "natural". If we think about it, we want to consider an isomorphism of $\mathcal{C}$ as a directed path, which is the case when $\mathcal{C}$ is a fundamental category. It makes sense geometrically to expect that all its subpaths are also dipaths i.e. are isomorphisms.

3.3. Components categories

We are now in position to fully define the main mathematical notion we are describing in this article. We have seen that *Yoneda* systems form a locale. Calling its greatest element $\top$:

DEFINITION 6. *The component category of a po-space $\overrightarrow{X}$ is defined as the category of fractions (see (*Gabriel and Zisman, 1967*)) $\overrightarrow{\pi_1}(\overrightarrow{X})[\top^{-1}]$.*

Because of the obvious analogy with the set of arc-wise connected components of a topological space, we denote the component category of a po-space as $\overrightarrow{\pi_0}(\overrightarrow{X})$.

4. Relevant reduction of the size of a loop-free category

The previous section gives a theoretically satisfactory definition of the component category. Still, it remains to show that it is a useful notion for our purposes. The component category is designed to reduce the size of the fundamental category without losing any "relevant" information. The following result formalizes this idea and confirms an intuition that was shared by the authors from the beginning.

Recalling in Section 4.1 the notion of generalized congruence (introduced originally in (Bednarczyk et al., 1999) applied in Section 4.2 to define quotients of small categories by one of its subcategories, we can finally re-define the component category as the quotient of the fundamental category by the biggest *Yoneda* system, Section 4.3.

4.1. Generalized congruences

This section is devoted to generalized congruences that have been formalized in (Bednarczyk et al., 1999).

DEFINITION 7 (Generalized Congruences). *A generalized congruence on a small category $\mathcal{C}$ is an equivalence relation $\sim_o$ on $Ob(\mathcal{C})$ and a partial equivalence relation $\sim_m$ on $Mo(\mathcal{C})^+$ (the set of all non-empty finite sequences of morphisms of $\mathcal{C}$), satisfying the following conditions ($\cdot$ is the usual concatenation, the α's, β's and γ's range over $Mo(\mathcal{C})$):*

- $(\beta_n, \ldots, \beta_0) \cdot (\alpha_p, \ldots, \alpha_0) \sim_m (\gamma_q, \ldots, \gamma_0) \Rightarrow tgt(\alpha_p) \sim_o src(\beta_0)$
- $(\beta_n, \ldots, \beta_0) \sim_m (\alpha_p, \ldots, \alpha_0) \Rightarrow tgt(\beta_n) \sim_o tgt(\alpha_p)$ *and* $src(\beta_0) \sim_o src(\alpha_0)$
- $x \sim_o y \Rightarrow id_x \sim_m id_y$
- $(\beta_n, \ldots, \beta_0) \sim_m (\alpha_p, \ldots, \alpha_0)$ *and* $(\delta_q, \ldots, \delta_0) \sim_m (\gamma_r, \ldots, \gamma_0)$ *and* $tgt(\beta_n) \sim_o src(\delta_0) \Rightarrow (\delta_q, \ldots, \delta_0) \cdot (\beta_n, \ldots, \beta_0) \sim_m (\gamma_r, \ldots, \gamma_0) \cdot (\alpha_p, \ldots, \alpha_0)$
- $src(\beta) = tgt(\alpha) \Rightarrow (\beta \circ \alpha) \sim_m (\beta, \alpha)$

PROPOSITION 1 (Quotient Category). *Given $(\sim_o, \sim_m)$ a generalized congruence on a small category $\mathcal{C}$, we define the* **quotient category** *$\mathcal{C}/_\sim$ by*

- $Ob(\mathcal{C}_{/\sim}) := \{[x]_{\sim_o} | x \in Ob(\mathcal{C})\}$
- $src([(\gamma_n, \ldots, \gamma_0)]_{\sim_m}) = [src(\gamma_0)]_{\sim_o}$
- $tgt([(\gamma_n, \ldots, \gamma_0)]_{\sim_m}) = [tgt(\gamma_n)]_{\sim_o}$
- $[(\beta_n, \ldots, \beta_0)]_{\sim_m} \circ [(\alpha_p, \ldots, \alpha_0)]_{\sim_m} = [(\beta_n, \ldots, \beta_0) \cdot (\alpha_p, \ldots, \alpha_0)]_{\sim_m}$

Moreover, there is a **quotient** *functor $Q_\sim : \mathcal{C} \to \mathcal{C}_\sim$, defined by $Q_\sim(x) = [x]_{\sim_o}$ and $Q_\sim(\gamma) = [\gamma]_{\sim_m}$. The functor $Q_\sim$ enjoys the following universal property, for any functor $f : \mathcal{C} \to \mathcal{C}_2$, if $\sim \subseteq \sim_f$ then there exists a unique $g : \mathcal{C}_{/\sim} \to \mathcal{C}_2$ such that $f = g \circ Q_\sim$. Still, we have the following facts :*

- *g is a monomorphism iff $\sim_f = \sim$,*
- $\sim_{Q_\sim} = \sim$,

– *$Q_\sim$ is an extremal epimorphism.*

LEMMA 6. *(*Bednarczyk et al., 1999*) Generalized congruences on a given small category, ordered by componentwise inclusion, form a complete lattice whose meets are componentwise intersections. The total relation which identifies all objects and all non-empty finite sequences of morphisms is a generalized congruence, precisely $\top$ of the lattice, while $(=_{Ob(\mathcal{C})}, \emptyset)$ is $\bot$. Thus, for an arbitrary pair of relations R_o on $Ob(\mathcal{C})$ and R_m on $Mo(\mathcal{C})^+$, there is a least generalized congruence containing (R_o, R_m).*

4.2. Quotient of a small category by one of its subcategories : $\mathcal{C}/_\Sigma$

Given Σ a subcategory of a small category $\mathcal{C}$, we can define $\mathcal{C}/\Sigma := \mathcal{C}/_\sim$ where $\sim$ is the least generalized congruence on $\mathcal{C}$ containing

$$(\emptyset, \{(id_{tgt(\sigma)}, \sigma), (\sigma, id_{src(\sigma)}) \mid \sigma \in Mo(\Sigma)\})$$

(by Lemma 6).

PROPOSITION 2 (Description and universal property of $\mathcal{C}/\Sigma$). *Given a small category $\mathcal{C}$ and $\Sigma \subseteq Mo(\mathcal{C})$, closed under composition (in fact, take Σ a subcategory of $\mathcal{C}$). Let $(\sim_{o,\Sigma}, \sim_{m,\Sigma})$ be the least generalized congruence containing $(\emptyset, \{(id_{tgt(\sigma)}, \sigma), (\sigma, id_{src(\sigma)}) | \sigma \in \Sigma\})$. Then:*

– *for all $x, y \in Ob(\mathcal{C})$, $x \sim_{o,\Sigma} y$ if and only if there is a Σ-zig-zag between x and y.*

– *for all $(\beta_n, \ldots, \beta_0), (\alpha_m, \ldots, \alpha_0)$ $\sim_{o,\Sigma}$-composable sequences (i.e. $src(\alpha_{i+1}) \sim_{o,\Sigma} tgt(\alpha_i)$ and $src(\alpha_{i+1}) \sim_{o,\Sigma} tgt(\alpha_i)$), we have*

$$(\beta_n, \ldots, \beta_0) \sim_{m,\Sigma} (\alpha_m, \ldots, \alpha_0)$$

if and only if there is a finite sequence of "elementary transformation" from $(\alpha_m, \ldots, \alpha_0)$ to $(\beta_n, \ldots, \beta_0)$, where an "elementary transformation" is either

- $(\alpha_n, \ldots, \alpha_{i+1}, \sigma, \alpha_{i-1}, \ldots, \alpha_0) \sim^1_{m,\Sigma} (\alpha_n, \ldots, \alpha_{i+1}, id_{src(\sigma)}, \alpha_{i-1}, \ldots, \alpha_0)$ *if* $\sigma \in \Sigma$

- $(\alpha_n, \ldots, \alpha_{i+1}, \sigma, \alpha_{i-1}, \ldots, \alpha_0) \sim^1_{m,\Sigma} (\alpha_n, \ldots, \alpha_{i+1}, id_{tgt(\sigma)}, \alpha_{i-1}, \ldots, \alpha_0)$ *if* $\sigma \in \Sigma$

- $(\alpha_n, \ldots, \alpha_{i+2}, \alpha_{i+1}, \alpha_i, \alpha_{i-1}, \ldots, \alpha_0) \sim^1_{m,\Sigma} (\alpha_n, \ldots, \alpha_{i+2}, \alpha_{i+1} \circ \alpha_i, \alpha_{i-1}, \ldots, \alpha_0)$ *if* $src(\alpha_{i+1}) = tgt(\alpha_i)$.

$\mathcal{C}/\Sigma$ is characterized by the following universal property:
for all $f \in \mathsf{Cat}[\mathcal{C}, \mathcal{C}']$, *if for all* $\sigma \in \Sigma, f(\sigma) = id$ *then there exists a unique* $g \in \mathsf{Cat}[\mathcal{C}/_\Sigma, \mathcal{C}']$ *such that* $f = g \circ Q_\Sigma$.

Moreover, if $\mathcal{C}_1 \xrightarrow{f} \mathcal{C}_2$ *satisfies* $f(\Sigma_1) \subseteq \Sigma_2$ *then there exists a unique* $h : \mathcal{C}_1/\Sigma_1 \longrightarrow \mathcal{C}_2/\Sigma_2$ *such that* $Q_{\Sigma_2} \circ f = h \circ Q_{\Sigma_1}$, *where* Q_Σ *is the quotient functor (refer to proposition 1) associated to the generalized congruence induced by* Σ.

The arrow h *is also denoted* $f_{/\Sigma_1,\Sigma_2}$, *and in the same stream of notation* g *is denoted* $f_{/\Sigma}$.

4.3. The component category as a quotient category

THEOREM 2. *Given a* <u>*loop-free*</u> *category* $\mathcal{L}$ *and* Σ *a* Yoneda-*system of* $\mathcal{L}$, $\mathcal{L}[\Sigma^{-1}]$ *is equivalent to* $\mathcal{L}/\Sigma$

Sketch of proof. By definition of calculus of fractions we have a (canonical) functor $I_\Sigma : \mathcal{L} \longrightarrow \mathcal{L}[\Sigma^{-1}]$ and by definition of generalized congruences, we have a (canonical) functor $Q_\Sigma : \mathcal{L} \longrightarrow \mathcal{L}/\Sigma$. By definition of Q_Σ, for all $\sigma \in \Sigma$ $Q_\Sigma(\sigma)$ is an identity of $\mathcal{L}/\Sigma$, it follows, by the universal property of I_Σ, that there is a unique functor $R_\Sigma : \mathcal{L}[\Sigma^{-1}] \longrightarrow \mathcal{L}/\Sigma$ such that $Q_\Sigma = R_\Sigma \circ I_\Sigma$.
Now we prove that R_Σ is an equivalence of category. Given an object $\overline{x}$ of $\mathcal{L}/\Sigma$ we "choose" (implicitly applying the axiom of choice) $x \in \overline{x}$ (we recall that, by definition, $\overline{x}$ is an equivalence class of $\mathcal{L}$), thus we have defined a mapping $J_\Sigma : Ob\Big(\mathcal{L}/\Sigma\Big) \longrightarrow Ob\Big(\mathcal{L}[\Sigma^{-1}]\Big)$. Given an object x of $\mathcal{L}[\Sigma^{-1}]$, $R_\Sigma(x) = Q_\Sigma(x) = \overline{x}$ since $Q_\Sigma = R_\Sigma \circ J_\Sigma$. It follows that given an object $\overline{x}$ of $\mathcal{L}/\Sigma$, $R_\Sigma(J_\Sigma(\overline{x})) = \overline{x}$. Moreover, it is a general fact that if $\mathcal{L}$ is a loop-free category and Σ is a *Yoneda*-system of $\mathcal{L}$, then $\mathcal{L}/\Sigma$ is still loop-free (see (Haucourt, 2005)). Thus, we conclude $\overline{x} \in Ob\Big(\mathcal{L}/\Sigma\Big)$. Hence the only morphism of $\mathcal{L}/\Sigma$ from $R_\Sigma(J_\Sigma(\overline{x})) = \overline{x}$ to $\overline{x}$ is $id_{\overline{x}}$.

Therefore the co-unit of the adjunction (if it exists) is necessarily $(\varepsilon_{\overline{x}} = id_{\overline{x}})_{\overline{x} \in \mathcal{L}/\Sigma}$. To check we actually have an adjunction, it suffices to prove that for all objects x and y of $\mathcal{L}$ with x satisfying $J_\Sigma(R_\Sigma(x)) = x$, the following mapping is a bijection:

$$\begin{array}{rcl} \mathcal{L}[\Sigma^{-1}][y,x] & \longrightarrow & \mathcal{L}/_\Sigma[\overline{y},\overline{x}] \\ f & \longmapsto & \varepsilon_{\overline{x}} \circ R_\Sigma(f) = R_\Sigma(f) \end{array}$$

It is obviously onto since Q_Σ is onto. The proof of the one-to-one property is omitted because of space limitations, but can be found in (Haucourt, 2005).

We thus actually have a bijection and the family $(\varepsilon_{\overline{x}} = id_{\overline{x}})_{\overline{x}\in\mathcal{L}/_\Sigma}$ is a natural transformation from $R_\Sigma \circ J_\Sigma$ to $Id_{\mathcal{L}/_\Sigma}$. To check that we have an equivalence of categories, it remains to see that the unit of the adjunction is also an isomorphism. Given $x \in Ob\Big(\pi_1(\overrightarrow{X})[\Sigma^{-1}]\Big)$, η_x is a morphism of $\pi_1(\overrightarrow{X})[\Sigma^{-1}]$ from x to $I_\Sigma(R_\Sigma(x))$ which are in the same Σ-component, hence η_x is an isomorphism of $\mathcal{L}[\Sigma^{-1}]$ (the pureness of Σ is implicitly involved).

This completes the proof except for the technicality that we signalled. ■

A complete proof of theorem 2 is available in (Haucourt, 2007) together with several corollaries and illustrating examples.

It would be interesting to know whether the component category construction is functorial or not, but it seems not to be so, as far as we know.

Finally we state:

COROLLARY 2. *Given a po-space $\overrightarrow{X}$ and Σ the biggest* Yoneda *system of $\overrightarrow{X}$, $\overrightarrow{\pi_1}(\overrightarrow{X})[\Sigma^{-1}]$ is equivalent to $\overrightarrow{\pi_1}(\overrightarrow{X})/_\Sigma$ (which is, by definition, the component category of $\overrightarrow{X}$).*

In the above Corollary, the quotient category $\overrightarrow{\pi_1}(\overrightarrow{X})/\top$, is in fact $\overrightarrow{\pi_1}(\overrightarrow{X})/\sim$ where $\sim$ is the generalized congruence generated by $\sigma \sim id$ for all $\sigma \in \top$.

From a computer science point of view, Corollary 2 is exactly what was expected of component categories. It gives a "smaller" model of X, as far as dipaths modulo dihomotopy are concerned.

For example, considering the example of the square with a hole, $\overrightarrow{\pi_1}(\overrightarrow{X})$ has the size of a continuum while its component category is finite. It remains to establish an algorithm to determine, at least in the cubical cases, the component category of a po-space (see (Goubault and Haucourt, 2005) for the first few steps in that direction).

5. van Kampen theorem for component categories

The following proposition shows that loop-freeness is well-behaved with respect to quotients. It is in fact necessary in the full proof of Theorem 2.

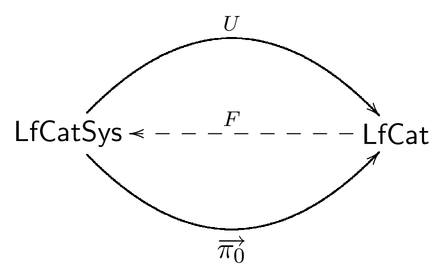
PROPOSITION 3. *Let* $\mathcal{C}$ *be a small category and* Σ *a wide subcategory of* $\mathcal{C}$. **If** $\mathcal{C}$ *is loop-free and* Σ *is a pure subcategory of* Yoneda *morphisms admitting the left and right extension properties* **then** $\mathcal{C}/\Sigma$ *is loop-free.*

The proof of Proposition 3 is given in (Haucourt, 2005). Theorem 3 gives the general framework in which the construction of the component category is **functorial**. The idea is to equip any small category $\mathcal{C}$ in our scope of interest with a subcategory of distinguished morphisms (called "inessential" in (Fajstrup et al., 2004)) which are informally those along which "nothing happens". The theorem will be applied with *Yoneda*-systems as subcategories of distinguished morphisms.

Let us denote LfCatSys the category whose objects are the couples $(\mathcal{C}, \Sigma)$ where Σ is a *Yoneda* system over $\mathcal{C}$ and the morphisms from $(\mathcal{C}_1, \Sigma_1)$ to $(\mathcal{C}_2, \Sigma_2)$ are the elements of $\mathsf{LfCat}[\mathcal{C}_1, \mathcal{C}_2]$ such that $f(\Sigma_1) \subseteq \Sigma_2$. In this case, for any couple $(\mathcal{C}, \Sigma)$, we choose a representative of the quotient of $\mathcal{C}$ by Σ that we denote $\overrightarrow{\pi_0}(\mathcal{C}, \Sigma)$; if f is a morphism of LfCatSys from $(\mathcal{C}_1, \Sigma_1)$ to $(\mathcal{C}_2, \Sigma_2)$, then by definition, $\overrightarrow{\pi_0}(f)$ is the unique small functor g from $\overrightarrow{\pi_0}(\mathcal{C}_1, \Sigma_1)$ to $\overrightarrow{\pi_0}(\mathcal{C}_2, \Sigma_2)$ such that $Q_{\Sigma_2} \circ f = g \circ Q_{\Sigma_1}$: the small functors Q_{Σ_1}, Q_{Σ_2} and g are given by the universal property of the quotients and the functoriality of $\overrightarrow{\pi_0}$ is thus a consequence of the uniqueness of g. Recall that the isomorphisms of a loop-free category are its identities. Let us denote U the forgetful functor from LfCatSys to LfCat.
Given any object $\mathcal{C}$ of LfCat, we set $F(\mathcal{C}) := (\mathcal{C}, Iso(\mathcal{C}))$ and $\overrightarrow{\pi_0}(\mathcal{C}, \Sigma) := \mathcal{C}/\Sigma$, defining thus the objects parts of the functors F and $\overrightarrow{\pi_0}$.
Given a morphism f of $\mathsf{LfCat}[\mathcal{C}_1, \mathcal{C}_2]$, $F(f)$ is the unique morphism in $\mathsf{LfCatSys}\Big[(\mathcal{C}_1, Iso(\mathcal{C}_1)), (\mathcal{C}_2, Iso(\mathcal{C}_2))\Big]$ induced by f, meaning $U(F(f)) = f$, and at last, for any morphism f of $\mathsf{LfCatSys}[(\mathcal{C}_1, \Sigma_1), (\mathcal{C}_2, \Sigma_2)]$, we set $\overrightarrow{\pi_0}(f) := f_{\Sigma_1, \Sigma_2}$. Then we have:

THEOREM 3 (The component category functor).

The functor $\overrightarrow{\pi_0}$ *is left adjoint to the functor* F*, which is left adjoint to the functor* U*.* $\overrightarrow{\pi_0} \dashv F \dashv U$



PROOF. From Theorem 1, the smallest *Yoneda* system of $\mathcal{C}$ is its set of isomorphisms. It follows that F is the left adjoint of U and the unit of this adjunction is an identity.

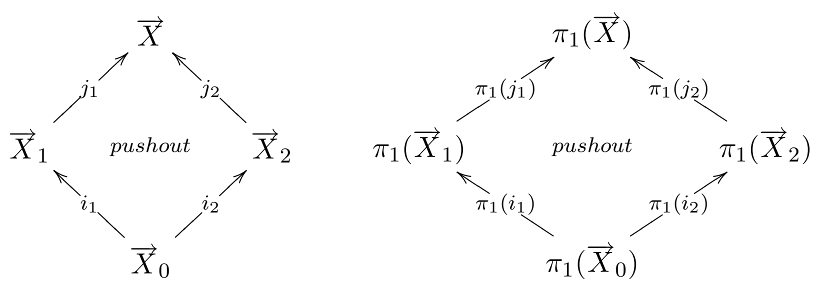
Let us prove that $\overrightarrow{\pi_0}$ is the left adjoint of F and that its unit is given by the collection of quotient functors Q_Σ from $\mathcal{C}$ to $\mathcal{C}/\Sigma$ (Proposition 2) that induce morphisms of $\mathsf{LfCatSys}$ from $(\mathcal{C}, \Sigma)$ to $(\mathcal{C}/\Sigma, Iso(\mathcal{C}/\Sigma))$.

We have here to check that $\mathcal{C}/\Sigma$ is actually loop-free, which is given by Proposition 3. If f is a morphism of $\mathsf{LfCatSys}$ from $(\mathcal{C}, \Sigma)$ to $(\mathcal{C}', Iso(\mathcal{C}'))$, then, still from Proposition 2, there exists a unique fonctor g from $\mathcal{C}/\Sigma$ to $\mathcal{C}'$ such that $f = g \circ Q_\Sigma$ and it is clear that g induces a unique morphism of $\mathsf{LfCatSys}[(\mathcal{C}, Iso(\mathcal{C})), (\mathcal{C}', Iso(\mathcal{C}'))]$, whence the expected adjunction. ■

Remember that a van Kampen theorem about a functor F is the statement that, under some particular conditions on a pushout square (algebraic topologists would rather say "glueing") this pushout square is preserved by F. We already have a directed van Kampen theorem for $\overrightarrow{\pi}_1$ and theorem 3 shows that $\overrightarrow{\pi_0}$ is a left adjoint, hence preserves all colimits and *a fortiori* pushout squares. It remains to find a way to "include" LfCat in $\mathsf{LfCatSys}$ in such a way that the pushout square we are interested in is preserved by the "inclusion". To do so, we will have to add hypotheses to the ones necessary already required by the directed van Kampen theorem for fundamental categories.

PROPOSITION 4 (van Kampen theorem for fundamental categories).

Let $\overrightarrow{X}_1, \overrightarrow{X}_2$ be sub-objects of $\overrightarrow{X}$ (object of PoSpc*) such that the underlying topological space of $\overrightarrow{X}$ is the union of the interiors*[6] *of the underlying topological spaces of $\overrightarrow{X}_1$ and $\overrightarrow{X}_2$. Let $\overrightarrow{X}_0$ be defined as $\overrightarrow{X}_1 \cap \overrightarrow{X}_2$ and $i_1 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_1, i_2 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_2, j_1 : \overrightarrow{X}_1 \hookrightarrow \overrightarrow{X}$ and $j_2 : \overrightarrow{X}_2 \hookrightarrow \overrightarrow{X}$ be the inclusion maps. Then we have the following pushout squares*



respectively in PoSpc *and* LfCat.

[6] with respect to the underlying topology of $\overrightarrow{X}$.

PROOF. The proof of Proposition 4 can be found in (Goubault, 2003)[7] or in (Grandis, 2003). It is an adaptation of the proof of van Kampen theorem for fundamental groupoid given in (Higgins, 1971). ■

The following result was a conjecture in Section 7 of (Fajstrup et al., 2004), and is the central result of this section:

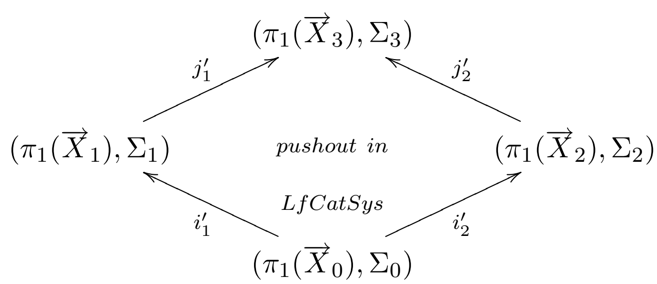
THEOREM 4 (van Kampen for component category).
Let $\overrightarrow{X}_1$, $\overrightarrow{X}_2$ *be sub-objects of* $\overrightarrow{X}_3$ *(object of* PoSpc*) such that the underlying topological space of* $\overrightarrow{X_3}$ *is the union of the interiors of the underlying topological spaces of* $\overrightarrow{X}_1$ *and* $\overrightarrow{X}_2$. *Also let* $\overrightarrow{X}_0 := \overrightarrow{X}_1 \cap \overrightarrow{X}_2$ *and* $i_1 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_1, i_2 : \overrightarrow{X}_0 \hookrightarrow \overrightarrow{X}_2, j_1 : \overrightarrow{X}_1 \hookrightarrow \overrightarrow{X_3}$ *and* $j_2 : \overrightarrow{X}_2 \hookrightarrow \overrightarrow{X_3}$ *be the respective inclusion maps.*
Moreover, suppose that

- Σ_1 *and* Σ_2 *are respectively* Yoneda-*systems of* $\pi_1(\overrightarrow{X}_1)$ *and* $\pi_1(\overrightarrow{X}_2)$,

- $\pi_1(j_1)(\Sigma_1) \uplus \pi_1(j_2)(\Sigma_2)$ *(also denoted* Σ_3*) is a* Yoneda *system of* $\pi_1(\overrightarrow{X}_3)$

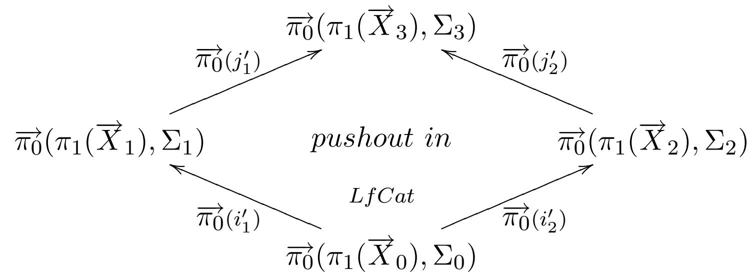
- $\pi_1(i_1)(\Sigma_0) \subseteq (\Sigma_1)$ *and* $\pi_1(i_2)(\Sigma_0) \subseteq (\Sigma_2)$ *(i.e.* $\pi_1(i_1), \pi_1(i_2)$ *are morphisms of* LfCatSys*).*

then
the inclusions i_1, i_2, j_1 *and* j_2 *give rise to* i'_1, i'_2, j'_1 *and* j'_2 *morphisms of* LfCatSys *and we have*

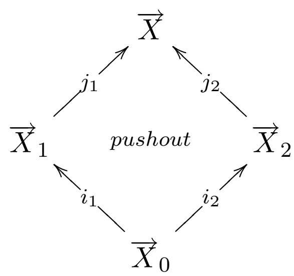


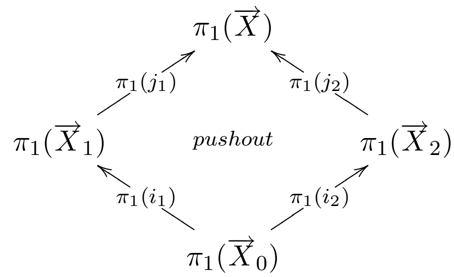
[7] In a more restrictive case than in (Higgins, 1971), but with a stronger dihomotopy relation.

and



PROOF. Theorem 4 gives us pushout squares in PoSpc and LfCat:





We have to prove that $\pi_1(\overrightarrow{X}_0)$, $\pi_1(\overrightarrow{X}_1)$, $\pi_1(\overrightarrow{X}_2)$ and $\pi_1(\overrightarrow{X}_3)$ respectively equiped with Σ_0, Σ_1, Σ_2 and Σ_3 give rise to a pushout square in LfCatSys.

Given $f_1 : (\pi_1(\overrightarrow{X}_1), \Sigma_1) \longrightarrow (\mathcal{L}, \Sigma)$ and $f_1 : (\pi_1(\overrightarrow{X}_2), \Sigma_2) \longrightarrow (\mathcal{L}, \Sigma)$ morphisms of LfCatSys such that $f_1 \circ i_1 = f_2 \circ i_2$, by hypothesis, there exists a unique $h : \pi_1(\overrightarrow{X}_3) \longrightarrow \mathcal{L}$ (morphism of LfCat) such that $f_1 = h \circ j_1$ and $f_2 = h \circ j_2$. It remains to see that h gives rise to a morphism of LfCatSys i.e. $h(\Sigma_3) \subseteq \Sigma$.

By hypothesis, $\Sigma_3 = j_1(\Sigma_1) \biguplus j_2(\Sigma_2)$ so any element of Σ_3 can be written $j_2(\alpha_{2n+1}) \cdot j_1(\alpha_{2n}) \cdot ... \cdot j_2(\alpha_1) \cdot j_1(\alpha_0)$ where for all $k \in \{0, ..., n\}$, $\alpha_{2k} \in \Sigma_1$ and $\alpha_{2k+1} \in \Sigma_2$. It follows that $h(j_2(\alpha_{2n+1}) \cdot j_1(\alpha_{2n}) \cdot ... \cdot j_2(\alpha_1) \cdot j_1(\alpha_0)) = (h \circ j_2)(\alpha_{2n+1}) \cdot (h \circ j_1)(\alpha_{2n}) \cdot ... \cdot (h \circ j_2)(\alpha_1) \cdot (h \circ j_1)(\alpha_0) = f_2(\alpha_{2n+1}) \cdot f_1(\alpha_{2n}) \cdot ... \cdot f_2(\alpha_1) \cdot f_1(\alpha_0) \in \Sigma$ since f_1, f_2 are morphisms of LfCatSys, hence h gives rise to a morphism of LfCatSys from $(\pi_1(\overrightarrow{X}_3), \Sigma_3)$ to $(\mathcal{L}, \Sigma)$. Thus we have a pushout square in LfCatSys. Now by Theorem 3, we know that $\overrightarrow{\pi_0}$ is a left adjoint hence (see for instance (Borceux, 1994a)) preserves colimits and, in particular, pushout squares.■

Theorem 4 does not necessarily give the biggest *Yoneda* system of $\pi_1(\overrightarrow{X}_3)$, so one has to guess what this biggest *Yoneda* system is in order to choose appropriate Σ_1 and Σ_2. The choice of Σ_0 is not relevant since once Σ_1 and Σ_2 are given, it is possible to take Σ_0

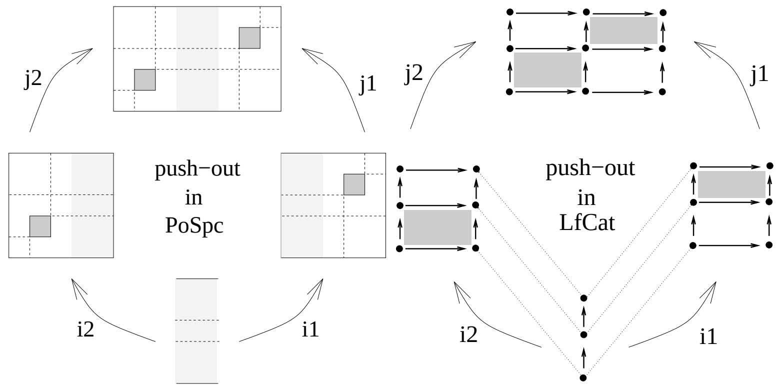


Figure 3. An application of the van Kampen theorem on component categories

as the biggest *Yoneda*-system of $\pi_1(\overrightarrow{X}_0)$ satisfying $\pi_1(i_1)(\Sigma_0) \subseteq (\Sigma_1)$ and $\pi_1(i_2)(\Sigma_0) \subseteq (\Sigma_2)$. Furthermore, given a po-space $\overrightarrow{X}$, we write $\overrightarrow{\pi_0}(\overrightarrow{X})$ instead of $\overrightarrow{\pi_0}\Big(\overrightarrow{\pi_1}(\overrightarrow{X}), \Sigma\Big)$ where Σ is the biggest *Yoneda* system of $\overrightarrow{\pi_1}(\overrightarrow{X})$. By definition, $\overrightarrow{\pi_0}(\overrightarrow{X})$ is the component category of $\overrightarrow{X}$.

Let us examine the example of the rectangles with two holes, see the left hand side of Figure 5, which gives, by Theorem 4 the category depicted on the right hand side figure below.

In this figure, rectangles in grey color are not commutative. The holes of the geometrical shape are represented by non-commutative squares in the component category. The dashed line represent the boundaries of the components. The problem of knowing which component these lines belong to is of topological nature and has been briefly studied in (Fajstrup et al., 2004), Section 5.3 and 6.

Applying Theorem 4 we can also prove that the component category of the cube with a centered cubical hole has 26 objects[8] as already noted in (Fajstrup et al., 2004). It can be represented in $\mathbb{R}^3$ putting an object in the "center" of each vertex, edge and face ($8\ vertices + 12\ edges + 6\ faces = 26\ objects$). Morphisms are generated by arrows from a point to its "closer neighbours in the future", for example those of $(0, 0, 0)$ are $(0, 0, \frac{1}{2}), (0, \frac{1}{2}, 0)$ and $(\frac{1}{2}, 0, 0)$ while $(1, 1, 1)$ has no such neighbours. In order to have the hypothesis of Theorem 4 satisfied, we split the cube into two parts so that, following notation of Theorem 4, $X_0 :=]\frac{1}{2} - \epsilon, \frac{1}{2} + \epsilon[\times [0, 1] \times [0, 1]$. It is the analog of the previous example in three dimensions.

[8] Geometrically, picture the Rubik's cube, the interior cube is the hole, all other cubes give an object.

6. Future and past components

In order to refine some of our results, and our understanding of components, we define a notion of component category in the future, as component categories of section 2, except we only keep "half" of the axioms:

DEFINITION 8. *Let* $\mathcal{C}$ *be a small category,* $\Sigma_+ \subseteq Mo(\mathcal{C})$ *is a* Yoneda-*f-system if and only if* Σ_+ *is stable under composition and satisfies*

(Af1) *for all objects* z *of* $\mathcal{C}$ *such that* $\mathcal{C}[y, z] \neq \emptyset$*, the map:*

$$\mathcal{C}[y, z] \xrightarrow{-\circ\sigma} \mathcal{C}[x, z] \text{ is a } \underline{\text{bijection}},$$

(Af2) *all* σ *in* Σ_+ *are monos in* $\mathcal{C}$*, and* Σ_+ *contains* $Iso(\mathcal{C})$

(Af3) Σ_+ *is stable under* <u>*pushouts*</u> *(with any morphim in* $\mathcal{C}$*)*

Remark that (Af1) of Definition 8 is equivalent to the preservation of the future cone, see the first part of Definition 3. The idea, that we will formalize in the rest of the section, is that we distinguish states, or objects in the fundamental category, only up to the possible futures. Dually we can define a *Yoneda*-system in the past as the other half of the axiom of Section 2:

DEFINITION 9. *Let* $\mathcal{C}$ *be a small category,* $\Sigma_- \subseteq Mo(\mathcal{C})$ *is a* Yoneda-*p-system if and only if* Σ_- *is stable under composition (of* $\mathcal{C}$*) and satisfies*

- *for all object* z *of* $\mathcal{C}$ *such that* $\mathcal{C}[z, x] \neq \emptyset$*, the map:*

$$\mathcal{C}[z, x] \xrightarrow{\sigma\circ-} \mathcal{C}[z, y] \text{ is a } \underline{\text{bijection}}.$$

- *all* σ *in* Σ_- *are epis in* $\mathcal{C}$*, and* Σ_- *contains* $Iso(\mathcal{C})$
- Σ_- *is stable under* <u>*pullbacks*</u> *(with any morphim in* $\mathcal{C}$*)*

We will always denote in the following *Yoneda*-f-systems by Σ_+ and *Yoneda*-p-systems by Σ_-. *Yoneda*-f-systems (respectively *Yoneda*-p-systems) form locales as *Yoneda*-systems did (similar arguments as those of Section 3.2 apply). Future component categories (respectively past component categories) are defined analogously as component categories, as the quotient of the fundamental category by the biggest *Yoneda*-f-system (respectively *Yoneda*-p-system).

The main result of (Fajstrup et al., 2004) is the following lifting property, as follows:

PROPOSITION 5. *(Proposition 7 of (*Fajstrup et al., 2004*))* *Let $\mathcal{C}$ be a category in which all endomorphisms are identities (this is true in particular if $\mathcal{C}$ is loop-free). Let Σ be any pure left and right calculus of* Yoneda *morphisms on $\mathcal{C}$ and $C_1, C_2 \subset Ob(\mathcal{C})$ be two objects of $\mathcal{C}[\Sigma^{-1}]$ such that the set of morphisms (in $\mathcal{C}[\Sigma^{-1}]$) is finite. Then, for every $x_1 \in C_1$ there exists $x_2 \in C_2$ such that the quotient map*

$$\mathcal{C}(x_1, x_2) \to \mathcal{C}/_{\Sigma}(C_1, C_2),\ f \mapsto [f]$$

is bijective.

Now we have similar results (which are then true also for component categories, since *Yoneda*-systems are in particular *Yoneda*-f-systems):

PROPOSITION 6. *Let $\mathcal{C}$ be a category in which all endomorphisms are identities (this is true in particular if $\mathcal{C}$ is loop-free). Let Σ_+ (respectively Σ_-) be any* Yoneda-*f-system (respectively* Yoneda-*p-system) on $\mathcal{C}$ and $C_1, C_2 \subset Ob(\mathcal{C})$ be two future components (respectively past components) such that the set of morphisms in $\mathcal{C}/\Sigma_+$ (or equivalently, by Theorem 2, in $\mathcal{C}[\Sigma_+^{-1}]$) is finite. Then, for every $x_1 \in C_1$ (respectively $x_2 \in C_2$) there exists $x_2 \in C_2$ (respectively $x_1 \in C_1$) such that the quotient map*

$$\mathcal{C}(x_1, x_2) \to \mathcal{C}/_{\Sigma_+}(C_1, C_2),\ f \mapsto [f]$$

is bijective.

PROOF. It suffices to go through each different step of the proof of Proposition 7 of (Fajstrup et al., 2004) for future components (the case of past components is similar).

Proposition 3 of (Fajstrup et al., 2004) still holds since future components form in particular a l-system (i.e. in the terminology of this paper, it has LEP with respect to $\mathcal{C}$, hence it has a left extension property):

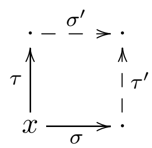
(B1) For every $f \in \mathcal{C}(x, y)$ and every $x' \sim_{o,\Sigma_+} x$ there exists $y' \sim_{o,\Sigma_+} y$ and $f' \in \mathcal{C}(x', y')$ such that $f' \sim_{m,\Sigma_+} f$.

(B2) Let $[f]_{m,\Sigma_+} \in \vec{\pi}_0(\mathcal{C}; \Sigma_+)([x]_{o,\Sigma_+}, [y]_{o,\Sigma_+})$ and let $x' \in [x]_{0,\Sigma_+}$, where $[.]_{o,\Sigma_+}$ denotes the equivalence class under $\sim_{o,\Sigma_+}$ for objects of $\mathcal{C}$, with the terminology of Section 4.2 (respectively, $[.]_{m,\Sigma_+}$ denotes the equivalence class under $\sim_{m,\Sigma_+}$ for morphisms of $\mathcal{C}$). Then there exists $y' \in [y]_{\Sigma_+}$ and $f' \in \mathcal{C}(x', y')$ such that $[f']_{\Sigma_+} = [f]_{\Sigma_+}$.

The proof of Proposition 4 of (Fajstrup et al., 2004) used in the proof of Proposition 7 of (Fajstrup et al., 2004) cannot be used here, because for *Yoneda*-f-systems, we do not have the pureness necessary for this proposition to be true. In fact, this will be not necessary here. We have a particular form of purity for future components that is sufficient. The same argument than for Lemma 1 shows (using the pushout property) that $f_2 \circ f_1 \in \Sigma_+$ implies $f_2 \in \Sigma_+$.

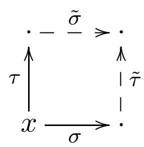
This is what we need to prove Proposition 5 of (Fajstrup et al., 2004) that we recast below:

Let $\sigma, \tau \in \Sigma_+(x, -)$. There exists a solution of the extension problem



with *both* morphisms $\sigma', \tau' \in \Sigma_+$.

Take $\sigma : \alpha \to \beta \in \Sigma_+$ and $\tau : \alpha \to \gamma \in \Sigma_+$. Take their pushout:



We know $\tilde{\sigma}$ is in Σ_+, and that $\tilde{\sigma} \circ \tau = \tilde{\tau} \circ \sigma$. With our "half-purity", this also implies that $\tilde{\tau} \in \Sigma_+$.

Let us now rephrase the proof of (Fajstrup et al., 2004).

Let $g_1, \ldots, g_n$ be the finitely many morphisms (by hypothesis) from C_1 to C_2 in $\mathcal{C}/\Sigma_+$. By repeated application of (B2), they can be lifted to $f_1, \ldots, f_n \in \overrightarrow{\pi_1}(\overrightarrow{X})$ with $f_i : x_1 \in C_1 \to y_i \in C_2$.

All y_i are in C_2, i.e. $y_1 \sim_{o,\Sigma_+} y_2 \sim_{o,\Sigma_+} \ldots \sim_{o,\Sigma_+} y_n$. By Proposition 2, we know that we have Σ_+-zig-zag morphisms between y_i and y_{i+1} $(1 \leq i \leq n-1)$. Taking repeated pushouts of these morphisms, we deduce that there exists $\lambda_i : y_i \to z_i$ and $\mu_i : y_i \to z_i$ both in Σ_+. Taking again repeated pushouts of the zig-zag morphisms constituted by $\lambda_1, \mu_1, \ldots, \lambda_{n-1}, \mu_{n-1}$, we find maps in Σ_+, $\tau_i : y_i \to x_2 \in C_2$.

The quotient map is onto, since $\tau_i \circ f_i \simeq f_i$, $1 \leq i \leq n$.

To prove injectivity, assume $f_i \in \mathcal{C}(x_1, x_2)$ with

$$[f_1] = [f_2] \in \vec{\pi}_0(\mathcal{C}; \Sigma_+)(C_1, C_2)$$

Then, there exist $x_0 \in C_1, x_3 \in C_2$ and morphisms $\sigma_i \in \Sigma_+(x_0, x_1), \tau_i \in \Sigma_+(x_2, x_3)$, $1 \leq i \leq 2$, such that $\tau_1 \circ f_1 \circ \sigma_1 = \tau_2 \circ f_2 \circ \sigma_2 \in \mathcal{C}(x_0, x_3)$. By (Af1), we find that there can only be a unique morphism between two fixed objects within the same component, hence $\sigma_1 = \sigma_2$ and $\tau_1 = \tau_2$. Since $.\circ\sigma_1$ is a bijection here by (Af1) again, we conclude $\tau_1 \circ f_1 = \tau_1 \circ f_2$. Finally, by (Af2), τ_1 is a mono, hence $f_1 = f_2 \in \mathcal{C}(x_1, x_2)$. ■

We can improve the result of Proposition 5, so that the hypothesis of finiteness can be suppressed. We need the following notion:

DEFINITION 10. *Given a set* $X \cup \{\perp\}$ *where* $\perp \notin X$*, we denote by* $\mathcal{D}_X$ *the poset* $X \cup \{\perp\}$ *ordered by the relation* $\{ (\perp, x) \mid x \in X \}$ *, so* $\perp$ *is the least element of the poset. Let* F *be a functor from* $\mathcal{D}_X$ *to a category* $\mathcal{C}$*. We name the colimit of* $F(\mathcal{D}_X)$ *in* $\mathcal{C}$*, when it exists, the* X*-pushout of* F*. An* X*-pushout is a natural generalization of a (binary) pushout.*

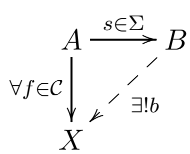
Given two sets X and Y, $\mathcal{D}_X$ and $\mathcal{D}_Y$ are isomorphic if and only if X and Y have the same cardinality, it follows that if X and Y have the same cardinality, $\mathcal{C}$ has X-pushouts if and only if it has Y-pushouts. So, for any cardinal κ, we will say that $\mathcal{C}$ has κ-pushouts when it has X-pushouts for some set X of cardinality κ. Given two cardinals κ_1 and κ_2 such that $\kappa_1 \leq \kappa_2$, if $\mathcal{C}$ has κ_2-pushouts, then it also has κ_1-pushouts. We say that $\mathcal{C}$ has infinite pushouts when $\mathcal{C}$ has κ-pushouts for any cardinal κ, one easily checks that given a small category $\mathcal{C}$, if $\mathcal{C}$ has $Ob(\mathcal{C})$-pushouts, then $\mathcal{C}$ has infinite pushouts.

Now, if we ask for the subcategory Σ_+ to have infinite pushouts then the lifting property of Proposition 5 *holds even if the set of morphisms* (in $\mathcal{C}/\Sigma_+$) between two objects is *not finite*.

We *conjecture* that in all "usual cases" of interest in concurrency semantics, such as PV models, future components have *automatically infinite pushouts* (respectively, past components have automatically infinite pullbacks).

Let us complete our knowledge of future and past component categories. We first recall a useful notion of category theory:

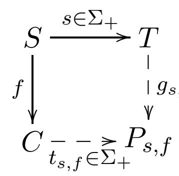
DEFINITION 11. *(*Borceux, 1994a*) Let* $\mathcal{C}$ *be a category and* Σ *a class of morphisms of* $\mathcal{C}$*. By the orthogonal subcategory of* $\mathcal{C}$ *determined by* Σ*, we mean the full subcategory* $\mathcal{C}_\Sigma$ *of* $\mathcal{C}$*, whose objects are those* $X \in \mathcal{C}$ *such that* $s \perp X$ *for every* $s \in \Sigma$*, i.e., such that for every* $s : A \to B \in \Sigma$*, for every morphism* $f : A \to X$*, there exists a unique morphism* $b : B \to X$ *such that* $b \circ s = f$*.*



THEOREM 5. *Let* Σ_+ *be a* Yoneda*-f-system in the small category* $\mathcal{C}$*. Suppose that* Σ_+ *has infinite pushouts, then* $\mathcal{C}_{\Sigma_+}$ *is reflective in* $\mathcal{C}$*.*

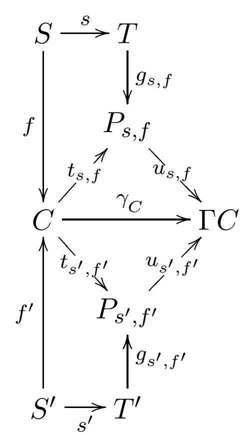
Note that Theorem 5.4.7 of Borceux's book (Borceux, 1994a) is of a similar nature. This theorem shows that in a cocomplete category $\mathcal{C}$ such that every object is presentable, for every set of morphism Σ_+, $\mathcal{C}_{\Sigma_+}$ is reflective in $\mathcal{C}$. But this does not apply in our context, since we want to apply it with $\mathcal{C} = \overrightarrow{\pi}_1$ which is in general not cocomplete nor well-complete. Also, few objects are presentable in general in $\overrightarrow{\pi}_1$ (just consider categorical sums). But if we have hypotheses on Σ_+ instead of hypotheses on $\mathcal{C}$, we get a very similar result.

Proof. We follow the proof of F. Borceux (Borceux, 1994a). By definition of the orthogonal subcategory, we have an obvious inclusion functor I from $\mathcal{C}_{\Sigma_+}$ to $\mathcal{C}$.
To prove that we have a reflective subcategory, we need to construct the left adjoint to I which is denoted Γ. Let $C \in \mathcal{C}$. For every pair (s, f) where $s : S \to T \in \Sigma_+$ and $f : S \to C \in \mathcal{C}$, we have a pushout diagram (call it a (s, f) pushout square) where $t_{s,f} \in \Sigma_+$ (because Σ_+ is a *Yoneda*-f-system).

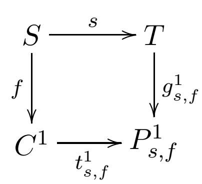


The diagram made of all the arrows $t_{s,f}$ is small because so are Σ_+ and $\mathcal{C}$. Since Σ_+ has infinite pushouts, the colimit of this diagram exists in Σ_+ and is denoted $\left(\Gamma C, (u_{s,f})_{s,f}\right)$, thus providing $\Gamma C \in \Sigma_+$.

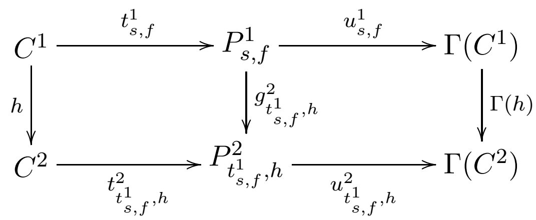
We have defined the object part of Γ. Now we construct a family of morphisms of $\mathcal{C}$, denoted $(\gamma_C)_{C \in Ob\mathcal{C}}$, that will be the unit of the adjunction. Let us determine $\gamma_C : C \longrightarrow \Gamma C$. Given two (s, f)-pushout squares, by definition of a colimit, we have $u_{s,f} \circ t_{s,f} = u_{s',f'} \circ t_{s',f'}$, hence we can set $\gamma_C := u_{s,f} \circ t_{s,f}$ since it does not depend on the (s, f)-pushout square we have chosen. Moreover, $\gamma_C \in \Sigma_+$ for it is given by the composite of two morphisms of Σ_+.



We determine the morphism part of Γ, this construction will implicitly prove that γ is a natural transformation from $Id_{\mathcal{C}}$ to $I \circ \Gamma$. Let $h : C^1 \longrightarrow C^2$. For each pushout square

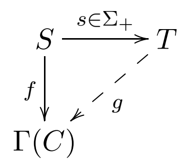


we have, by hypothesis on Σ_+ and since $t_{s,f} \in \Sigma_+$, the commutative diagram:

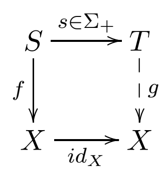


where the left square is a pushout square as in the beginning of the proof with $C := C^2$, $s := t^1_{s,f}$ and $f := h$, whence the - heavy but coherent - notation. The immediate consequence of this setting is that γ is actually a natural transformation.

Now we prove that for all $C \in \mathcal{C}$, $\Gamma(C) \in \mathcal{C}_{\Sigma_+}$. Given (s, f) with $s : S \to T \in \Sigma_+$, $src(f) = S$ and $tgt(f) = \Gamma(C)$, we have a unique g making the right side diagram commutative. Precisely, $g := u_{s,f} \circ g_{s,f}$, indeed, $u_{s,f} \circ g_{s,f} \circ s = u_{s,f} \circ t_{s,f} \circ f = \gamma_C \circ f$. The uniqueness is due to the bijectivity of $\gamma \in \mathcal{C}[S, T] \longrightarrow \gamma \circ s \in \mathcal{C}[S, \Gamma(C)]$, because s preserves the future cone (see Definitions 8 and 3). Thus, $s \perp \Gamma(C)$ and $\Gamma(C)$ is in the orthogonal subcategory determined by Σ_+.



Conversely, suppose that $X \in \mathcal{C}_{\Sigma_+}$. Given (s, f) with $s : S \to T \in \Sigma_+$, $src(f) = S$ and $tgt(f) = X$, we have a unique g making the right side diagram commutative, which is in fact a pushout square since s is an epi. With the notation introduced at the beginning of the proof, $g_{s,f} = g$ and $t_{s,f} = id_X$. Then the colimit $(\Gamma(X), u_{s,f})$ is the colimit of the family $(id_X)_{\{(s,f) \text{ with } s:S\to T\in\Sigma_+,\ src(f)=S \text{ and } tgt(f)=X\}}$. Hence $u_{s,f} \cong id_X$ for all such pairs (s, f) and $\Gamma(X) \cong X$ in $\mathcal{C}$.



The last part of the proof consists of seeing that

$$\alpha \in \mathcal{C}_{\Sigma_+}[\Gamma(C), D] \longmapsto I(\alpha) \circ \gamma_C \in \mathcal{C}[C, I(D)]$$

is a bijection. Let us consider the canonical morphism $\gamma_C : C \to \Gamma C$ of the colimit. Given $D \in \mathcal{C}_{\Sigma_+}$ and $m : C \to D$, we have to find a unique $n : \Gamma C \to D$ such that $n \circ \gamma_C = m$.

As $D \in \mathcal{C}_{\Sigma_+}$, it is orthogonal to all morphisms $s \in \Sigma_+$. In particular, for all pairs (s, f) as above, there exists a unique $b_{s,f}$ such that $b_{s,f} \circ s = m \circ f$. By the pushout property defining $P_{s,f}$ we deduce that there is a unique morphism $a_{s,f} : P_{s,f} \to D$ such that $a_{s,f} \circ t_{s,f} = m$ and $a_{s,f} \circ g_{s,f} = b_{s,f}$. This is done for all pairs (s, f). Hence by the colimit property defining ΓC, we find a unique morphism $n : \Gamma_C \to D$ such that $n \circ u_{s,f} = a_{s,f}$. Hence

$$\begin{aligned} n \circ \gamma_C &= n \circ u_{s,f} \circ t_{s,f} \\ &= a_{s,f} \circ t_{s,f} \\ &= m \end{aligned}$$

which ends the proof. ■

Similarly for *Yoneda*-p-systems, if we suppose Σ_- to have infinite pullbacks (take the dual of the definition of infinite pushouts), then $\mathcal{C}_{\Sigma_-}$ (the dual orthogonal category) is coreflective in $\overrightarrow{\pi}_1(X)$. In that case, points that represent past components are the minimal points of these components.

Note that, the construction of Γ done above is almost exactly the same as the one that proves the adjoint functor theorem see (Borceux, 1994a) for details. The construction is made up to isomorphism, to have a "concrete" Γ we would have to "choose" a family of $u_{s,f}$. In fact, $\mathcal{C}_{\Sigma_+}$ is the image of Γ up to isomorphism, that is to say any object of $\mathcal{C}_{\Sigma_+}$ is isomorphic to an element of the image of a "concrete" Γ. This remark is made to emphasize the fact that $\mathcal{C}_{\Sigma_+}$ is a replete subcategory of $\mathcal{C}$, i.e. any object of $\mathcal{C}$ isomorphic to an object of $\mathcal{C}_{\Sigma_+}$ is also in $\mathcal{C}_{\Sigma_+}$.

Refering to (Fajstrup et al., 2004), the co-completeness of the *Yoneda*-f-system of $\pi_1(\overrightarrow{X})$, namely Σ_+, exactly implies that all the components of $\overrightarrow{X}$ have a reachable maximum element. Then intuitively, $\mathcal{C}_{\Sigma_+}$ and $\mathcal{C}/\Sigma_+$ should be isomorphic, $\mathcal{C}_{\Sigma_+}$ being just a representation in $\mathcal{C}$ of $\mathcal{C}/\Sigma_+$.

7. Conclusion, related and future work

The material of last section has to be compared with the notion of future equivalence (and dually, past equivalence) due to Marco Grandis (Grandis, 2005). Our approach consist in determining criteria that tell which points should be identified according to some notion of "preservation of the choices". In Marco Grandis' setting, one tries to find, for any point p, a point p^+[9] which is the further point in the future before the first choice: intuitively the first crossroad on the path. The point p^+ should be, in our framework, the "last" (think about it as the greatest)

[9] Think about p^+ as Γp where the notation Γ refers to the proof of theorem 5.

Figure 4 illustrates the construction and the geometrical meaning of theorem 5. The "•'s" represent the maximum elements of the - two - components of the po-space and the - two - wide arrows are the two arrows of $\mathcal{C}_{\Sigma_+}$. The thin arrows starting from C correspond to the $t_{s,f}$ of the proof, while the thin dotted arrows are the $u_{s,f}$. It clearly appears that $\Gamma(C)$ is the maximum element of the component containing C.

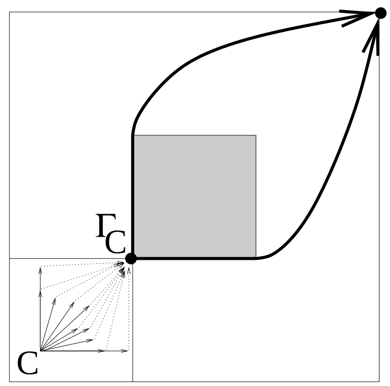


Figure 4. Ilustration of theorem 5

point of the <u>future</u> component of p. If one considers a Σ-component C for some Σ as in Definition 5, it might happen that C neither has greatest nor least element: see the components 5 and 6 of the swiss flag on Figure 2. Even if C is a future component, the point p^+ does not necessarily exist as one can see on the example of the directed real line, the pathology comes from the fact that the underlying topological space is not "bounded".
However, in the case where the underlying topology of a po-space $\overrightarrow{X}$ is <u>compact</u>, we hope (it is actually a conjecture) that any future component C of $\overrightarrow{\pi_1}(\overrightarrow{X})$ has a greatest element. In fact, we even expect that p^+ is the colimit (in $\overrightarrow{\pi_1}(\overrightarrow{X})$) of the full subcategory of $\overrightarrow{\pi_1}(\overrightarrow{X})$ whose objects are the points of C, in other words the infinite pushout of the family $(f_{s,f})$ introduced in the proof of Theorem 5. While this property is weaker than the existence of all infinite pushouts, it should be strong enough to prove Theorem 5 as well as (it is another conjecture) to guarantee the existence of the future spectra of $\overrightarrow{\pi_1}(\overrightarrow{X})$ in the sense of (Grandis, 2005).

Let us mention also that the careful study of the structure of components is of primary importance in practice. As an example, we refer the reader to (Goubault and Haucourt, 2005), where a (still naive) inductive computation of components, in the particular case of PV models, is used for static analysis of concurrent programs, and shows very good performances. The more we understand the structure of the fundamental category (and of higher-order fundamental categories), the better we can design practical methods for validation of concurrent programs, and the better we can understand the structure of concurrent and distributed computations.

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