Boucles dans le modèle XOR-Ising

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Journée Cartes
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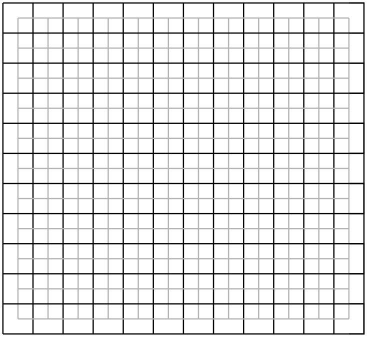
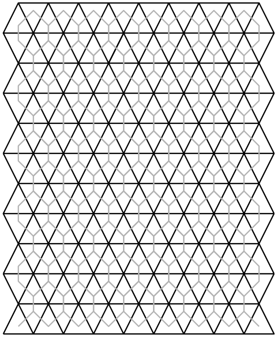
2-dimensional Ising model

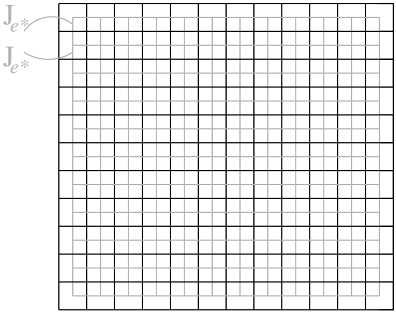
- Planar graph $G = (V, E)$, its dual graph $G^* = (V^*, E^*)$.

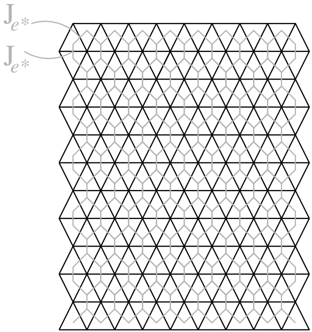




2-dimensional Ising model

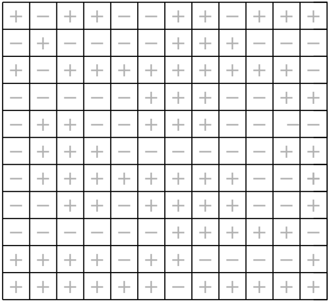
- Planar graph $G = (V, E)$, its dual graph $G^* = (V^*, E^*)$.
- Edges of G^* are assigned positive coupling constant, $(J_{e^*})_{e^* \in E^*}$.

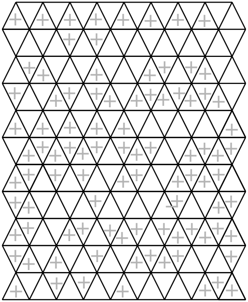




2-dimensional Ising model

- Spin configuration: $\sigma \in \{-1, 1\}^{V^*}$.



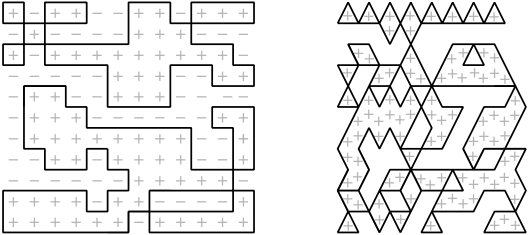


- Ising Boltzmann measure:

$$\mathbb{P}_{\text{Ising}}(\sigma) = \frac{1}{Z_{\text{Ising}}(J)} \exp\Big(\sum_{\{e^* = u^* v^* \in E^*\}} J_{e^*} \sigma_{u^*} \sigma_{v^*} \Big),$$

Partition function expansions (Kramers-Wannier)

- Normalizing constant is the partition function:
  $Z_{\text{Ising}}(J) = \sum_{\sigma \in \{-1,1\}^{V^*}} \exp\Big(\sum_{\{e^*=u^*v^* \in E\}} J_{e^*} \sigma_{u^*} \sigma_{v^*} \Big).$
- Low temperature expansion (LT)
  - Based on the identity:
    $$e^{J_{e^*}\sigma_{u^*}\sigma_{v^*}} = e^{J_{e^*}} (\delta_{\{\sigma_{u^*}=\sigma_{v^*}\}} + e^{-2J_{e^*}} \delta_{\{\sigma_{u^*} \neq \sigma_{v^*}\}}).$$
  - Geometric interpretation: polygons separate ± 1 clusters.



Partition function expansions (Kramers-Wannier)

- Partition function can be rewritten as:

$$\boxed{Z_{\text{Ising}}(J) = 2\Big(\prod_{e^*\in E^*} e^{J_{e^*}}\Big) \sum_{P\in\mathcal{P}(G)} \Big(\prod_{e\in P} e^{-2J_{e^*}}\Big),}$$

- $\mathcal{P}(G)$ is the set of polygonal configuration of G: edge config. P such that each vertex of G is incident to an even number of edges of P.

- High temperature expansion (HT)
  - Based on the identity:

$$e^{J_{e^*}\sigma_{u^*}\sigma_{v^*}} = \cosh(J_{e^*})(1+\sigma_{u^*}\sigma_{v^*}\tanh(J_{e^*})).$$

  - Partition function can be rewritten as:

$$\boxed{Z_{\text{Ising}}(J) = 2^{|V^*|}\Big(\prod_{e^*\in E^*} \cosh(J_{e^*})\Big) \sum_{P^*\in\mathcal{P}(G^*)} \Big(\prod_{e^*\in E^*} \tanh(J_{e^*})\Big).}$$

Partition function expansions (Kramers-Wannier)

- Writing

$$Z_{\mathrm{LT}}(G,J) = \sum_{P\in\mathcal{P}(G)} \Big(\prod_{e\in P} e^{-2J_{e^*}}\Big),$$

$$Z_{\mathrm{HT}}(G^*,J) = \sum_{P^*\in\mathcal{P}(G^*)} \Big(\prod_{e^*\in E^*} \tanh(J_{e^*})\Big),$$

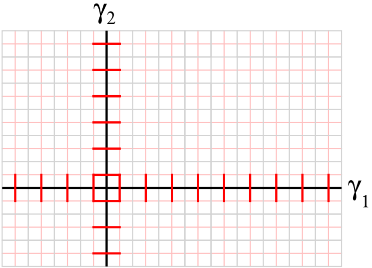
yields Kramers & Wannier duality relation:

$$Z_{\mathrm{LT}}(G,J) = \mathcal{C}(G) Z_{\mathrm{HT}}(G^*,J).$$

2-dimensional Ising model with defects

- Assume G is embedded on the torus $\mathbb{T}$.
- γ_1, γ_2 are cycles in the graph G, winding around the torus in the 2 possible directions.

  (γ_1, γ_2): representative of a basis of $H_1(\mathbb{T}, \mathbb{Z}/2\mathbb{Z}) \simeq (\mathbb{Z}/2\mathbb{Z})^2$.



Fix $\varepsilon = (\varepsilon_1, \varepsilon_2) \in \{0, 1\}^2$.

- Change sign of coupling constants of edges crossing γ_i iff $\varepsilon_i = 1$.

  $\Rightarrow$ Ising model with coupling constants (J_{e^*}) and defect ε.

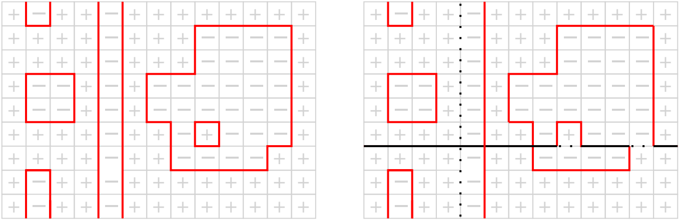
  Partition function: $Z^{\varepsilon}_{\text{Ising}}(J)$.

Low temperature expansion of $Z^{\varepsilon}_{\text{Ising}}(J)$

- Polygonal configurations, $\mathcal{P}^{\varepsilon}(G)$: edge configurations P of G s.t.
  - each vertex is incident to even number of edges of P,
  - P has homology class ε in $H_1(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$.
- Low temperature expansion

$$\forall \varepsilon \in \{0,1\}^2, \quad Z^{\varepsilon}_{\text{Ising}}(J) = \Big(\prod_{e^* \in E^*} e^{J_{e^*}}\Big) \sum_{P \in \mathcal{P}^{\varepsilon}(G)} \prod_{e \in P} e^{-2J_{e^*}}.$$

- Geometric interpretation: polygons 'separate' clusters of ± 1 spins.



$\varepsilon = (0,0)$ $\qquad$ $\varepsilon = (1,1)$

Low temperature representation of σ.

Double Ising model

- Take 2 copies (red/blue) of an Ising model on G^*, with coupling constants (J_{e^*}), having the *same* defect condition.
- LT representation of the partition function:

$$Z_{2\text{-Ising}}(J) = (Z_{\text{Ising}}^{(0,0)})^2 + (Z_{\text{Ising}}^{(1,0)})^2 + (Z_{\text{Ising}}^{(0,1)})^2 + (Z_{\text{Ising}}^{(1,1)})^2.$$

- Interested in the probability measure $\mathbb{P}_{2\text{-Ising}}$:
  - defined on, $\mathcal{P}^2 := \bigcup_{\varepsilon \in \{0,1\}^2} \mathcal{P}^\varepsilon(G) \times \mathcal{P}^\varepsilon(G)$,
  - by, $\mathbb{P}_{2\text{-Ising}}(P, P) = \dfrac{\mathcal{C}^2 \big(\prod_{e \in P} e^{-2J_{e^*}} \big) \big(\prod_{e \in P} e^{-2J_{e^*}} \big)}{Z_{2\text{-Ising}}(J)}.$

XOR-configurations

Let $(P, P) \in \mathcal{P}^2$, and consider spin configurations (σ, σ) corresponding to (P, P). Define XOR-spin configuration ξ:

$$\forall v^* \in V^*,\ \xi_{v^*} = \sigma_{v^*}\sigma_{v^*}.$$

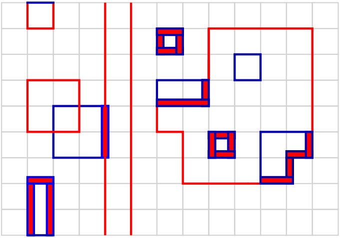
Conjecture (Wilson, Ikhlef, Picco, Santachiara)

The scaling limit of polygonal configurations separating ± 1 clusters of the critical XOR model are contour lines of the Gaussian free field, with the heights of the contours spaced $\sqrt{2}$ times as far apart as they are for [...] the double dimer model on the square lattice.

Goal: partial proof of this conjecture.

Double Ising model

Let $(P, P) \in \mathcal{P}^2$, and consider the superimposition $P \cup P$.

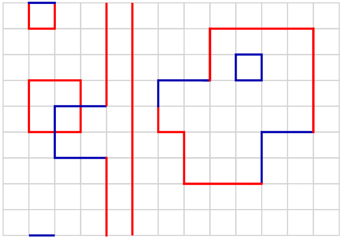


Superimposition of 2 polygonal configurations $P \cup P$.

Define 2 new edge configurations:

- $\mathrm{Mono}(P, P)$: monochromatic edges.
- $\mathrm{Bi}(P, P)$: bichromatic edges.

Monochromatic edges



Monochromatic edge configuration of $\color{red}{P} \cup \color{blue}{P}$.

Lemma

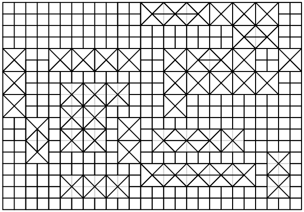
Mono(P, P) *is the polygonal configuration separating* ± 1 *clusters of the corresponding XOR-configuration.*

$$\Rightarrow \text{Mono}(P, P) \in \mathcal{P}^{(0,0)}(G).$$

Goal: *understand the law of monochromatic edge configurations.*

Theorem (Boutillier, dT)

1. *Monochromatic edge configurations have the same distribution as a family of 'contours' in a bipartite dimer model.*
2. *This family of 'contours' are the* 1/2*-integer level lines of a restriction of the height function of this bipartite dimer model.*



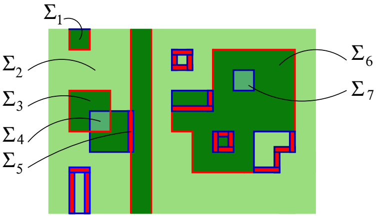
- When the double Ising model is *critical*, so is the bipartite dimer model.
- Using results of [dT] on the convergence of the height function, we give a partial proof of Wilson's conjecture.

Bichromatic edge configurations

Let $\varepsilon \in \{0,1\}^2$, and let $(P, P) \in \mathcal{P}^{\varepsilon}(G) \times \mathcal{P}^{\varepsilon}(G)$.

Mono(P, P) separates the torus into connected components

$$\Sigma_1, \cdots, \Sigma_{n_P}$$



Lemma

For every i, the restriction of Bi(P, P) *to Σ_i is the LTE of an Ising configuration on $G^*_{\Sigma_i}$, with coupling constants $(2J_{e^*})$, and defect condition $\Pi_{\Sigma_i}(\varepsilon)$.*

$$\Rightarrow \text{Bi}(P, P)_i \in \mathcal{P}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}).$$

Probability of monochromatic config.

Lemma

Let $P \in \mathcal{P}^{(0,0)}(G)$, and for every $i \in \{1, \cdots, n_P\}$ let P_i be a polygon configuration of $\mathcal{P}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i})$.
Then, there are 2^{n_P} pairs $(P, P) \in \mathcal{P}^{\varepsilon}(G) \times \mathcal{P}^{\varepsilon}(G)$ having P as monochromatic edges, and $P_1, \cdots, P_{n_P}$ as bichromatic edges.

For $P \in \mathcal{P}^{(0,0)}(G)$, denote by $W^{\varepsilon}(P)$ the contribution of:
$\{(P, P) \in \mathcal{P}^{\varepsilon}(G) \times \mathcal{P}^{\varepsilon}(G) : \text{Mono}(P, P) = P\}$,
to $(Z^{\varepsilon}_{\text{Ising}}(J))^2$.

Corollary

- $W^{\varepsilon}(P) = \mathcal{C}\left(\prod_{e \in P} e^{-2J_{e^*}}\right) \prod_{i=1}^{n_P} \left(2 Z_{\text{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J)\right)$.
- $Z_{2\text{-Ising}}(J) = \sum_{P \in \mathcal{P}^{(0,0)}(G)} \sum_{\varepsilon \in \{0,1\}^2} W^{\varepsilon}(P)$.

$$\forall P \in \mathcal{P}^{(0,0)}(G),\ \mathbb{P}_{2\text{-Ising}}(\text{Mono} = P) = \frac{\sum_{\varepsilon \in \{0,1\}^2} W^{\varepsilon}(P)}{Z_{2\text{-Ising}}(J)}.$$

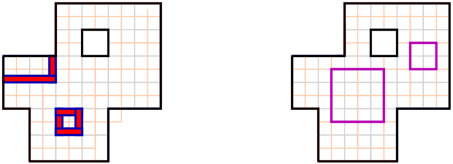
Mixed contour expansion

$$W^{\varepsilon}(P) = \mathcal{C}\big(\prod_{e\in P} e^{-2J_{e^*}}\big)\prod_{i=1}^{n_P}\Big(2Z_{\mathrm{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J)\Big).$$

Idea (Nienhuis): use Kramers and Wannier duality in each connected component Σ_i.

$$Z_{\mathrm{LT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G_{\Sigma_i}, 2J) = \mathcal{C}(\Sigma_i)\,\underbrace{Z_{\mathrm{HT}}^{\Pi_{\Sigma_i}(\varepsilon)}(G^*_{\Sigma_i}, 2J)}_{(*)},$$

where $(*) = \sum_{\tau\in H_1(\Sigma_i,\mathbb{Z}/2\mathbb{Z})} (-1)^{(\tau|\Pi_{\Sigma_i}(\varepsilon))} \sum_{P^*\in\mathcal{P}^\tau(G^*_{\Sigma_i})} \Big(\prod_{e^*\in P^*} \tanh(2J_{e^*})\Big).$



Low temp. expansion on G_{Σ_i} High temp. expansion on $G^*_{\Sigma_i}$

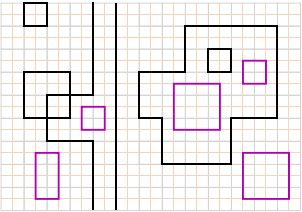
Mixed contour expansion

Combining terms, and taking the sum over $\varepsilon \in \{0,1\}^2$, yields:

Proposition

For all monochromatic configuration $P \in \mathcal{P}^{(0,0)}(G)$,

$$\sum_{\varepsilon\in\{0,1\}^2} W^{\varepsilon}(P) = \mathcal{C}\prod_{e\in P}\left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right)\sum_{\{P^*\in\mathcal{P}^{(0,0)}(G^*):\, P\cap P^*=\emptyset\}}\prod_{e^*\in P^*}\left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)$$



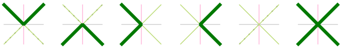
$$\mathbb{P}_{\text{2-Ising}}(\text{Mono}=P) = \frac{\prod_{e\in P}\left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right)\sum_{\{P^*\in\mathcal{P}^{(0,0)}(G^*):\, P\cap P^*=\emptyset\}}\prod_{e^*\in P^*}\left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)}{\sum_{P\in\mathcal{P}^{(0,0)}(G)}\cdots}$$

6-VERTEX MODEL ON THE MEDIAL GRAPH

- Medial graph G^M of G (or G^*) [Ore]:
  - vertices of G^M $\leftrightarrow$ edges of G (or G^*),
  - vertices are joined by an edge if corresponding edges are incident.

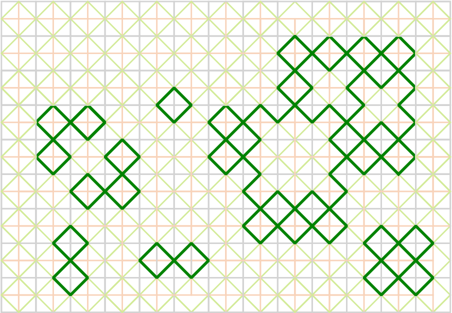
Remark: vertices of the medial graph have degree 4.

- 6-vertex configuration: at each vertex, one of the following:



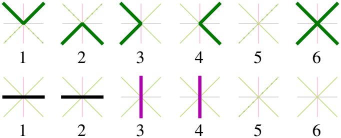
- Weights: $\omega_{12} = \frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}$, $\omega_{34} = \frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}$, $\omega_{56} = 1$.

6-VERTEX MODEL ON THE MEDIAL GRAPH



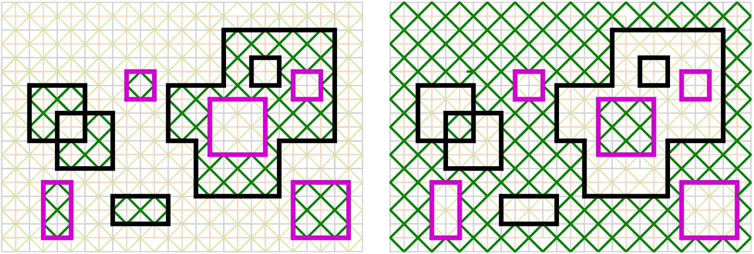
6-vertex configuration

Mapping:



Lemma (Nienhuis)

- *6-vertex configuration* $\rightarrow$ *a pair* (P, P^*) *of primal and dual polygonal configurations of* G *and* G^**:*
  - P *and* P^* *do not intersect,*
  - *homology class of* $P \cup P^*$ *in* $H_1(\mathbb{T}, \mathbb{Z}/2\mathbb{Z})$ *is* $(0, 0)$.
- *Given* (P, P^*) *as above* $\rightarrow$ *two 6-vertex configurations.*



Let $\mathcal{P}^{(0,0)}(G,G^*) = \{(P,P^*) \in \mathcal{P}(G) \times \mathcal{P}(G) : \hom(P \cup P^*) = (0,0)\}$.

As a consequence of the above mapping, we have:

Corollary

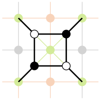
The 6-vertex partition function can be expressed as:

$$Z_{6-\text{vertex}}(J) =$$

$$2^{-1} \sum_{\{(P,P^*) \in \mathcal{P}^{(0,0)}(G,G^*) :\, P \cap P^* = \emptyset\}} \prod_{e \in P} \left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}} \right) \prod_{e^* \in P^*} \left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}} \right).$$

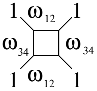
Quadri-tilings

- Quadri-tiling graph G^Q of G (or G^*): each vertex of the medial graph is replaced by a quadrangle. It is *bipartite*.



- Dimer configuration: subset of edges M such that each vertex is incident to exactly on edge of M. Denote by $\mathcal{M}(G^Q)$ the set of dimer configurations. At each decoration, one of the following:

- Weights assigned to edges.



Quadri-tilings

- Remark: weights satisfy the free fermion condition :

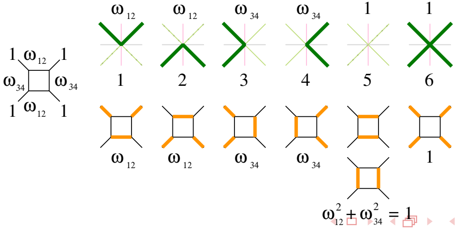
$$\omega_{12}^2+\omega_{34}^2=\left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right)^2+\left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)^2=\omega_{56}^2=1.$$

- Dimer Boltzmann measure:

$$\forall M\in\mathcal{M}(G),\quad \mathbb{P}_{\mathrm{Q}}(M)=\frac{1}{Z_{\mathrm{Q}}}\prod_{e\in M}\mathrm{weight}_e,$$

where Z_{Q} is the partition function.

- Mapping [Wu & Lin, Dubédat]



- Let M be a dimer configuration of G^Q:
  - $\mathrm{Poly}(M) = (\mathrm{Poly}_1(M), \mathrm{Poly}_2(M))$ is the pair of polygonal configurations assigned by the mappings.
- Let $(P, P^*) \in \mathcal{P}^{(0,0)}(G, G^*)$ be such that $P \cap P^* = \emptyset$.
  - Denote by $W_Q(P, P^*)$ the contribution to Z_{Q} of:

$$\{M \in \mathcal{M}(G^Q) : \mathrm{Poly}(M) = (P, P^*)\}.$$

As a consequence of the mappings we have:

COROLLARY

- $W_Q(P, P^*) = 2^{-1} \prod_{e \in P} \left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}} \right) \prod_{e^* \in P^*} \left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}} \right)$
- $Z_{\mathrm{Q}} = \sum_{(\{P,P^*) \in \mathcal{P}^{(0,0)}(G \cup G^*) : \, P \cap P^* = \emptyset\}} W_Q(P, P^*).$

$$\mathbb{P}_{\mathrm{Q}}(\mathrm{Poly} = (P, P^*)) = \frac{\prod_{e \in P} \left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}} \right) \prod_{e^* \in P^*} \left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}} \right)}{\sum_{(P,P^*)} \cdots}$$

Wrapping up

- $\mathcal{M}^{(0,0)}(G^Q)$: dimer configurations of G^Q such that $\mathrm{Poly}_1(M)$ has $(0,0)$ homology class.
- $\mathbb{P}_{\mathrm{Q}}^{(0,0)}$: dimer probability measure on $\mathcal{M}^{(0,0)}(G^Q)$. Then:

$$\forall (P,P^*) \in \mathcal{P}^{(0,0)}(G) \times \mathcal{P}^{(0,0)}(G^*) \text{ such that } P \cap P^* = \emptyset,$$

$$\mathbb{P}_{\mathrm{Q}}^{(0,0)}(\mathrm{Poly} = (P,P^*)) = \frac{\prod_{e\in P}\left(\frac{2e^{-2J_{e^*}}}{1+e^{-4J_{e^*}}}\right)\prod_{e^*\in P^*}\left(\frac{1-e^{-4J_{e^*}}}{1+e^{-4J_{e^*}}}\right)}{\sum_{\{(P,P^*)\in\mathcal{P}^{(0,0)}(G)\times\mathcal{P}^{(0,0)}(G^*):\, P\cap P^*=\emptyset\}} \cdots}$$

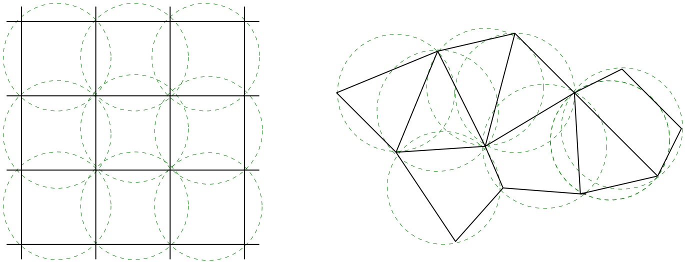
Wrapping everything together we obtain, the following:

Theorem

$$\boxed{\forall P \in \mathcal{P}^{(0,0)}(G), \quad \mathbb{P}_{\text{2-Ising}}(\mathrm{Mono} = P) = \mathbb{P}_{\mathrm{Q}}^{(0,0)}(\mathrm{Poly}_1 = P)}$$

Isoradial graphs

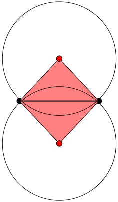
- A graph $G = (V, E)$ is isoradial if it can be embedded in the plane such that all faces are inscribed in a circle of radius 1.

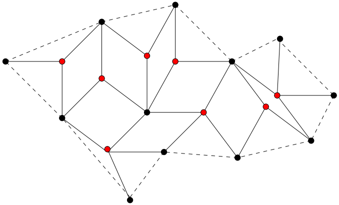


- G isoradial $\Rightarrow$ G^* isoradial: vertices of G^* = center of the circumcircles.

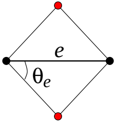
Isoradial graphs

- $G^\diamond$: associated rhombus graph, $\begin{cases} \text{vertices: } V(G) \cup V(G^*) \\ \text{edges: radii of the circles} \end{cases}$





- Edge $e \longrightarrow \begin{cases} \text{rhombus (or 1/2)} \\ \text{angle } \theta_e \end{cases}$



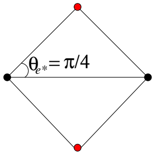
Critical 2-dimensional Ising model

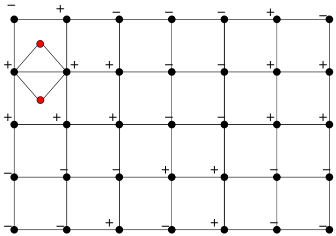
- The Ising model on an isoradial graph G^* is critical if the coupling constants are given by, for every edge e^*:

$$J_{e^*} = \frac{1}{2} \log \left(\frac{1 + \sin \theta_{e^*}}{\cos \theta_{e^*}} \right).$$

(Z-invariance + duality [Baxter], proof in period. case [Cimasoni & Duminil-Copin, Li])

Example: $G = \mathbb{Z}^2$: $\forall$ edge e^*, $\theta_{e^*} = \frac{\pi}{4}$, $J_{e^*} = \frac{1}{2}\log(1+\sqrt{2})$.

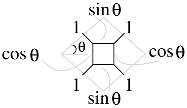




$\Rightarrow$ critical temperature computed by Kramers and Wannier.

Infinite volume measures

Corresponding quadri-tiling graph G^Q is also isoradial, and corresponding weights are the critical dimer weights (Kenyon):



Theorem (dT,Boutillier-dT)

- *There exists a probability measure* $\mathbb{P}^{\infty}_{\mathrm{Q}}$ *on* $\mathcal{M}(G^Q)$*, respectively* $\mathbb{P}^{\infty}_{\text{2-Ising}}$ *on* $\mathcal{P}(G)\times\mathcal{P}(G)$*, having explict expressions on cylinder sets.*
- *When* G *is* $\mathbb{Z}^2$*-periodic, each of these measures is obtained as weak limit of the Boltzmann measures.*

Corollary

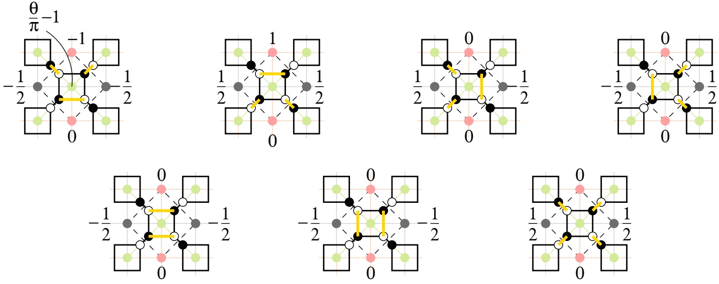
For every finite subset of edges $\mathcal{E}$ *of* E*, we have:*

$$\mathbb{P}^{\infty}_{\text{2-Ising}}(\mathcal{E}\subset \mathrm{Mono}) = \mathbb{P}^{\infty}_{\mathrm{Q}}(\mathcal{E}\subset \mathrm{Poly}_1).$$

Height function on quadri-tilings

- The height function is defined on faces of G^Q:
  - If f and f' are incident along an edge wb having rhombus half-angle θ_{wb}, if from f to f', b is on the left:

$$h^M(f') - h^M(f) = \frac{\theta_{wb}}{\pi} - \mathbb{I}_{\{wb\}}(M).$$

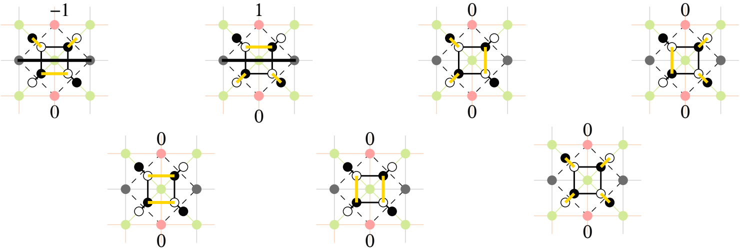


  - If f and f' are not incident, h^M is defined inductively. Well defined up to choice of value at base point.

Height function on quadri-tilings

- Faces of G^Q are vertices of $(G^Q)^*$, which consist of vertices of:
  - the medial graph V^M (green),
  - the primal graph V (gray),
  - the dual graph V (pink).
- Consider the restriction of h^M to vertices of V^* (pink).
- $\frac{1}{2}$-integer level lines of the above restriction (live on G):

  draw a primal edge $e = uv$ iff $h^M(v^*) - h^M(u^*) = \pm 1$.



Level lines and monochromatic configurations

Lemma

Let M be a quadri-tiling of G^Q, then $1/2$ integer level lines of the restriction of h^M to V^* exactly consist of $\mathrm{Poly}_1(M)$.

Corollary

Monochromatic configurations of the double critical Ising model have the same law as $1/2$-integer level lines of the restriction to V^* of the height function of quadri-tilings.

$$\text{Define, } H^\varepsilon : \begin{array}{ccc} C^\infty_{c,0}(\mathbb{R}^2) & \longrightarrow & \mathbb{R} \\ \phi & \mapsto & H^\varepsilon \phi = \varepsilon^2 \sum_{f \in V(G^*_Q)} a(f^*)\phi(f)h(f). \end{array}$$

Theorem (dT)

The random distribution H^ε converges weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field in the plane.

The same holds for the restricted height function if $a(f^*)$ is replaced by the area of the corresponding face of G^*.

Back to Wilson's conjecture

Suppose we had strong form of convergence, allowing for convergence of level lines. Then:

level lines of h^ε	$\longrightarrow$	level lines of GFF	
$(k, k \in \mathbb{Z})$ $(k+\frac{1}{2}, k \in \mathbb{Z})$		$(\sqrt{\pi}k, k \in \mathbb{Z})$ $(\frac{\sqrt{\pi}}{2}(2k+1), k \in \mathbb{Z})$	 XOR loops

For the critical double dimer model. The height function is $h_1^\varepsilon - h_2^\varepsilon$, where h_1 and h_2 are independent, and each converges weakly in distribution to $\frac{1}{\sqrt{\pi}}$ a Gaussian free field. Thus, $h_1 - h_2$ converges weakly in distribution to $\frac{\sqrt{2}}{\sqrt{\pi}}$ a Gaussian free field.

level lines of $h_1^\varepsilon - h_2^\varepsilon$	$\longrightarrow$	level lines of GFF	
$(k, k \in \mathbb{Z})$ $(k+\frac{1}{2}, k \in \mathbb{Z})$		$(\frac{\sqrt{\pi}}{\sqrt{2}}k, k \in \mathbb{Z})$ $(\frac{\sqrt{\pi}}{2\sqrt{2}}(2k+1), k \in \mathbb{Z})$	 d-dimer loops