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Multiplicity hunting and approximating multiple roots of polynomials systems*

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Abstract. The computation of the multiplicity and the approximation of isolated multiple roots of polynomial systems is a difficult problem. In recent years, there has been an increase of activity in this area. Our goal is to translate the theoretical background developed in the last century on the theory of singularities in terms of computation and complexity. This paper presents several different views that are relevant to address the following issues : predict the multiplicity of a root and/or determine the number of roots in a ball, approximate fast a multiple root and give complexity results for such problems. Finally, we propose a new method to determine a regular system, called equivalent but deflated, i.e., admitting the same root as the initial singular one.

1. Introduction

Let $x \in \mathbf{C}^n$ and $f(x) = (f_1(x), \ldots, f_m(x)) \in \mathbf{C}[x]^m$. We denote by I the ideal generated by f. A multiple isolated root w of $f(x)$ is by definition the only root w of $f(x)$ in a certain ball at which its Jacobian matrix $Df(w)$ is not full rank. We use equally in the text singular root and multiple root. It is well known that the quadratic convergence of the Newton's method is lost in the neighbourhood of a multiple root. From starting points close to such roots, Newton's method is found to converge linearly or to diverge. For example the behaviour of the Newton sequence associated to the system $x - y^2 = 0$, $2cy^3 - 2xy = 0$ studied by Griewank and Osborne in [23] close to the root $(0, 0)$ of multiplicity 3 depends on the parameter c. For $c = 5/32$ there is linear convergence and for $c = 29/32$ we can observe

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divergence (see Fig. 1 and Fig. 2).

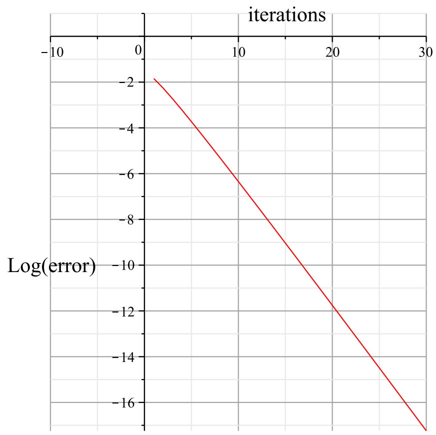

Fig. 1. Linear convergence of Newton sequence from $(0.1, -0.2)$ with $c = 5/32$.

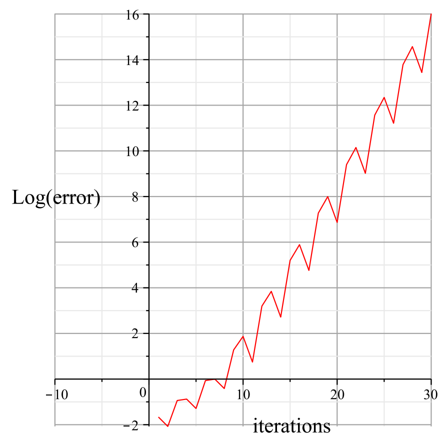

Fig. 2. Divergence of Newton sequence from $(0.1, -0.2)$ with $c = 29/32$.

Our purpose is to recover this quadratic convergence. In the example above, it is easy to determine a regular system admitting the same root as the initial one (we say an equivalent system). For that we remark the gradient of $2cy^3 - 2xy$ is zero at $(0, 0)$. Hence we can replace the polynomial $2cy^3 - 2xy$ by the two partial derivatives : y and $3cy^2 - x$. It turns out that the system $(x - y^2, y, 3cy^2 - x)$ is now regular at $(0, 0)$. We will develop this idea in section 6 to propose a new method to compute an equivalent system. More formally, from the initial system we compute a sequence of systems and stop when appears a regular system. A step in this iterative method consists of two operations called respectively *deflating* and *kerneling* [42]. The deflating operation replaces the polynomials by their gradient when the latter vanishes at the root. After the deflating operation we have ensured that all the rows of the Jacobian matrix evaluated at the root are non-zero. If this Jacobian matrix is not full rank, the kerneling operation consists to add the numerators of coefficients of a formal Schur complement of this Jacobian matrix. The multiplicity of the root obtained after a step decreases in the number of distinct polynomials added by the deflating and kerneling operations.

The goal of hunting the multiplicity is ambitious. This is a long standing challenge in many areas as optimization, dynamical systems, computer algebra and numerical algorithms dealing with polynomial or analytic systems. The univariate case is well understood : the Taylor series is a useful tool to describe the multiplicity of a root. For instance two iterations of Newton's method close to a multiple root are enough to predict the multiplicity. In fact the Newton sequence converges to the multiple root following a quasi straight line. More precisely, if $N_f(x) = x - \dfrac{f(x)}{f'(x)}$ is the Newton operator associated to a univariate function f, the iterate $x_{k+1} = N_f(x_k)$, $(k \geq 0)$, defining the Newton sequence starting from an initial point x_0, it is easy to see that

$$x_{k+1} - w = \left(1 - \frac{1}{m}\right)^k (x_0 - w) + O((x_0 - w)^2), \quad k \geq 0.$$

Schröder points out in [51] that the quadratic convergence is recovered using the generalized Newton operator

$$S_{f,m}(x) = x - m\frac{f(x)}{f'(x)}.$$

This has been hugely studied in the literature see Ostrowski [45], Rall [47], Householder [26], Traub [59]. α-theory in the spirit of Smale [55] for multiple roots in the univariate case has been done by Giusti-Lecerf-Salvy-Yakoubsohn in [18] and Yakoubsohn in [62], [63] : the links between Rouché's theorem and Schröder-Newton's method for multiple roots are precisely studied. To sum up, the order of Taylor series at the neighbourhood of the root defines the multiplicity in the univariate case. But unfortunately, Taylor series are not sufficient to determine the multiplicity in the multivariate case. In order to recover the quadratic convergence, the behaviour of Newton's method has been extensively investigated by Reddien [48], [49], Decker-Keller-Kelley in [12], [13], [11], Griewank in [20], [21], Griewank-Osborne in [22], [23], Rabier-Reddien [46]. These papers give characterizations of certain singular points and assumptions to get convergence. Sometimes the authors propose modifications to accelerate the convergence. In areas other than numerical analysis, the question of the multiplicity theory has also been intensively studied. There are many different way to introduce the concept of multiple root but, this is a more complicated matter than it is in one dimension : this requires background from algebra and analysis. The elimination theory provides algebraic objects like standard bases and the introduction of local rings reduces the multiplicity to the dimension of a quotient space. From an algebraic point of view, Fulton [16] chapter 7 gives a more general framework and explain different approaches. Milnor in Appendix B of [37] defines the multiplicity as the degree of a certain map. Using a similar approach Arnold, Varchenko, Gusein-Zade [5] rely the multiplicity to the index of a holomorphic germ. Another presentation is treated by Aizenberg and Yuzhakov in [1] where the multiplicity is defined via a perturbation of an analytic map. This last definition is directly linked to homotopy continuation methods which can be a reliable and an efficient way to numerically approximate isolated roots. After these theoretical studies on the multiplicity, we don't forget the heuristic book of Stetter, *Numerical Polynomial Algebra*, [57] and especially the chapter nine including the work of Thallinger.

The paper is organized as follows, first a survey part: in section 2 we present the algebraic geometric point of view on the multiplicity. Next, via the notion of duality, we give relationship to linear algebra where the multiplicity appears as the dimension of the kernel of a Macaulay matrix. In section 3, we explain how the multiplicity comes numerically from Rouché's theorem and recall some results. We also state an open problem concerning an efficient Rouché's theorem. In section 4, we justify why the homotopy methods work in the regular case and discuss the complexity of the linear homotopy in the singular case. The section 5 is devoted to describe the theoretical background of some deflation methods which are implemented in ApaTools of Zhonggang Zeng (recently upgraded to NAClab) `http://www.neiu.edu/~zzeng/NAClab.html` [64] and, PHCpack of Jan Verschelde `http://www.math.uic.edu/~jan/download.html` [60].

Section 6 is original. We propose a new way to determine an equivalent regular system from an initial singular system. We end by examples to show how this new method works.

2. Multiplicity. Algebraic geometric point of view

This theoretical material belongs to folklore. An exposition can be found e.g. in Cox, Little, O'Shea in [8], among others.

2.1. Number of roots and dimension. Let $x = (x_1, \dots, x_n) \in \mathbf{C}^n$ and I be the ideal generated by the polynomials $f_1(x), \dots f_m(x)$ of $\mathbf{C}[x]$. The first question is the number of isolated roots of a polynomial system. This is given by the following Bézout's Theorem which is the equivalent of the fundamental theorem of algebra for univariate polynomials:

Theorem 1. *The number of isolated roots of a polynomial system is less than the product of degrees of each polynomial.*

We refer to Heintz [25] for a proof using the dimension theory. Evidently the bound of theorem 1 is reached. If $V(I)$ means the variety associated to I then the following theorem gives a necessary and sufficient condition for $V(I)$ to be a set of isolated points. In this case the cardinal of $V(I)$ is the dimension of a quotient space. More precisely :

Theorem 2. *Under the previous notations we have :*

1– *The dimension of $\mathbf{C}[x]/I$ is finite if and only if the dimension of $V(I)$ is zero.*

2– *In the finite dimension case we have :*

$$\dim \mathbf{C}[x]/I \geq \# V(I)$$

where $\#V(I)$ is the number of distinct points of $V(I)$. This equality holds if and only if the ideal I is radical.

In fact we will see below that when the ideal I is not radical we can associate a multiplicity at each point of $V(I)$ so that the sum of multiplicities equals the dimension of $C[x]/I$. A way to determine $\dim \mathbf{C}[x]/I$ is to compute a Gröbner basis of the ideal I.

Theorem 3. *Let G a Gröbner basis of an ideal I. Let $LT(G)$ the ideal generated by the leading terms of G. Define $SM(G) = \{\textsf{monomials} \notin LT(G)\ \}$. Then*

$$\dim \mathbf{C}[x]/I = \#SM(G).$$

Example 1.

Let $f_1(x,y) = x^2+x^3$, $f_2(x,y) = x^3+y^2$. Then $V(I) = \{(0,0), (-1,1), (-1,-1)\}$. Let us choose the lexicographic ordering induced by $x > y$; the leading term is the Sup. A Gröbner basis of I is $\{y^4 - y^2, xy^2 + y^2, x^2 - y^2\}$ and $SM(G) = \{1, x, y, y^2, y^3, xy\}$. We deduce $\dim \mathbf{C}[x]/I = 6$. We will see that the root $(0,0)$ has multiplicity 4. $\circ$

Some computer algebra systems compute Gröbner bases, among them Maple, Magma, Singular. For instance, most classical algorithms are implemented in Maple.

2.2. Multiplicity and dimension. A way to define the multiplicity at a point of $w = (w_1, \dots, w_n) \in V(I)$ is to consider the local ring $\mathbf{C}\{x-w\}$ of convergent series in n variables with the maximal ideal generated by $x_1 - w_1, \dots x_n - w_n$. We denote by $I\mathbf{C}\{x-w\}$ the ideal generated by I in $\mathbf{C}\{x-w\}$. Finally we consider the local quotient space $A_w = \mathbf{C}\{x-w\}/I\mathbf{C}\{x-w\}$. The link between the local quotient spaces associated to points of $V(I)$ and the quotient space $C[x]/I$ is given by the :

Theorem 4. *Let* $V(I) = \{w^{(1)}, \dots, w^{(N)}\}$. *Then*

1– $\mathbf{C}[x]/I \sim A_{w^{(1)}} \times \dots \times A_{w^{(N)}}$.

2– $\dim \mathbf{C}[x]/I = \displaystyle\sum_{i=1}^{N} \dim A_{w^{(i)}}$.

We then can define the algebraic multiplicity.

Definition 1. *Let* $w \in V(I)$. *The dimension of local quotient space* A_w *is the algebraic multiplicity of* w.

To determine the dimension of A_w, a similar way to the affine global setting is to compute a standard basis of A_w. We then have an equivalent result to the theorem 3.

Theorem 5. *Let* S *a standard basis of the ideal* $I\mathbf{C}\{x - w\}$. *Let* $LT(S)$ *the ideal generated by the leading terms of* S. *Define* $SM(S) = \{\textsf{monomials} \notin LT(S)\ \}$. *Then*

$$\dim A_w = \#SM(S).$$

Example 2.

Let $f_1(x, y)$ and $f_2(x, y)$ be as the example 1. We are interested first in the root $(0, 0)$. Let us choose an ordering refining the valuation; the leading term will be the Inf. A standard basis of $I\mathbf{C}\{(x, y)\}$ is $S = \{x^2, y^2\}$. Hence $SM(S) = \{1, x, y, xy\}$ and $\dim A_{(0,0)} = 4$.
In the same way a standard basis of $I\mathbf{C}\{(x, y) - (-1, 1)\}$ (respectively $I\mathbf{C}\{(x, y) - (-1, -1)\}$) is $S = \{x, y\}$. Hence $SM(S) = \{1\}$ and $\dim A_{(-1,1)} = \dim A_{(-1,-1)} = 1$.
The identity $\dim \mathbf{C}[x]/I = \dim A_{(0,0)} + \dim A_{(-1,1)} + \dim A_{(-1,-1)}$ is satisfied. ◦
The tangent cone algorithm [38] allows to compute standard bases. An improved version of this algorithm is implemented in Singular by Greuel and Pfister [19].

2.3. Multiplicity and Duality. The link between multiplicity and duality is described first by Macaulay in [34] and perhaps also Gröbner [24]. A modern exposition is done by Emsalem [15]. More recent developments are given by Marinara, Möller, Mora in [36], Alonso, Marinari, Mora in [3], [4]. Also improvements concerning complexity are proposed by Mantzaflaris, Mourrain [35], [41]. For a multiple index $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbf{N}^n$, we denote by ∂^α the differential operator $g \to \dfrac{\partial^\alpha g(x)}{\partial x^\alpha}$. The operator ∂_w^α is the evaluation operator of ∂^α at a point w of $\mathbf{C}^n$. Also, if $L = \displaystyle\sum_{|\alpha| \le k} L_\alpha \partial^\alpha$ then $L_w = \displaystyle\sum_{|\alpha| \le k} L_\alpha \partial_w^\alpha$.
It is classical that there is an isomorphism between the dual space $\mathbf{C}[x]^*$ of $\mathbf{C}[x]$ and the set of formal series in ∂_w. Macaulay in [34] introduce the inverse system of the ideal I

$$I^\perp = \{L \in \mathbf{C}[x]^* \ : \forall g \in I,\, L(g) = 0\}$$

The result is that we can identify $I^\perp$ and the dual of $\mathbf{C}[x]/I$:

Theorem 6. *There is a canonical* $\mathbf{C}$*-isomorphism between* $I^\perp$ *and the dual of* $\mathbf{C}[x]/I$.

The link between the duality and the multiplicity is explained by the relation between the quotient rings A_w and the subspaces

$$D_w^k(I) = \{L = \sum_{|\alpha| \le k} L_\alpha \partial^\alpha \ : \forall g \in I,\, L_w(g) = 0\}.$$

We will write D_w^k for $D_w^k(I)$. We have :

Theorem 7. *A root w of f is isolated if and only if there exists an integer δ satisfying $\mathcal{D}_w^{\delta-1} = \mathcal{D}_w^\delta$. In this case D_w^δ is the dual space of A_w and the dimension of $\mathcal{D}_w^\delta$ is equal to the multiplicity of w. In other words*

$$\dim A_w = \dim D_w^\delta.$$

We call δ the thickness of the multiple root w.

Remark 1.

We adopt the term *thickness* which is the translation of the french word *épaisseur* introduced by Ensalem in [**15**] rather than the term *depth* more recently used by Mourrain, Matzaflaris in [**35**] or Dayton, Li, Zeng [**10**], [**9**]. ◦

To compute the dimension of the vector space D_w^k, let us introduce the Macaulay matrices

$$S_k = \left(\partial_\alpha[w]((x-w)^\alpha f_i(x)) \right)_{\substack{|\alpha| \le k-1 \\ 1 \le i \le m}}$$

Theorem 8. *The vector space D_w^k is isomorphic to the kernel of S_k.*

Consequently the multiplicity μ of w satisfies $\mu = \dim Ker(S_{\delta-1}) = \dim Ker(S_\delta)$.

Example 3.

Let $f_1 = x^2 + y^2 - 2$, $f_2 = xy - 1$. $w = (1,1)$. Let us construct the Macaulay matrices in $w = (1,1)$:

		∂_{00}	∂_{10}	∂_{01}	∂_{20}	∂_{11}	∂_{02}
S_0	f_1	0	2	2	2	0	2
S_1	f_2	0	1	1	0	1	0
	$(x-1)f_1$	0	0	0	4	2	0
S_2	$(x-1)f_2$	0	0	0	2	1	0
	$(y-1)f_1$	0	0	0	0	2	4
	$(y-1)f_2$	0	0	0	0	1	2

We have successively $\mathsf{rank}(S_0) = 0$, $\mathsf{rank}(S_1) = 1$, $\mathsf{rank}(S_2) = 4$. Hence $\mathsf{corank}(S1) = \mathsf{corank}(S_2) = 2$. It follows the multiplicity of $(1,1)$ is 2. ◦

We now explain how the knowledge of the structure of the dual space permits to find a regular system at w. Let μ the dimension of D_w^k and $\Lambda = \{\Lambda_1, \dots, \Lambda_\mu\}$ a basis of D_w^k. We introduce the polynomial system of $m\mu$ equations and n variables :

$$\Lambda(f) = (\Lambda_1(f), \dots, \Lambda_\mu(f))$$

with $\Lambda_k(f) = (\Lambda_k(f_1), \dots, \Lambda_k(f_m))$. Mantzaflaris and Mourrain state the following :

Theorem 9. [**35**] *The polynomial system $\Lambda(f)$ is regular at w.*

Example 4.

A basis of the kernel of the Macaulay matrix S_2 of the example 3 is $\{(1,0,0,0,0,0),(0,1,-1,0,0,0)\}$. Hence the set $\{\partial^{(0,0)}, \partial^{(1,0)} - \partial^{(0,1)}\}$ is a basis of $D^2_{(1,1)}$. Consequently

$$\Lambda(f_1, f_2) = (x^2 + y^2 - 2, xy - 1, 2x - 2y, y - x).$$

It is easy to see the Jacobian of $\Lambda(f_1, f_2)$ has rank 2.

3. Multiplicity. Numerical point of view

3.1. Multiplicity and perturbation. From a numerical point of view an exact multiple root makes no sense. We must think of a cluster of roots which comes from perturbations of the data. In this way we can consider the initial system as close to another system which admits an exact multiple root.

Definition 2. *A root w of $f = (f_1, \dots, f_m)$ is regular if the Jacobian matrix $Df(w)$ has full rank (in the opposite case w is a singular root).*

The link to the algebraic multiplicity is given by the following.

Proposition 1. *The algebraic multiplicity of a regular root is equal to* 1.

Proof. We denote by $Df(w)^*$ the adjoint of $Df(w)$. Let I the ideal generated by f. Since $Df(w)$ has full rank $Df(w)^*Df(w)$ is invertible. Hence the ideal generated by $g(x) = (Df(w)^*Df(w))^{-1}f(x)$ is equal to I. But

$$(Df(w)^*Df(w))^{-1}f(x) = x - w + \sum_{k\geq 2} \frac{1}{k!}(Df(w)^*Df(w))^{-1}D^kf(w)(x-w)^k.$$

Consequently $LT(g)$ is generated by $x - w$. Its follows that $\dim A_w = 1$. $\square$

A very useful result is the Rouché's theorem [50] which links a perturbation of analytic functions to the number of roots in a ball, see also Lojasiewicz for a version in several variables [33].

Theorem 10. *Let f and g two analytic functions defined in a real ball $B(x,r) \subset \mathbf{C}^n$. If for all $z \in \partial B(x,r)$ we have*

$$||f(z) - g(z)|| < ||f(z)||$$

then f and g have the same number of roots in $B(x,r)$ where each root is counted as many times as its multiplicity.

Proposition 2. *w is a singular isolated root of f if and only if the multiplicity of w is strictly greater than 1.*

Proof. Since w is an isolated root there exists a ball $B(w,r)$ where f admits only this root. There exists $z_0 \in \partial B(w,r)$ such that for all $z \in \partial B(w,r)$ one has $||f(z)|| \geq ||f(z_0)||$. Then the function $g(z) = f(z) + y$ with $||y|| < ||f(z_0)||/2$ satisfies the inequality of Rouché's theorem on $\partial B(w,r)$. Consequently the number of roots of g in $B(w,r)$ is the multiplicity, say μ, of w. Moreover for almost every y, Sard's theorem insures that $Dg(z)$ has full rank at each of the roots. Hence the roots of g, say $w^{(1)}, \dots, w^{(\mu)}$, are regular in the ball $B(w,r)$. Let us consider the homotopy

$$h(z,t) = (1-t)g(z) + tf(z) = f(z) - (1-t)y.$$

We have $h(w^{(k)},0)=0$ for every k and $h(w,1)=0$. For almost every y, from implicit function theorem there exists μ regular curves $x^{(k)}(t)$: $[0,1[\to B(w,r)$ such that $f(x^{(k)}(t))=(1-t)y$ and $x^{(k)}(t)'=-Df(x^{(k)}(t))^{-1}y$. Hence if $\mu>1$ the quantities $x^{(k)}(1)'$ make no sense and the root w is singular. $\square$

The link between Rouché's theorem and the local ring theory can be summarized by the identity

$$\dim A_w^f=\sum_{\bar{w}\in B(w,r)\cap g^{-1}(0)}\dim A_{\bar{w}}^g$$

where A_w^f (respectively $A_{\bar{w}}^g$) is the local quotient ring associated to f (respectively g). Here we find again the classical idea from a numerical point of view that we deal with clusters of roots rather than exact multiple roots.
In the case where the system has no root or only one regular root in a ball, it is possible to give an effective version of Rouché's theorem : this is obtained from the Taylor series of f. It is also valid when the system f is analytic.

Theorem 11. [17] *Let us consider a ball $B(x,r)$.*

1– *If*

$$||f(x)||>\sum_{k\geq 1}\frac{1}{k!}||D^kf(x)||r^k$$

there is no root in $B(x,r)$.

2– *Let r be a positive real number smaller than the radius of convergence of $\sum_{k\geq 0}\frac{1}{k!}||D^kf(x)||r^k$. If*

$$||Df(x)^{-1}f(x)||<r-\sum_{k\geq 2}\frac{1}{k!}||Df(x)^{-1}D^kf(x)||r^k$$

there is only one regular root of f in $B(x,r)$.

The case of a simple double root has been studied by Dedieu-Shub [14].

Theorem 12. *Let $c=0.19830\ldots$. For $v,x\in\mathbf{C}^n$, $||v||=1$, we define the linear operator :*

$$A(x,f,v)=Df(x)+\frac{1}{2}D^2f(x)(v,\Pi_v)$$

where Π_v is the projection on the space spanned by v. Let L be the linear operator defined by $L(v)=Df(x)v$ and $L(w)=0$ if w is orthogonal at v. Let $B(x,f,v)=A(x,f,v)-L$. We introduce the quantity

$$\gamma_2(f,x,v)=\max\left(1,\sup_{k\geq 2}\left|\left|\frac{1}{k!}B(f,x,v)^{-1}D^kf(x)\right|\right|^{\frac{1}{k-1}}\right).$$

If we have

$$||f(x)||+||Df(x)v||\frac{c}{2\gamma_2(f,x,v)^2}<\frac{c^3}{4||B(f,x,v)^{-1}||\gamma_2(f,x,v)^4}$$

then f has two zeros (counting multiplicities) in the ball of radius $\dfrac{c}{2\gamma_2(f,x,v)^2}$ around x.

In fact the previous case describes double roots of corank one : they are clusters of two roots of embedding dimension one. A quantitative version of Rouché's theorem in the embedding dimension 1 case is given by Giusti, Lecerf, Salvy, Yakoubsohn in [17] but, the statement is technically too difficult to appear here.

Open problem 1.

Find a qualitative version of Rouché's theorem for clusters of roots of analytic systems. ◦

Let us remark that the theorem 11 applied to the dual system $\Lambda(f)$ of theorem 9 can prove the existence of a (regular) root of $\Lambda(f)$.

4. Multiplicity and homotopy methods

Homotopy methods consist to deform smoothly a system with known roots to the initial system with unknown roots. These methods are currently used to solve systems of equations : the textbook of Allgower and Georg [2] or Morgan [39] are classical references. The homotopy used in this section is the linear homotopy h :$[0,1]\times\mathbf{C}^n\to\mathbf{C}^n$ defined by

$$h(x,t)=(1-t)g_{a,b}(x)+tf(x)$$

where $g_{a,b}(x)=(a_1x_1^{d_1}-b_1,\dots,a_nx_n^{d_n}-b_n)$. There are three kinds of curves $x(t)$ solutions of $h(t,x(t))=0$. First, the regular curves defined on $[0,1]$ which correspond to a regular root of $f(x)$. Next, the curves which are only regular on $[0,1[$ due to the existence of a multiple root of $f(x)$. Finally, the curves which go to infinity as $t\to1$ and which correspond to infinite roots of $f(x)$. Infinite roots are explicitly described using complex projective space $\mathbf{CP}^n$. Wright in [61] give a proof of Bézout's theorem using the linear homotopy. More precisely

Theorem 13. [61] *Let* $F(x_0,x)=\Big(x_0^{d_1}f_1(x/x_0),\dots,x_0^{d_n}f_n(x/x_0)\Big)$, $G_{a,b}(x_0,x)=(a_1x_1^{d_1}-b_1x_0^{d_1},\dots,a_nx_n^{d_n}-b_nx_0^{d_n})$ *and*

$$H_{a,b}(t,x_0,x)=(1-t)G_{a,b}(x_0,x)+tF(x_0,x).$$

Let $Z_{a,b}=\{(t,x_0,x)\in[0,1[\times\mathbf{CP}^n\;:\;H_{a,b}(t,x_0,x)=0\}$. *For almost* $(a,b)\in\mathbf{C}^{2n}$ *we have :*

1– $0\in\mathbf{C}^n$ *is a regular value of* $H_{a,b}(t,1,x)=0$, *i.e,* $D_xH(t,1,x)$ *has full rank of for all* $(t,x)\in[0,1[\times\mathbf{C}^n$ *such that* $H_{a,b}(t,1,x)=0$.

2– $Z_{a,b}$ *consists of* $d_1\dots d_n$ *disjoint half-open arcs in* $\mathbf{CP}^n\times[0,1)$, *where the endpoint of each arc is a known root of* $G_{a,b}(x_0,x)$ *in* $\mathbf{CP}^n\times\{0\}$, *and where the limit of the other end of the arc is a root of* $F(x_0,x)$.

In fact linear homotopy methods are useful to prove Bézout's theorem : see Blum, Cucker, Shub, Smale [7] page 199 and references inside.

A straightforward consequence of this result is the multiplicity can be computed thanks to homotopy methods. More precisely

Corollary 1. *Let us consider the linear homotopy of the theorem 13. Each isolated root (respectively root at infinity) of multiplicity* μ *generates* μ *homotopy paths* $x(t)$ *converging towards it.*

To find one regular root, the complexity and the analysis of this homotopy method is studied by Shub and Smale in [53] and [54]. A better complexity bound is given by Shub [52]. We give a simplified version of this complexity result in the linear homotopy case.

Theorem 14. [52], [6] *The number of numerical homotopy steps performed by the projective Newton's method to yield an approximate zero of the initial system is bounded by*

$$71d^{3/2}L$$

where d is the maximum of degrees of f_i's and L is the condition length of the linear homotopy (see the references above for this definition).

The paper of T.Y Li [32] gives a good review on homotopy continuation methods and their improvement for deficient polynomial systems, i.e., for which the isolated solutions are fewer than the Bézout's number.

Open problem 2.

Estimate the complexity to approximate a multiple root using linear homotopy.
$\circ$
In the chapter 10 of [56], Sommese and Wampler give some numerical heuristics to deal with *singular end games* based on power series, Cauchy integral and trace theorem. In the same vein, Huber and Verschelde in [27] explore links between *polyhedral end game* and power series to give some refinements. Another interesting way is proposed by Kobayashi, Suzuki and Sakai in [28] using Zeuthen's rule but unfortunately without study of complexity.

5. Recovering the quadratic convergence

The idea is to compute from the initial system another one which is regular at the singularity. The theorem 9 gives an augmented system computed from the kernel of the Macaulay matrices S_k. But the size of S_k is very huge i.e., $m\sum_{j=0}^{k}\binom{n+j-1}{j}\times\binom{n+k+1}{n}$. In the sequel, we describe two kinds of what is called a *deflation* method.

5.1. Lecerf deflation method. [29] The idea is to differentiate well chosen equations and to select new equations at each step of the method in order to obtain a regular system at the root w. From now we adopt the Matlab notation : $x_{i:j}$ is the vector $(x_i,\dots x_j)$.
Initial Step : the system $f=(f_1,\dots,f_m)$ is considered as a subset of $\mathbf{C}\{x-w\}$. We set $\Phi_1=f$ and $R_1=1$.
Step $k\geq 1$. We compute a new system Φ_{k+1} and a new integer R_{k+1} from Φ_k and R_k. Let m_k be the valuation of Φ_k and

$$\tilde{\Phi}_k=\frac{\partial^{m_k-1}}{\partial x_{R_k}^{m_k-1}}\Phi_k:=\left\{\frac{\partial^j}{\partial x_{R_k}^j}\Phi_k\ :\ 1\leq j<m_k\right\}$$

Let r_k the rank of Jacobian of $\tilde{\Phi}_k$ with respect to the variables $x_{R_k:n}$ evaluated at $w_{R_k:n}$. Then we set $R_{k+1}=r_k+R_k$. Next we extract a subset Ω_k from $\tilde{\Phi}_k$ such that the gradient of Ω_k has rank r_k at $w_{R_k:n}$.
Finally, thanks to the implicit function theorem, there exist r_k power series $y_{R_k:R_{k+1}-1}$ in $\mathbf{C}\{x_{R_{k+1}:n}-w_{R_{k+1}:n}\}$ expressing $x_{R_k:R_{k+1}-1}$ in terms of $x_{R_{k+1}:n}$ such that $\Omega_k(y_{R_k:R_{k+1}-1},x_{R_{k+1}:n})=0$. Then

$$\Phi_{k+1}(x_{R_{k+1}:n})=\Phi_k(y_{R_k:R_{k+1}-1},x_{R_{k+1}:n}).$$

Stopping criterion. The above construction stops when $R_{k+1} = n + 1$.
Output of the method. Let us suppose that there are ν steps. The output is the system $\Omega = (\Omega_1(x_{R_1:n}, \dots, \Omega_\nu(x_{R_\nu:n}))$. The properties of this deflation sequence are given by

Theorem 15. *Without loss of generality we can assume that at each step of the deflation process the variable x_{R_k} is in Weierstrass position with respect to the ideal generated by Φ_k (i.e. there exists an element of this ideal of valuation m_k having $x_{R_k}^{m_k}$ in its support). The construction above works up to a permutation of the variables. Moreover :*

1– $1 \le r_k \le n - R_k + 1$.
2– $1 \le m_k \dim\left(\mathbf{C}\{x_{R_k:n} - w_{R_k:n}\}/\tilde{\Phi}_k\right) \le \dim\left(\mathbf{C}\{x_{R_k:n} - w_{R_k:n}\}/\Phi_k\right)$.
3– *The system Ω is regular at the root w.*
4– $m_1 \dots m_\mu \le \dim A_w$.

Example 5.

Let $f := (f_1, f_2, f_3) = (x^2 + x + y + z, y^2 + y + x + z, z^2 + z + x + y)$. The root $w = (0, 0, 0)$ has multiplicity 4.
We denote by O_k a generic power series $\sum_{|\alpha| \ge k} a_\alpha (x - w)^\alpha$.
Let $\Phi_1 = \{f_1, f_2, f_3\}$ and $R_1 = 1$. The rank of the Jacobian matrix of f is $r_1 = 1$ at x. We find $m_1 = 1$ and $\tilde{\Phi}_1 = \Phi_1$ and $R_2 = 2$. We choose $\Omega_1 = \{f_1\}$. The power series solution of $f_1(y_1, y, z) = 0$ is

$$y_1(y, z) = -y - z - y^2 - 2\, zy - z^2 + O_3.$$

Substituting x by y_1 in Φ_1 we find

$$\Phi_2 = \{O_3, -2zy - z^2 + O_3, -y^2 - 2zy + O_3\}.$$

For the next step $m_2 = 2$ and

$$\tilde{\Phi}_2 := \frac{\partial \Phi_2}{\partial y} = \{\Phi_2, -2z + O_2, -2y - 2z + O_2\}.$$

The rank of the Jacobian matrix of $\tilde{\Phi}_2$ is $r_2 = 2$. We choose $\Omega_2 = \{-2z + O_2, -2y - 2z + O_2\}$ Since $R_3 = R_2 + r_2 = 4$. The deflation construction stops. The regular system at w is

$$\Omega = \{f_1, -2z + O_2, -2y - 2z + O_2\}.$$

We refer to [29] for the study of the complexity of this construction. Another type of deflation method mixing symbolic and numerical computations have been considered by Ojika, Watanabe, and Mitsui in [44], [43] : the new equations are generated by symbolic Gaussian eliminations but it remains to perform the numerical analysis and to study the complexity of this *modified deflation* method.

5.2. Augmented systems and deflation methods. From the knowledge of the structure of the local quotient algebra, Mantzaflaris and Mourrain determine a regular system given in the theorem 9. We sketch now another construction of deflation sequence based on a augmentation of the number of equations and of the number of variables. First, one defines a deflation operator which associates to the initial system f, a new system $Defl(f,x,y)$ where $(x,y)\in \mathbf{C}^{n+j}$. Next, one iterates this operator to obtain the deflation sequence :

$$x^0=x,\ y^0=y,\ F_0=f,\quad x^{k+1}=(x^k,y^k),\quad F_{k+1}=Defl(f_k,x^k,y^k),\ k\geq 0.$$

The *length* of the deflation is the vector $(n_0,\ldots,n_k,\ldots)$ where n_k is the dimension of the kernel of the Jacobian matrix $DF_k(x^k)$. The thickness of the deflation is the number N such that $n_{N+1}=0$.
In this way such a type of deflation operator had been proposed by Leykin, Verschelde, Zhao in [30], and extended in [31]. From an original system $f=(f_1,\ldots,f_m)$ with $\mathsf{rank}\, Df(w)=r$ they define the following :

$$LVZ(f,x,y):=LVZ(f,B,h,x,y)=\left\{\begin{array}{c} f(x)\\ Df(x)By\\ h^*y-1\end{array}\right.$$

where B is a random $n\times(r+1)$ matrix and h a random $r+1$ vector. The matrix $Df(x)B$ has generically a rank equal to r and the dimension of $\mathsf{Kernel}\, Df(w)B$ is 1. Hence there exists a unique $\lambda\in\mathbf{C}^{r+1}$ such that $Df(w)B\lambda=0$ and $h^*\lambda-1=0$.
Theorem 16. [30], [31] *The multiplicity of the root (w,λ) of the system $LVZ(f,x,y)$ is strictly less than the multiplicity of the root w of the system f.*

Unfortunately the deflated system $LVZ(f,x,y)$ is not regular at its root (w,λ). In this case the method consists to deflate more until to find a regular system. We have
Theorem 17. [30], [31] *The number of deflation steps to obtain a regular system is bounded by the multiplicity of w. If N is the number of deflations, the regular system has $n+N+\sum_{k=1}^{N} r_k$ variables and $2^N(n+1)-1$ equations.*

Example 6. [11]

Let $f(x,y)=(x+y^3,x^2y-y^4)$ with $(0,0)$ has multiplicity 3. The number of deflations steps is 3 and the coranks of the Jacobian matrices of the deflated systems are equal to 1. The regular system has 16 variables and 23 equations. $\circ$

Example 7. [10]

Let $f=(x^4-yzt,y^4-zxt,z^4-xyt,t^4-xyz)$. The root has multiplicity 131. Two steps of LVZ deflation are needed with length $(4,4)$. The regular system has 7 variables and 19 equations. $\circ$
Another way to construct deflated systems by adding variables and equations has been proposed par Dayton and Zeng in [10] for the polynomial case and Dayton, Li, Zeng in [9] for the analytic case. The deflation operator proposed by these authors is

$$DLZ(f,x,y):=DLZ(f,R,e_1,x,y)=\left\{\begin{array}{l} f(x)\\ Df(x)y\\ Ry-e_1\end{array}\right.$$

where R is $p\times n$ random matrix in order that $\left[\begin{array}{c} Df(w)\\ R\end{array}\right]$ has full rank and $e_1=(1,0,\ldots,1)^T$ with size p is the dimension of the kernel of $Df(x)$.
Theorem 18. [10], [9] *The number of steps of the DLZ deflation is bounded by the thickness δ of the root w defined in theorem 7. The last deflated system has 2^δ variables and $2^\delta n+\displaystyle\sum_{k=0}^{\delta-1}2^kp_k$ where p_k is the corank of DLZ system k.*

Example 8. [10]

Let $f = (x^4 - yzt, y^4 - zxt, z^4 - xyt, t^4 - xyz)$. The root has multiplicity 131. Two steps of DLZ deflation are needed with length $(4, 4)$. The regular system has 16 variables and 28 equations. $\circ$

The example 6 lies to the class of systems of "breadth one" as defined by Dayton and Zeng in [10], i.e., the length is $(1, \ldots, 1)$. Note that this notation corresponds to the embedding dimension 1 as introduced by Giusti, Lecerf, Salvy, Yakoubsohn in [17]. For this class the DLZ deflation can be modified in order to obtain μn variables and μm equations.

6. Deflating and kerneling

We propose a new construction to deflate a system without adding new variables. It is based on two operations we called deflating and kerneling in the introduction.

6.1. Deflating. This operation consists to replace an equation $g(x) = 0$ by the n equations $\partial_i g(x) = 0, i = 1 : n$ when we have simultaneously $g(w) = 0$ and $\partial_i g(w) = 0$, $i = 1 : n$. We then can define the following recursive algorithm.

deflating$(f, \bar{w}, \epsilon)$

- Input : $f = (f_1, \ldots, f_m)$, $\bar{w}$ a point close to a multiple root w of f, and ϵ a precision.
- Let $J := Df(x)$ and $J_{\bar{w}} := Df(\bar{w})$.
- Let m_J the number of lines of J.
- $f_{deflated} = \emptyset$
- for $k = 1 : m_J$
- if $\max_{1 \le j \le n} |J_{\bar{w}}(k, j)| \le \epsilon$ then
- deflating$(J(k, :), \bar{w}, \epsilon)$
- else
- $f_{deflated} = f_{deflated} \cup \{f_k / LT(f_k)\}$
- end if
- end for
- Output $f_{deflated}$

Remark 2.

The assignment $f_{deflated} = f_{deflated} \cup \{f_k / LT(f_k)\}$ must be understood in the following way : the polynomial $f_k / LT(f_k)$ is added if it is not already an element of the set $f_{deflated}$.$\circ$

6.2. Kerneling. Let us consider a system $f = (f_1, \ldots, f_m)$ such that each line of $Df(w)$ is non zero and $Df(w)$ has a rank $r < n$. Without loss of generality we can write

$$Df(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix} \in \mathbf{C}^{m \times n}$$

where $A(w)$ is an invertible matrix of size $r \times r$. Then the Schur complement $D(w) - C(w)A^{-1}B(w)$ is zero. Hence w is a root of the system

$$D(x) - C(x)A^{-1}(x)B(x) = 0.$$

The kerneling operation consists of adding to the initial system at most the $(m -$

$r) \times (n-r)$ polynomials given by the non zero numerators of the coefficients of the Schur complement. We then can define the following algorithm.
$\textsf{\textbf{kerneling}}(f, \bar{w}, \epsilon)$

- Input : $f = (f_1, \ldots, f_m)$, $\bar{w}$ a point close to a multiple root w of f, and ϵ a precision. Each line of $Df(\bar{w})$ is non zero.
- Determine r the numerical rank of $Df(\bar{w})$.
- Determine an invertible submatrix $A(\bar{w})$ of $Df(\bar{w})$ of size $r \times r$.
- Compute $S(x) = det(A(x))D(x) - det(A(x))C(x)A^{-1}B(x)$.
- $f_{deflated} = f \cup \{\text{elements of } S(x)\}$
- Output $f_{deflated}$

6.3. Equivalent system. Combining deflating and kerneling operations we compute a equivalent system of n variables and n equations.
$\textsf{\textbf{equivalent}}(f, \bar{w}, \epsilon)$

- Inputs : $f = (f_1, \ldots, f_m)$, $\bar{w}$ a point close to a multiple root w of f and ϵ a precision.
- $f_{deflated} = f$.
- while $Df_{delated}(\bar{w})$ is not numerically full rank
-     $f_{deflated} = \textsf{deflating}(f_{deflated}, \bar{w}, \epsilon)$
-     $f_{deflated} = \textsf{kerneling}(f_{deflated}, \bar{w}, \epsilon)$
- end while
- $f_{deflated} = \{n \textsf{ equations of full rank from } f_{deflated}\}$
- Output $f_{deflated}$

6.4. Example. Let us consider

$$f(x,y) = (x^3/3 + xy^2 + x^2 + 2xy + y^2,\ x^2y + x^2 + 2xy + y^2)$$

The point $(0,0)$ is a root of $f(x,y) = 0$ with multiplicity 6. The deflating algorithm applied with $w = (0,0)$ gives :

$$\begin{array}{cc|cc} \partial_1 & \partial_2 & \partial_1 & \partial_2 \\ x^2+y^2+2x+2y & 2yx+2x+2y & 2xy+2x+2y & x^2+2x+2y \end{array}$$

All these previous quantities vanish at w. An additional step of deflating operation gives

$$\begin{array}{cc|cc||cc|cc} \partial_{11} & \partial_{12} & \partial_{21} & \partial_{22} & \partial_{11} & \partial_{12} & \partial_{11} & \partial_{12} \\ 2x+2 & 2y+2 & 2y+2 & 2x+2 & 2y+2 & 2x+2 & 2x+2 & 2 \end{array}$$

All these quantities are non zero at w. Hence the deflated system is :

$$f_{deflated}(x,y) = (x^2 + y^2 + 2x + 2y, \quad xy + x + y, \quad x^2 + 2x + 2y)$$

Now we can use the kerneling algorithm of this new system.

$$Df_{deflated}(x,,y) = \begin{pmatrix} 2x+2 & 2y+2 \\ y+1 & x+1 \\ 2x+2 & 2 \end{pmatrix}.$$

Then $Df_{deflated}(0,0)$ has rank one. We can consider $A(x) = 2x+2$. The Schur complement of $Df_{deflated}(x,y)$ associated to $2x+2$ is

$$\begin{pmatrix} x+1 \\ 2 \end{pmatrix} - \frac{2y+2}{2x+2}\begin{pmatrix} y+1 \\ 2x+2 \end{pmatrix} = \frac{1}{x+1}\begin{pmatrix} x^2+2x-y^2-2y \\ -2xy-2y \end{pmatrix}.$$

Finally from the system

$$(x^2+y^2+2x+2y, \quad xy+x+y, \quad x^2+2x+2y,\, x^2+2x-y^2-2y,\, y)$$

we can choose

$$f_{deflated}(x,y) = (x+y+xy, y)$$

which is regular at w.

6.5. Why the multiplicity decreases? Let I be the ideal generated by $f_1, \ldots, f_m$ and w a multiple isolated root of $f_1 = \ldots = f_m = 0$. We deal with $\mathbf{C}\{x-w\}$ the local ring of convergent power series at w and $I\mathbf{C}\{x-w\}$ the ideal generated by I in $\mathbf{C}\{x-w\}$. Then the multiplicity of w is the dimension of the local quotient algebra $\mathbf{C}\{x-w\}/I\mathbf{C}\{x-w\}$. This dimension is finite if and only if the root w is isolated. We denote by $\{g_1, \ldots, g_p\}$ a local standard basis of $I\mathbf{C}\{x-w\}$. Let $< LT(I\mathbf{C}\{x-w\}) >$ the ideal generated by the leading monomials of IA. Then the multiplicity is the number of monomial that are not contained in $< LT(I\mathbf{C}\{x-w\}) >$. This number is independent of the chosen order on the monomials. We have the two classical results :

Lemma 1. *Let h not in IA and $h(w) = 0$. Then the multiplicity of w as root of $f_1 = \ldots = f_m = 0$ is strictly greater than the multiplicity of w as root of $h = f_1 = f_2 = \ldots = f_m = 0$.*

Proof. Since the leading term of h is not in $I\mathbf{C}\{x-w\}$ the lemma follows easily. □

The result we use to explain why the the multiplicity decreases under the action of the algorithm deflated is stated by Arnold, Gusein-Zade and Varchenko in [5] page 100 :

Lemma 2. *Let $g = (g_1, \ldots, g_n) \in \mathbf{C}[x]^n$. Then the Jacobian $det(Dg(x))$ is not in the ideal $< g_1, \ldots, g_n >$.*

The two lemmas below explain why the multiplicity decreases under the operations of deflating and kerneling.

Lemma 3. *Let w a multiple root of a system $f_1 = \ldots = f_m = 0$ such that* $\operatorname{grad} f_1(w) = 0$. *Then the multiplicity of w as root of $f_1 = \ldots = f_m = 0$ is strictly greater than the multiplicity of w as root of $\partial_1 f_1 = \ldots = \partial_n f_1 = f_2 = \ldots = f_m = 0$.*

Proof. Let $g = (f_1, g_2 \ldots, g_n)$, the $g_i's$ being selected from the $f_2, \ldots, f_m$. Since the jacobian of g is not is the ideal generated by g, see lemma 2, then each line of the jacobian matrix of g has at least one element which is not in $< g >$. In particular at least one of $\partial_i f_1$'s is not in $< g >$. Following the lemma 1 we are done. □

Lemma 4. *Let w a multiple root of $f_1 = \ldots = f_m = 0$ such that* $\operatorname{grad} f_i(w) \neq 0$, $i = 1 : m$. *Let r be the rank of $Df(w)$ and*

$$Df(w) = \begin{pmatrix} A(w) & B(w) \\ C(w) & D(w) \end{pmatrix}$$

where $A(w)$ is an invertible matrix of size $r \times r$. Let $S(x) = det(A(x))D(x) - C(x)\Delta(x)B(x)$ where $\Delta(x) = det(A(x))A(x)^{-1}$. Then the multiplicity of w as root of $f_1 = \ldots = f_m = 0$ is strictly greater than the multiplicity of w as root of $S_{11} = \ldots = S_{m-r,n-r} = f_1 = f_2 \ldots = f_m = 0$.

Proof. It is sufficient to prove that one of S_{ij}'s is not in the ideal $< f_1, \dots, f_m >$. Then, by lemma 1, the multiplicity of w as root of $f_1 = \dots = f_m = 0$ is strictly greater than the multiplicity of w as root of $S_{ij} = f_1 = f_2 \dots = f_m = 0$.
Let $F = (f_1, \dots, f_r, h_1, \dots, h_{n-r})$ with $h_i \in \{f_{r+1}, \dots, f_m\}$.
We have $det(DF(x)) = det(A(x)\, det(S_F(x))$ where $S_F(x)$ is the Schur complement of $DF(x)$ associated to $A(x)$. From lemma 2, $det(DF(x))$ is not in the ideal $< F >$. So it is the same for $det(A(x)$ and $det(S_F(x))$ which divide $det(DF(x))$. Hence there exists at least $n-r$ coefficients of the matrix $S_F(x)$ which are not in the ideal $< F >$. Since the coefficients of $S_F(x)$ are also coefficients of the matrix $S(x)$ the conclusion follows. □
How much the multiplicity drops at each step of the equivalent algorithm ?

Theorem 19. *For $k \geq 1$, let $F_0 = f$ and F_{k-1} the deflated system obtained at the step $k-1$ of equivalent algorithm and m_{k-1} the number of polynomials of F_{k-1}. Let p_k be the number of polynomials we add by deflating operation at the step k. We note by G_k the system F_k augmented by these p_k polynomials. Let r_k be the rank of the jacobian matrix of G_k at w. Then the number N of steps of the algorithm stops is equal to*

$$\min\{k \,:\, r_k = n \quad \textit{or} \quad \sum_{k=1}^{N} s_k + t_k \leq \mu\}$$

where $\max(0, \min(1, p_k)) \leq s_k \leq p_k$ *and* $1 \leq t_k \leq p_k(n - r_k)$.

Proof. From the lemmas 3 and 4 the multiplicity decreases at least by one. But we can be more precise. Let μ_k be the multiplicity of w as root of F_k. The deflating algorithm gives p_k polynomials. Then the multiplicity of the root w of G_k drops by $\mu_{k-1} - s_k$ where $\max(0, \min(1, p_k) \leq s_k \leq p_k$. Next, if the jacobian matrix of G_k at w has rank $r_k = n$ the equivalent algorithm stops. Otherwise, the multiplicity of w as root of F_k is $\mu_{k-1} - s_k - t_k$ where $1 \leq t_k \leq p_k(n - r_k)$. This bound is justified because all the polynomials of the Schur complement computed by the kerneling algorithm can be equal. □

7. Examples

We first treat three examples given in [65]. These examples show it is not necessary to know the complete structure of the local quotient algebra to determine a regular equivalent system from the initial one with a multiple root.
Example 9. [65]

$$f_k(x_1, \dots, x_n) = x_1 + \dots + x_n + x_k^2, \quad k = 1 : n.$$

The jacobian matrix $Df(x) = \begin{pmatrix} 2x_1 + 1 & 1 & \dots & 1 \\ 1 & 2x_2 + 1 & \dots & 1 \\ \vdots & & & \\ 1 & 1 & \dots & 2x_n + 1 \end{pmatrix}$ has rank one at $(0, \dots, 0)$. The Schur complement associated to $2x_1 + 1$ gives the equations :

$$x_1 = 0$$
$$(2x_1 + 1)(2x_k + 1) - 1 = 0, \quad k \geq 2.$$

Example 10. [65]

$$\begin{aligned} f_k(x_1,\dots,x_n) &= x_k^3 - x_{k+1}x_{k+2}, \quad k = 1:n-2 \\ f_{n-1}(x_1,\dots,x_n) &= x_{n-1}^3 - x_n x_1 \\ f_n(x_1,\dots,x_n) &= x_n^3 - x_1 x_2 \end{aligned}$$

A multiple root is $(0,\dots,0)$. In the first deflation step we replace the f_k's by their gradients. We obtain the equations :

$$x_1 = \ldots = x_n = 0.$$

Example 11. [65]

$$\begin{aligned} f_k(x_1,\dots,x_n) &= x_k + \ldots + x_{n-2}, \quad k = 1:n-2 \\ f_{n-1}(x_1,\dots,x_n) &= x_1 + \ldots + x_{n-2} + x_{n-1}^5 + x_n^2 \\ f_n(x_1,\dots,x_n) &= x_1 + \ldots + x_{n-2} + x_n^2 \end{aligned}$$

A multiple zero is $(0,\dots,0)$. The Jacobian matrix $Df(x) = \begin{pmatrix} I_{n-2} & 0 & 0 \\ 1\ldots1 & 5x_{n-1}^4 & 2x_n \\ 1\ldots1 & 0 & 2x_n \end{pmatrix}$ has rank $n-2$ at the multiple root $(0,\dots,0)$. The Schur complement associated to I_{n-2} furnishes the equations

$$5x_{n-1}^4 = 2x_n = 0.$$

After one step of deflation we obtain the system

$$f_1 = \ldots = f_{n-2} = x_{n-1} = x_n = 0.$$

Example 12. cmbs1 [58]

$$f(x,y,z) = (x^3 - yz,\ y^3 - xz,\ z^3 - xy).$$

A multiple root is $(0,0,0)$. A first of deflation gives the equations $x = y = z = 0$.

Example 13. cmbs2 [58]

$$\begin{aligned} f(x,y,z) =&(x^3 - 3x^2y + 3xy^2 - y^3 - z^2, \\ & z^3 - 3z^2x + 3zx^2 - x^3 - y^2, \\ & y^3 - 3y^2z + 3yz^2 - z^3 - x^2). \end{aligned}$$

A multiple root is $(0,0,0)$. A first step of deflation gives the equations $x = y = z = x - y = x - z = y - z = 0$.

Example 14. caprasse [40]

$$\begin{aligned} f(x,y,z,t) =&(-x^3z + 4\,xy^2z + 4\,x^2yt + 2\,y^3t + 4\,x^2 - 10\,y^2 + 4\,xz - 10\,yt + 2, \\ & -xz^3 + 4\,yz^2t + 4\,xzt^2 + 2\,yt^3 + 4\,xz + 4\,z^2 - 10\,yt - 10\,t^2 + 2 \\ & y^2z + 2\,xyt - 2\,x - z, \\ & 2\,yzt + xt^2 - x - 2\,z). \end{aligned}$$

The multiple root is $(2, -i\sqrt{3}, 2, i\sqrt{3})$. The gradient of each f_k is non zero at w and the jacobian matrix $Df(w)$ has rank 2. The step of kerneling adds the four polynomials before we get a regular system at w.

$$-10\,xt-5\,xy-5\,zt+\tfrac{17}{4}\,xyt^2z^2-7/2\,yt^2x^2z^3+11/4\,yt^4x^2z+\tfrac{17}{4}\,yt^2x^2z-2\,y^2txz^4+\tfrac{47}{8}\,y^2txz^2+\tfrac{49}{8}\,xt^3z^2y^2-7\,x^2z^3t-3/4\,x^2zt^3$$
$$+\tfrac{31}{4}\,x^2zt+\tfrac{37}{4}\,y^2z^3t-5\,y^2zt^3-25\,y^2zt+5\,xyt^4+\tfrac{103}{8}\,xz^2t+xyz^4+\tfrac{15}{4}\,xyz^2+15\,yzt^2-xt^3+\tfrac{19}{4}\,z^3t+11\,zt^3-5/2\,yz^3+5/4\,y^3t^2z^3$$
$$-y^3t^4z+11\,y^3t^2z-3\,y^2t^5x+7\,y^2t^3x+\tfrac{13}{4}\,yt^2z^3-yt^4z-1/2\,x^3t^3z^2-\tfrac{7}{8}\,xt^3z^2-3/2\,x^3tz^2+4\,xty^2+x^3tz^4-xtz^4-3/2\,x^2yz^3+2\,x^2yz$$
$$-\tfrac{9}{8}\,x^3t^5+3\,xt^5+3/4\,x^3t^3+3/8\,x^3t,$$

$$5/2\,xy^2z+\tfrac{25}{2}\,xzt^2+5/4\,y^3t-\tfrac{25}{4}\,y^2-\tfrac{25}{4}\,t^2-\tfrac{25}{2}\,yt+\tfrac{55}{4}\,yt^3+15\,yzxt+1/2\,yz^4x^2t-5/4\,yz^2x^2t^3-4\,yz^2x^2t-19/2\,y^2zxt^2-5/4\,yz^3xt$$
$$+3\,yzxt^3-\tfrac{13}{8}\,xt^2y^2z^3+\tfrac{25}{4}\,t^4-\tfrac{23}{4}\,t^2z^2x^2+15/2\,t^2z^2y^2+1/4\,y^3z^4t-3/2\,y^3z^2t^3-3/2\,y^3z^2t+1/2\,y^2z^5x-3\,y^2z^3x-yz^2t^3+x^3t^2z^3$$
$$-3/8\,x^3t^4z-1/8\,x^3t^2z+3/2\,x^2t^5y-11/2\,x^2t^3y-\tfrac{13}{8}\,xt^2z^3+1/2\,xt^4z+5\,t^2z^2-3/2\,t^4x^2-15/2\,t^4y^2-5/2\,t^2x^2+\tfrac{55}{4}\,t^2y^2+15/2\,z^2y^2$$
$$-5/4\,z^4y^2+y^3t^5-5/4\,yt^5-9/4\,y^3t^3,$$

$$-5\,xt-10\,xy-15\,yz-10\,zt+9/4\,xyt^2z^2+24\,yt^2x^2z+\tfrac{25}{4}\,y^2txz^2-7/2\,y^2x^2z^3t+\tfrac{43}{8}\,y^2x^2zt^3-\tfrac{7}{8}\,y^2x^2zt+\tfrac{33}{4}\,y^3t^2xz^2-6\,yt^2x^3z^2$$
$$-3/2\,x^2z^3t-\tfrac{11}{8}\,x^2zt^3+\tfrac{103}{8}\,x^2zt+5\,y^2z^3t+6\,y^2zt^3+2\,y^2zt+7\,xyt^4-15\,xyt^2+\tfrac{31}{4}\,xz^2t-1/2\,xyz^4+\tfrac{43}{2}\,xyz^2+5\,yzt^2-5\,xt^3$$
$$+3/8\,z^3t+3\,zt^3+11/4\,yz^3+15\,y^3t^2z+11\,y^2t^3x-7\,x^3tz^2+23\,xty^2-9\,x^2yz^3+8\,x^2yz+\tfrac{21}{8}\,y^4z^3t-5\,y^4zt^3+4\,y^4zt-3\,y^3xt^4$$
$$-3\,y^3xt^2-3/2\,y^3xz^4+2\,y^3xz^2-yz^2x^3+\tfrac{25}{4}\,x^3t^3+\tfrac{19}{4}\,x^3t-5\,y^3z+\tfrac{21}{4}\,y^3z^3+\tfrac{13}{4}\,yt^2x^3-x^4tz+7/4\,yt^4x^3+yz^4x^3+x^4z^3t-zx^4t^3,$$

$$10\,xy^2z+10/3\,yz^2t+\tfrac{20}{3}\,xzt^2+10\,y^3t-\tfrac{25}{3}\,y^2-\tfrac{25}{3}\,t^2-\tfrac{50}{3}\,yt+10\,yt^3+20\,yzxt-\tfrac{47}{6}\,yz^2x^2t-6\,y^2zxt^2-4/3\,yz^3xt-4\,yzxt^3+x^3yz^3t$$
$$-5/6\,x^3yzt^3-7/6\,xy^3z^3t+2/3\,xy^3zt^3-2/3\,xy^3zt-5/4\,y^2t^2x^2z^2-4/3\,t^2z^2x^2-8/3\,t^2z^2y^2-2/3\,y^3z^2t-7/2\,y^2z^3x-13/2\,x^3t^2z$$
$$+4/3\,x^2t^3y-5/3\,t^4y^2+10\,t^2x^2+\tfrac{80}{3}\,t^2y^2+10/3\,z^2y^2-1/4\,z^4y^2-\tfrac{34}{3}\,y^3t^3-5/3\,y^4t^2+2\,y^4z^2+4/3\,y^4t^4-3/4\,y^4z^4-2/3\,x^2y^2t^4$$
$$+2/3\,x^2y^2z^4-\tfrac{23}{12}\,x^2y^2z^2+x^4t^2z^2$$

Example 15. decker2 [11]

$$f(x,y) = (x+y^3, x^2y-y^4).$$

A multiple root is $(0,0,0)$. A first step of deflation gives the equations $x+y^3 = x = y = 0$.

Example 16. mth191 [30]

$$f(x,y) = (x^3+y^2+z^2-1, x^2+y^3+z^2-1, x^2+y^2+z^3-1).$$

A multiple root is $w = (0,1,0)$. The gradients of each polynomials are non zero at w. The jacobian matrix has rank 1. The step of kerneling adds the four polynomials :

$$x\,(9\,xy-4)$$
$$z\,(3\,y-2)$$
$$x\,(3\,y-2)$$
$$z\,(9\,zy-4)$$

The system $f_1 = x\,(9\,xy-4) = z\,(3\,y-2) = 0$ is regular at w.

Example 17. DZ1 [10]

$$f(x,y,z,t) = (x^4-yzt,\ y^4-xzt,\ z^4-xyt,\ t^4-xyz).$$

A multiple root is $w = (0,0,0,0)$. A step of deflation gives $x = y = z = t = 0$.

Example 18. DZ2 [10]

$$f(x,y,z) = (x^4,\ x^2y + y^4,\ z + z^2 - 7x^3 - 8x^2).$$

A multiple root is $w = (0, 0, -1)$. A step of deflation adds the equation $x = y = 0$.

Example 19. DZ3 [10]

$$\begin{aligned} f(x,y) = &(14\,x+33\,y-3\,\sqrt{5}\left(x^2+4\,xy+4\,y^2+2\right)+\sqrt{7}+x^3+6\,x^2y+12\,xy^2+8\,y^3 \\ &\tfrac{41}{8}\,x-9/4\,y-1/8\,\sqrt{5}+x^3-3/2\,x^2y+3/4\,xy^2-1/8\,y^3+3/8\,\sqrt{7}\left(4\,xy-4\,x^2-y^2-2\right). \end{aligned}$$

A multiple root is $w = ((2\sqrt{7} + \sqrt{5})/5,\ (2\sqrt{5} - \sqrt{7})/5)$. The gradients of each polynomials are non zero. The step of kerneling adds the polynomial

$$\begin{aligned} &-360\,x^2\sqrt{5}y+630\,xy^2\sqrt{5}+240\,xy-180\,\sqrt{7}x^3+360\,\sqrt{7}y^3+1260\,x^2+1440\,y^2-360\,x^3\sqrt{5}+540\,x^3y+45\,x^2y^2-540\,xy^3-180\,y^3\sqrt{5} \\ &+540\,\sqrt{7}x\sqrt{5}y+180\,x^4+180\,y^4+1605-960\,\sqrt{7}x+480\,\sqrt{7}y-600\,\sqrt{5}x-1200\,\sqrt{5}y+360\,\sqrt{7}\sqrt{5}x^2-630\,\sqrt{7}x^2y-360\,\sqrt{7}xy^2 \\ &-360\,\sqrt{7}\sqrt{5}y^2 \end{aligned}$$

Its gradient is zero at w. The step of deflation replaces it by the two following polynomials :

$$\begin{aligned} &1/3\,y+7/2\,x+x^3+3/4\,\sqrt{7}\sqrt{5}y-4/3\,\sqrt{7}-5/6\,\sqrt{5}-3/4\,\sqrt{7}x^2-3/2\,x^2\sqrt{5}+9/4\,x^2y+1/8\,xy^2-3/4\,y^3-x\sqrt{5}y+\tfrac{7}{8}\,y^2\sqrt{5}+\sqrt{7}\sqrt{5}x \\ &-7/4\,\sqrt{7}xy-1/2\,\sqrt{7}y^2 \end{aligned}$$

$$\begin{aligned} &4/9\,x+16/3\,y+4/3\,y^3+\sqrt{7}\sqrt{5}x+\tfrac{8}{9}\,\sqrt{7}-\tfrac{20}{9}\,\sqrt{5}+2\,\sqrt{7}y^2+x^3+1/6\,x^2y-3\,xy^2-y^2\sqrt{5}-2/3\,x^2\sqrt{5}+7/3\,x\sqrt{5}y-7/6\,\sqrt{7}x^2 \\ &-4/3\,\sqrt{7}xy-4/3\,\sqrt{7}\sqrt{5}y \end{aligned}$$

The system build from $f_1,\ f_2$ and from the two previous polynomials is regular at w.

Example 20. Ojika2 [43]

$$f(x,y,z) = (x^2 + y + z - 1,\ x + y^2 + z - 1,\ x + y + z^2 - 1).$$

A multiple root is $w = (1, 0, 0)$. The rank of $Df(w)$ is 2. The step of kerneling adds the equation $4xyz - x - y - z + 1 = 0$.

Example 21. Ojika3 [43]

$$f(x,y,z) = (x + y + z - 1,\ 2x^3 + 5y^2 - 10z + 5z^3 + 5,\ 2x + 2y + z^2 - 1).$$

A multiple root is $w = (-5/2, 5/2, 1)$. The rank of $Df(w)$ is 2. The step of kerneling adds the equation $3x^2z - 5yz + 5y - 3x^2 = 0$.

Example 22. Lecerf [29]

$$\begin{aligned} f(x,y,z) = (\quad &2\,x + 2\,x^2 + 2\,y + 2\,y^2 + z^2 - 1, \\ &(x + y - z - 1)^3 - x^3, \\ &\left(2\,x^3 + 2\,y^2 + 10\,z + 5\,z^2 + 5\right)^3 - 1000\,x^5 \quad). \end{aligned}$$

A multiple root is $w = (0, 0, -1)$. The rank of $Df(w)$ is one. There is only one step of deflation to obtain the regular system

$$\begin{aligned}
&x + x^2 + y + y^2 + 1/2\, z^2 - 1/2,\\
&y - z - 1,\\
&x + y - z - 1,\\
&\frac{9}{14}\, x^5 + \frac{5}{28}\, \left(2\, x^3 + 2\, y^2 + 10\, z + 5\, z^2 + 5\right) x^2 - \frac{625}{126}\, x,\\
&y,\\
&x,\\
&1 + z.
\end{aligned}$$

8. Conclusion and future work

We have shown how to derive an equivalent regular system from a singular initial one, when we know the root. The stability of this process will be done in a future work and we describe briefly how to proceed. But from a numerical point of view a multiple root makes no sense and it is more realistic to speak of a cluster of roots : a m–cluster of roots is a open ball which contains m isolated regular roots of the system. Moreover we would hope for results with a “small” size of the cluster. The operation of deflating is based on the evaluation of the gradient of a function, say $g(x)$, at given point $\bar{w}$. To decide whether there exists a root (or a cluster of roots) of this gradient closed to $\bar{w}$ we need to know if there exists $\bar{x}_1$ such that $(\bar{x}_1, \bar{w}_2, \ldots, \bar{w}_n)$ is closed to $\bar{w}$ and cancels the gradient of g. This can be done with the theoretical background developed in [**18**] where the words ”closed to” and ”small” are quantified.

The operation of kerneling requires more attention since we must discover the numerical rank of a jacobian matrix at a point $\bar{w}$ “closed to” the multiple root or the cluster of roots. The difficulty is that the rank drops only at the multiple root or in the cluster of roots. We propose to fix a coordinate, say x_1, and to perform a LU decomposition of the jacobian evaluated at $(x_1, \bar{w}_2, \ldots, \bar{w}_n)$. Each element of the diagonal of the matrix U of the LU decomposition is a polynomial in x_1. The numerical rank of the jacobian matrix is the number of these polynomials having a root “closed to” $\bar{w}_1$.

We illustrate these principles on Lecerf’s example 22 [**29**]. We first show how to numerically discover that there is probably a point w near $(x_0, y_0, z_0) = (0.1, 0.09, -1.1+0.1i)$ where the jacobian matrix has a rank one. For that we determine the matrix U of the LU decomposition at (x_0, y_0, z). The diagonal of U is given by

$$\begin{aligned}
&2.4\\
&2.28 + 4.51\, z + 2.28\, z^2,\\
&3633.58 + 25322.98\, z + 75771.82\, z^2 + 126177.32\, z^3 + 126276.08\, z^4 + 75944.48\, z^5 + 25413.69\, z^6 + 3650.4\, z^7.
\end{aligned}$$

The Newton iteration (or more generally the Schröder iteration) initialized to z_0 and applied respectively to the polynomials $U_{22}(z)$ and $U_{33}(z)$ converges respectively to $-0.99 + 0.14i$ and $-0.98 + 0.05i$. The initial point z_0 is an approximated zero of $U_{22}(z)$ and $U_{33}(z)$. This is the meaning given to the word “closed to”. We will deduce that the numerical rank of the jacobian is one.

In this example we can numerically prove that there exists a point w where the two last lines of the jacobian matrix are zero. In fact the evaluation of the gradients of

f_2 and f_3 at (x_0, y_0, z) gives

$$\begin{aligned}\nabla f_2(x_0,y_0,z) =&(2.91+5.94\,z+3.0\,z^2, \quad 0.0003\ (99+100\,z)^2, \quad -0.0003\ (99+100\,z)^2),\\ \nabla f_3(x_0,y_0,z) =&(4.03+18.06\,z+27.03\,z^2+18.0\,z^3+4.5\,z^4,\\ &-0.0000000432\ \left(25091+50000\,z+25000\,z^2\right)^2,\\ &0.0000012\ \left(25091+50000\,z+25000.\,z^2\right)^2(1+z)).\end{aligned}$$

Thanks to Newton iteration initialized at z_0 and applied successively to each polynomial coordinate of these two gradients we find a root closed to z_0. From this we can prove the existence of a perturbed system of the initial one with the two last lines of the jacobian matrix are zero. With this information we deflate the two corresponding equations of the initial system. This heuristic approach will be completely justified in a future work.

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