A simplicial complex model for epistemic logic

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Abstract

The usual epistemic S5 model for a multi-agent system is based on a Kripke frame, which is a graph whose edges are labeled with agents that do not distinguish between two states. We propose to uncover the higher dimensional information implicit in this structure, by considering a dual, *simplicial complex model*. We show that there is an equivalence of categories between the usual Kripke models and our simplicial models. Thus, desirable properties of Kripke models like soundness and completeness are preserved. What we gain is that we can now study the topological properties of these models, and try to interpret them in terms of knowledge.

Keywords: Epistemic logic, Distributed computing, Simplicial complexes

1 Introduction

The usual Kripke model for epistemic logic **S5** is based on a graph whose nodes are the possible worlds and edges are labeled with agents that do not distinguish between two worlds. We introduce a new kind of model based on *simplicial complexes*. Now, the possible worlds are represented by higher-dimensional simplexes, and the indistinguishability relation corresponds to how the simplexes are glued together. Thus, these models have a topological flavor.

We prove that these simplicial models are very closely related to the usual Kripke models: there is an equivalence of categories between the two structures. This means that both kinds of model actually contain the same information. By going from Kripke models to simplicial models, we uncover the topological structure which is already present, but hidden, in Kripke models. Thus, simplicial models retain the nice properties of Kripke models, such as soundness and completeness w.r.t. (a slightly modified version of) the logic **S5**.

Simplicial models have first been introduced in the context of distributed computing, in order to prove that some distributed *tasks* cannot be solved when processes can crash [6]. Work on knowledge and distributed systems is of course one of the inspirations of the present work [4], especially where connectivity [2,3] is used. In [5], we extend these simplicial models to the setting of dynamic epistemic logic [1,7], and study more in-depth the relationship between knowledge, topology, and distributed computing.

2 A simplicial model for epistemic logic

We describe here the new kind of model for epistemic logic, based on chromatic simplicial complexes.

Syntax. Let AP be a countable set of propositional variables and A a finite set of agents. The language $\mathcal{L}_K$ is generated by the following BNF grammar:

$$\varphi ::= p \mid \neg\varphi \mid (\varphi \wedge \varphi) \mid K_a\varphi \qquad p \in AP,\ a \in A$$

In the following, we work with $n+1$ agents, and write $A = \{a_0, \ldots, a_n\}$.

Kripke frames. A *Kripke frame* $M = \langle S, \sim \rangle$ over a set A of agents consists of a set of states S and a family of equivalence relations on S, written $\sim_a$ for every $a \in A$. Two states $u, v \in S$ such that $u \sim_a v$ are said to be *indistinguishable* by a. A Kripke frame is *proper* if any two states can be distinguished by at least one agent. Let $M = \langle S, \sim \rangle$ and $N = \langle T, \sim' \rangle$ be two Kripke frames. A *morphism* from M to N is a function f from S to T such that for all $u, v \in S$, for all $a \in A$, $u \sim_a v$ implies $f(u) \sim'_a f(v)$. We write $\mathcal{K}_A$ the category of proper Kripke frames, with morphisms of Kripke frames as arrows.

Simplicial complexes. Given a base set V, a *simplicial* complex C is a family of non-empty finite subsets of V such that for all $X \in C$, $Y \subseteq X$ implies $Y \in C$. We say Y is a *face* of X. Elements of V (identified with singletons) are called *vertices*. Elements of C are *simplexes*, and those which are maximal w.r.t. inclusion are *facets*. The set of vertices of C is noted $\mathcal{V}(C)$, and the set of facets $\mathcal{F}(C)$. The *dimension* of a simplex $X \in C$ is $|X| - 1$. A simplicial complex C is *pure* if all its facets are of the same dimension, n. In this case, we say C is of dimension n. Given the set A of agents (that we will represent as colors), a *chromatic simplicial complex* $\langle C, \chi \rangle$ consists of a simplicial complex C and a coloring map $\chi : \mathcal{V}(C) \to A$, such that for all $X \in C$, all the vertices of X have distinct colors.

Let C and D be two simplicial complexes. A *simplicial map* $f : C \to D$ maps the vertices of C to vertices of D, such that if X is a simplex of C, $f(X)$ is a simplex of D. A *chromatic simplicial map* between two chromatic simplicial complexes is a simplicial map that preserves colors. Let $\mathcal{S}_A$ be the category of pure chromatic simplicial complexes on A.

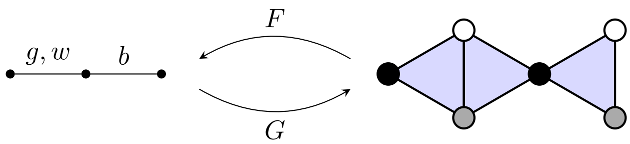
Theorem 2.1 *$\mathcal{S}_A$ and $\mathcal{K}_A$ are equivalent categories.*

Proof (Sketch). We can canonically associate a Kripke frame to a pure chromatic simplicial complex, and vice versa. Let C be a pure chromatic simplicial complex on the set of agents A. We associate the Kripke frame $F(C) = \langle S, \sim \rangle$ with S being the set of facets of C and the equivalence relation $\sim_a$, for all $a \in A$, generated by the relations $X \sim_a Y$ (for X and Y facets of C) if $a \in \chi(X \cap Y)$.

Conversely, consider a Kripke frame $M = \langle S, \sim \rangle$ on the set of agents $A = \{a_0, \dots, a_n\}$. Intuitively, what we want to do is take one n-simplex $\{v_0^s, \dots, v_n^s\}$ for each $s \in S$, and glue them together according to the indistiguishability relation. Formally, let $V = \{v_i^s \mid s \in S, 0 \leq i \leq n\}$, and equip it with the equivalence relation R defined by $v_i^s \; R \; v_i^{s'}$ if and only if $s \sim_{a_i} s'$. Then define $G(M)$ whose vertices are the equivalence classes $[v_i^s] \in V/R$, and whose simplexes are of the form $\{[v_0^s], \dots, [v_n^s]\}$ for $s \in S$, as well as their sub-simplexes. The coloring map is given by $\chi([v_i^s]) = a_i$.

The equivalence is given by the two maps F and G defined above, that we extend to functors between the two categories. □

Example 2.2 The picture below shows a Kripke frame (left) and its associated chromatic simplicial complex (right). The three agents, named b, g, w, are represented as colors black, grey and white on the vertices of the simplicial complex. The three worlds of the Kripke frame correspond to the three triangles (i.e., 2-dimensional simplexes) of the simplicial complex. The two worlds indistinguishable by agent b, are glued along their black vertex; the two worlds indistinguishable by agents g and w are glued along the grey-and-white edge.



We now decorate our simplicial complexes with atomic propositions to get a notion of simplicial model. For technical reasons, we restrict to models where all the atomic propositions are saying something about some local value held by one particular agent. All the examples that we are interested in will fit in that framework. Let $\mathcal{V}$ be some countable set of values, and $AP = \{p_{a,x} \mid a \in A, x \in \mathcal{V}\}$ be the set of *atomic propositions*. Intuitively, $p_{a,x}$ is true if agent a holds the value x. We write AP_a for the atomic propositions concerning agent a.

Kripke models. A *Kripke model* $M = \langle S, \sim, L \rangle$ consists of a Kripke frame $\langle S, \sim \rangle$ and a function $L : S \to \mathscr{P}(AP)$. Intuitively, $L(s)$ is the set of atomic propositions that are true in the state s. A Kripke model is *proper* if the underlying Kripke frame is proper. A Kripke model is *local* if for every agent $a \in A$, $s \sim_a s'$ implies $L(s) \cap AP_a = L(s') \cap AP_a$, i.e., an agent always knows its own values. Let $M = \langle S, \sim, L \rangle$ and $M' = \langle S', \sim', L' \rangle$ be two Kripke models on the same set AP. A *morphism of Kripke models* $f : M \to M'$ is a morphism of the underlying Kripke frames such that $L'(f(s)) = L(s)$ for every state s in S. We write $\mathcal{KM}_{\mathcal{A},AP}$ for the category of local proper Kripke models.

Simplicial models. A *simplicial model* $M = \langle C, \chi, \ell \rangle$ consists of a pure chromatic simplicial complex $\langle C, \chi \rangle$ of dimension n, and a labeling $\ell : \mathcal{V}(C) \to \mathscr{P}(AP)$ that associates to each vertex $v \in \mathcal{V}(C)$ a set of atomic propositions concerning agent $\chi(v)$, i.e., such that $\ell(v) \subseteq AP_{\chi(v)}$. Given a facet $X = \{v_0, \dots, v_n\} \in C$, we write $\ell(X) = \bigcup_{i=0}^{n} \ell(v_i)$. A *morphism* of simplicial models $f : M \to M'$ is a chromatic simplicial map that preserves the labeling:

$\ell'(f(v)) = \ell(v)$ (and χ). We write $\mathcal{SM}_{A,AP}$ the category of simplicial models over the set of agents A and atomic propositions AP.

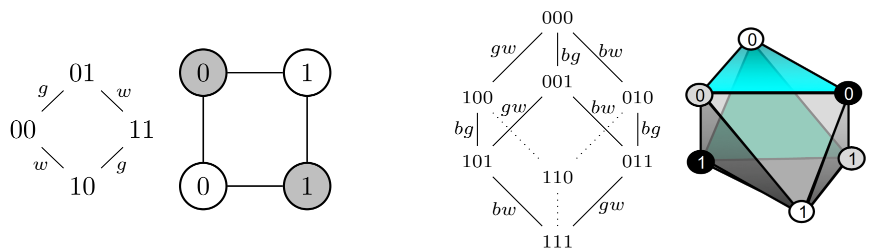
Theorem 2.3 *$\mathcal{SM}_{A,AP}$ and $\mathcal{KM}_{A,AP}$ are equivalent categories.*

Proof (Sketch). We extend the two maps F and G of Theorem 2.1 so that they preserve the labeling ℓ and L of atomic propositions accordingly. □

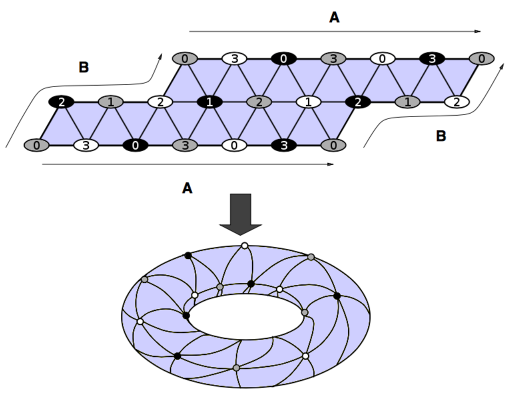
Example 2.4 The figure below shows the so-called binary input complex and its associated Kripke model, for 2 and 3 agents. Each agent gets a binary value 0 or 1, but doesn't know which value has been received by the other agents. So, every possible combination of 0's and 1's is a possible world.

In the Kripke model, the agents are called b, g, w, and the labeling L of the possible worlds is represented as a sequence of values, e.g., 101, representing the values chosen by the agents b, g, w (in that order).

In the simplicial model, the agents are represented as colors (black, grey, and white). The labeling ℓ is represented as a single value in a vertex, e.g., the value 1 in a grey vertex means agent g has chosen value 1. The possible worlds correspond to edges in the 2-agents case, and triangles in the 3-agents case.



Example 2.5 Consider now three agents, and a deck of four cards, $\{0, 1, 2, 3\}$. One card is given to each agent, and the last card is kept hidden. The simplicial model corresponding to that situation is depicted below on the left. The color of vertices indicate the corresponding agent, and the labeling is its card. In the planar drawing, vertices that appear several times with the same color and value should be identified: what we obtain is a triangulated torus.



Keeping that translation in mind, we can reformulate the usual semantics of formulas in Kripke models, in terms of simplicial models.

Definition 2.6 We define the truth of a formula φ in some epistemic state (M, X) with $M = \langle C, \chi, \ell \rangle$ a simplicial model, $X \in \mathcal{F}(C)$ a facet of C and $\varphi \in \mathcal{L}_K(A, AP)$. The satisfaction relation, determining when a formula is true in an epistemic state, is defined as:

$$
\begin{array}{lll}
M, X \models p & \text{iff} & p \in \ell(X) \\
M, X \models \neg\varphi & \text{iff} & M, X \not\models \varphi \\
M, X \models \varphi \wedge \psi & \text{iff} & M, X \models \varphi \text{ and } M, X \models \psi \\
M, X \models K_a\varphi & \text{iff} & \text{for all } Y \in \mathcal{F}(C), a \in \chi(X \cap Y) \text{ implies } M, Y \models \varphi
\end{array}
$$

We can show that this definition of truth agrees with the usual one (which we write $\models_{\mathcal{K}}$ to avoid confusion) on the corresponding Kripke model.

Proposition 2.7 *Given a simplicial model M and a facet X, $M, X \models \varphi$ iff $F(M), X \models_{\mathcal{K}} \varphi$. Conversely, given a local proper Kripke model N and state s, $N, s \models_{\mathcal{K}} \varphi$ iff $G(N), G(s) \models \varphi$, where $G(s)$ is the facet $\{v_0^s, \ldots, v_n^s\}$ of $G(N)$.*

Proof. This is straightforward by induction on the formula φ. $\square$

It is well-known that the axiom system **S5** is sound and complete with respect to the class of Kripke models [7]. Since we restrict here to local Kripke models, we need to add the following axiom (or axiom schema, if $\mathcal{V}$ is infinite), saying that every agent knows which values it holds:

$$\mathbf{Loc} \;=\; \bigwedge_{a \in A, x \in \mathcal{V}} K_a(p_{a,x}) \vee K_a(\neg p_{a,x})$$

Corollary 2.8 *The axiom system* $\mathbf{S5} + \mathbf{Loc}$ *is sound and complete w.r.t. the class of simplicial models.*

Proof. Adapting the proof of [7] for **S5**, it can be shown that $\mathbf{S5} + \mathbf{Loc}$ is sound and complete w.r.t. the class of local proper Kripke models, adapting the usual proof techniques. Then we transpose it to simplicial models using Proposition 2.7. Indeed, suppose a formula φ is true for every local proper Kripke model and any state. Then given a simplicial model and facet (M, X), since by assumption $F(M), X \models_{\mathcal{K}} \varphi$, we also have $M, X \models \varphi$ by Proposition 2.7. So φ is true in every simplicial model. Similarly, the converse also holds. $\square$

3 Conclusions

We have defined a new kind of model for epistemic logic, which uncovers the topological structure hidden in the usual Kripke models. We have hope that studying the topological properties of these models will give us information on the knowledge of the agents; extending the well-known relationship between common knowledge and connectedness to higher-dimensional properties. In a companion paper [5], we explore the relationship with distributed computing, where topological arguments have been used to prove the impossibility of solving some distributed tasks.

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