Lecture 1

General Introduction to Differential Equations

COMASIC (M2)
January 11th, 2017

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Functional equation with derivatives

$$(x', y') = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = (\dot{x}, \dot{y}) = (-y, x - y)$$

Local description of motion
Functional equation with derivatives

\[ (x', y') = \left( \frac{dx}{dt}, \frac{dy}{dt} \right) = (\dot{x}, \dot{y}) = (-y, x - y) \]

Local description of motion
Ordinary Differential Equations

Very Useful

Ordinary (or Total) vs Partial \( \frac{\partial}{\partial u} \)

Sophus Lie

"Among all of the mathematical disciplines the theory of differential equations is the most important(...) It furnishes the explanation of all those elementary manifestations of nature which involve time."

Convenient modeling language

• Continuous dynamics (vs discrete)
• No Boundary conditions (entire space)
• No memory (next state completely determined from the current)
Ordinary (or Total) vs Partial \( \left( \frac{\partial}{\partial u} \right) \)

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**Convenient modeling language**

- Continuous dynamics (vs discrete)
- No Boundary conditions (entire space)
- No memory (next state completely determined from the current)
An ant starts somewhere on a black and white squared plane

- if the square is white, the ant turns right then move forward
- if the square is black, the ant turns left then move forward
- the ant flips the color of its square before moving
• Cauchy-Lipschitz theorem: Local existence and unicity theorem (assuming Lipschitz continuity) $f : \mathbb{R} \rightarrow \mathbb{R}$ is Lipschitz continuous in $X \subset \mathbb{R}$ if and only if there exists $K \geq 0$ such that

$$|f(y) - f(x)| \leq K|y - x|, \quad \forall x, y \in X$$

• Solutions often involve transcendental functions (sine, exp, etc.) For instance the first-order homogeneous equation $y' = ay$:

$$y = y_0 \exp(at)$$

• Liouville theorem: No closed form solutions in general $x' = \exp(-t^2)$ then

$$x(t) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} \exp(-t^2) \, dt$$
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\]
Finite Time Explosion Problems

- \( x' = x^2, \ x(0) = x_0 \) (Only locally Lipschitz)
- \( x(t) = \frac{1}{\frac{1}{x_0} - t} \)
- Singularity at \( t = \frac{1}{x_0} \), maximum interval \((-\infty, \frac{1}{x_0})\)
Numerical Integration

Euler Integration Schemes $x' = f(x)$

- Explicit $x^* = x + f(x)\delta$
- Implicit $x^* = x + f(x^*)\delta$

Other similar Integration Schemes: the Runge-Kutta family
Numerical Integration

Euler Integration Schemes \( x' = f(x) \)

\[
\begin{align*}
\dot{x} & = x + f(x)\delta \quad \text{Explicit} \\
\dot{x} & = x + f(x\cdot)\delta \quad \text{Implicit}
\end{align*}
\]

Other similar Integration Schemes: the Runge-Kutta family

Picard Iterations

\[
\dot{x} = x + \int_0^\delta f(x)dt
\]

It boils down to approximate the integral term
Numerical Integration: Convergence and Stability

Numerical Analysis

- **Convergence**: does the numerical scheme approximates the solution when the discrete step goes toward zero? The **order** gives the **local** quality of convergence.
- **Stability**: the propagation of errors (stiffness).
Numerical Analysis

- **Convergence**: does the numerical scheme approximate the solution when the discrete step goes toward zero? The **order** gives the **local** quality of convergence.

- **Stability**: the propagation of errors (stiffness).

\[
(x', y') = (-y, x)
\]

Euler (order 1)  Runge-Kutta (order 4)
Other Methods

- **Geometrical Integration**: invariant-aware integration (e.g. Symplectic Methods)
- **Quantized State Systems (QSS) Methods**: efficient when simulating sparse systems
• **Geometrical Integration**: invariant-aware integration (e.g. Symplectic Methods)

• **Quantized State Systems (QSS) Methods**: efficient when simulating sparse systems

\[ (x', y') = (-y, x) \]

Symplectic Integration
Qualitative Analysis

\[
(\dot{x}_1, \dot{x}_2) = (x_1 - x_1^3 - x_2 - x_1x_2^2, x_1 + x_2 - x_1^2x_2 - x_2^3)
\]

The solution for \( x_0 = (1, 0) \) respects \( x_1(t)^2 + x_2(t)^2 - 1 = 0 \) for all \( t \).
Why Are Invariants Important?

Numerical Integration & Qualitative Analysis

- More precise numerical integration (Geometrical Integration)
- Better understanding of the dynamics without solving the problem (some invariants represent conserved quantities like momentum or energy)

Formal Verification

- Formal verification for dynamical and hybrid systems
- Static Analysis (as templates to statically analyze an implementation)
- Safety, Reachability, Stability
Problem I. Checking Invariance of Algebraic Equations

Given \( \dot{x} = (-2x_2, -2x_1 - 3x_1^2) \), \( p(x_0) = 0 \), is \( p(x(t)) = 0 \) for all \( t \)?

\[
p(x_1, x_2) = x_1^2 + x_1^3 - x_2^2
\]

\[
p(x_1, x_2) = x_1 - x_2^2
\]
Problem I. Checking Invariance of Algebraic Equations

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\[
p(x_1, x_2) = x_1^2 + x_1^3 - x_2^2
\]

True

\[
p(x_1, x_2) = x_1 - x_2^2
\]

False
Problem II. *Generate Algebraic Invariant Equations*

Given \( \dot{x} = (-x_1 + 2x_1^2x_2, -x_2) \), how to generate \( p \) such that \( p(x(t)) = 0 \)?
Problem II. *Generate Algebraic Invariant Equations*

Given $\dot{x} = (-x_1 + 2x_1^2x_2, -x_2)$, how to generate $p$ such that $p(x(t)) = 0$?

$p(x_1(0), x_2(0))(x_1, x_2) = (x_2(0) - x_1(0)x_2(0)^2)x_1 - x_1(0)(x_2 - x_1x_2^2) = 0$
Problem II. \textit{Generate Algebraic Invariant Equations}

Given \( \dot{x} = (-x_1 + 2x_1^2x_2, -x_2) \), how to generate \( p \) such that \( p(x(t)) = 0 \)?

\[
\begin{align*}
p(x_1(0), x_2(0))(x_1, x_2) &= \left( 2(0)x_2(0) - x_1(0)x_2(0)^2 \right) x_1 - x_1(0)(x_2 - x_1x_2^2) = 0 \\
\frac{x_1}{x_2 - x_1x_2^2} & \text{ is an invariant rational function.}
\end{align*}
\]
1 Ordinary Differential Equations:
   • Cauchy-Lipschitz theorem: existence and uniqueness of solutions
   • Liouville theorem: no closed form solutions in general
   • Numerical integration: convergence and stability
   • Qualitative analysis: invariant regions

2 Next: Differential-Algebraic Equations (Examples)
\[ \ddot{x} = -\lambda x \]
\[ \ddot{y} = -\lambda y - g \]  
\[ \text{(Newton's law)} \]
\[ 0 = L^2 - x^2 - y^2 \]  
\[ \text{(Algebraic constraint)} \]

State variables: \( (x, y, \dot{x}, \dot{y}) \):
- \( x, y \): differential variables
- \( \lambda \): algebraic variable

(Photograph source: Wolfram)
\[ \ddot{x} = -\lambda x \]
\[ \ddot{y} = -\lambda y - g \]  \hspace{1cm} (Newton's law)
\[ 0 = L^2 - x^2 - y^2 \]  \hspace{1cm} (Algebraic constraint)
\[ \ddot{x} = -\lambda x \]

\[ \ddot{y} = -\lambda y - g \]  \quad \text{(Newton’s law)}

\[ 0 = L^2 - x^2 - y^2 \]  \quad \text{(Algebraic constraint)}

State variables: \((x, y, \dot{x}, \dot{y})\):

\[
\begin{align*}
x, y & \quad \text{differential variables} \\
\lambda & \quad \text{algebraic variable}
\end{align*}
\]
Lagrange Equations

\[ \frac{d}{dt} \left( \frac{\partial L}{\partial \dot{q}} \right) - \frac{\partial L}{\partial q} = Q + F^t \lambda \]

- Lagrangian: \( L = T - U \) (Kinetic and potential Energies)
- Generalized coordinates \( q = (q_1, \ldots, q_n) \)
- Holonomic constraints: \( f(q) = 0 \)
- Nonconservative forces: \( Q \)
- \( F^t \): the transpose of the Jacobian of \( f \)
- \( \lambda \): vector of Lagrange multipliers
\[ \dot{V}_C = I_C \]
\[ \dot{I} = V_L \]
\[ L_0 = V_R - R I \]

**Ohm's Law**

\[ 0 = V_S - V_R - V_L - V_C \]

**Algebraic Constraint**

**State variables:** \((V_R, V_C, V_L, I)\)
RLC Circuit

\[ \dot{V}_C = \frac{I}{C} \]
\[ I = \frac{V_L}{L} \]
\[ 0 = V_R - RI \] \hspace{1cm} \text{Ohm’s Law}
\[ 0 = V_S - V_R - V_L - V_C \] \hspace{1cm} \text{Algebraic Constraint}

State variables: \((V_R, V_C, V_L, I)\)
State variables: \((\theta, x)\)

\[
\begin{align*}
L &= \frac{1}{2} Mv_1^2 + \frac{1}{2} mv_2^2 - mg\ell \cos(\theta) \\
v_1 &= \left( \frac{d}{dt} x, 0 \right) \\
v_2 &= \left( \frac{d}{dt} (x - \ell \sin(\theta)), \frac{d}{dt} (\ell \cos(\theta)) \right)
\end{align*}
\]
Lagrange Equations

\[ F = (M + m)\ddot{x} - m\ell\ddot{\theta}\cos(\theta) + m\ell\dot{\theta}^2\sin(\theta) \]

\[ \ddot{x}\cos(\theta) = \ell\ddot{\theta} - g\sin(\theta) \]
Lagrange Equations

\[ F = (M + m)\ddot{x} - m\ell\ddot{\theta}\cos(\theta) + m\ell\dot{\theta}^2 \sin(\theta) \]
\[ \ddot{x}\cos(\theta) = \ell\ddot{\theta} - g \sin(\theta) \]

Control Problem: Find \( F \) such that

\[ \theta \in [\theta_r - \epsilon, \theta_r + \epsilon] \]

for some given reference value \( \theta_r \)
Some Forms of DAEs

- Non-Linear (inverted pendulum):
  \[ f(\dot{x}, x, t) = 0, \quad (f \text{ nonlinear}) \]

- Linear (RLC circuit):
  \[ A(t)\dot{x} + B(t)x + c(t) = 0 \]

- Semi-Explicit (pendulum):
  \[
  \begin{cases}
  \dot{x} = f(x, y, t) \\
  0 = g(x, y, t)
  \end{cases}
  \]
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  0 &= g(x, y, t)
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  \]
To conclude

Next Lecture: More on DAEs

- Index reduction
- Numerical integration
- Modelling tools

Some References