Voronoi tesselations in large random trees and random maps of finite excess

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joint work with Louigi Addario-Berry, Omer Angel,
Guillaume Chapuy and Christina Goldschmidt

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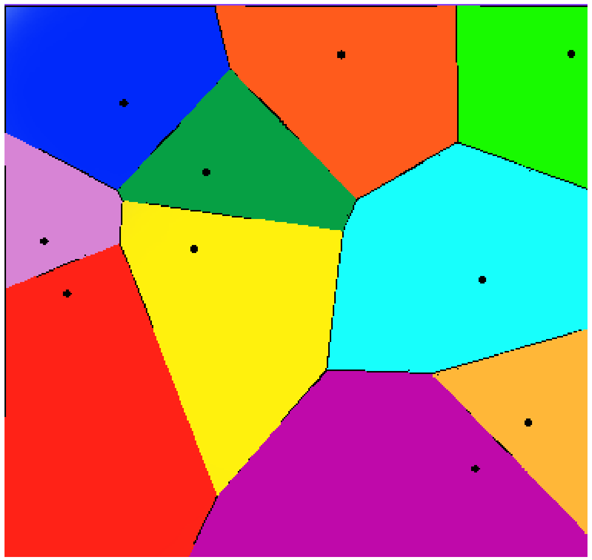
Voronoi tesselations in measured metric spaces

Let $X = (E, d, \mu)$ be a measured metric space (with $\mu(E) = 1$)

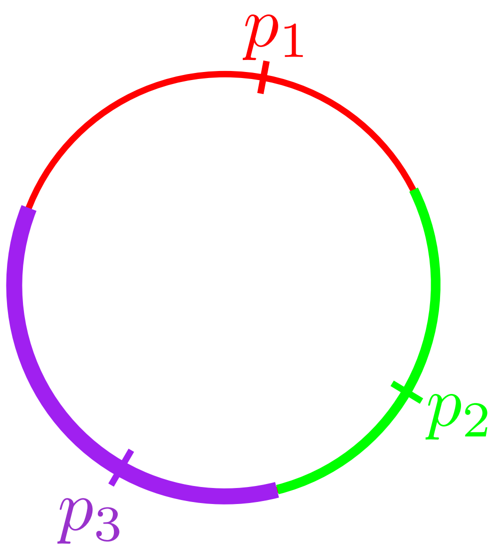
Consider k points $p_1, \ldots, p_k$ in E

The space E is 'partitioned' into cells $C_1, \ldots, C_k$ where

$$C_i = \{p \in X,\ d(p, p_i) = \min\ d(p, p_j)_{j \in [1..k]}\}$$



on unit square (10 cells)



on unit circle (3 cells)

The corresponding **Voronoi vector** is $\mathrm{Vor}^{(k)} := (\mu(C_1), \ldots, \mu(C_k))$

(Rk: $\mu(C_1) + \cdots + \mu(C_k) = 1$ when cell intersections have zero measure)

The discrete case

Graph G = discrete metric space

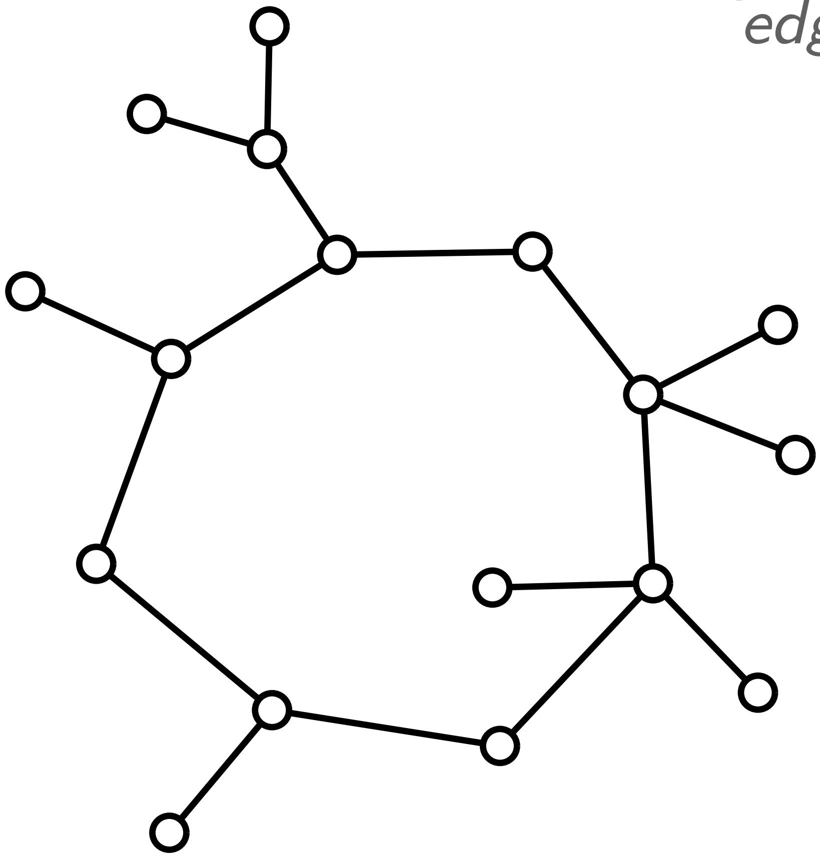
E = vertex-set

d = graph distance

possibly with edge-lengths

$$\mu = \frac{1}{|V|}\sum_{v\in V}\delta_v$$

uniform distribution on vertex-set



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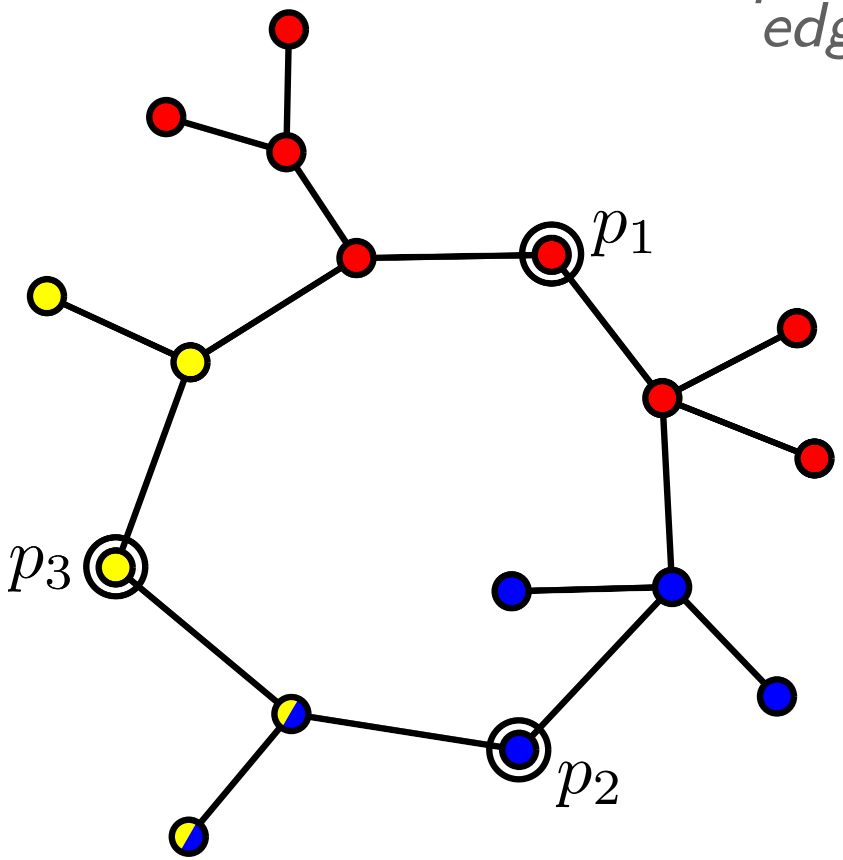


$$\mathrm{Vor}^{(3)} = \left(\frac{8}{17}, \frac{6}{17}, \frac{5}{17} \right)$$

Voronoi vector for random points in a metric space

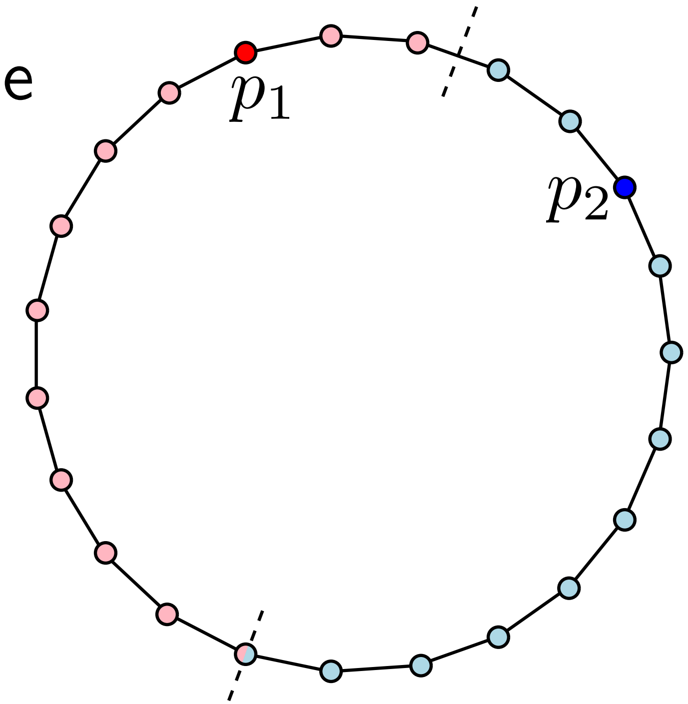
Let $X = (E, d, \mu)$ be a **fixed** measured metric space

Consider k **random** points $p_1, \ldots, p_k$ in E (chosen under μ)

What is the distribution of the corresponding (random) vector $\mathrm{Vor}^{(k)}$?

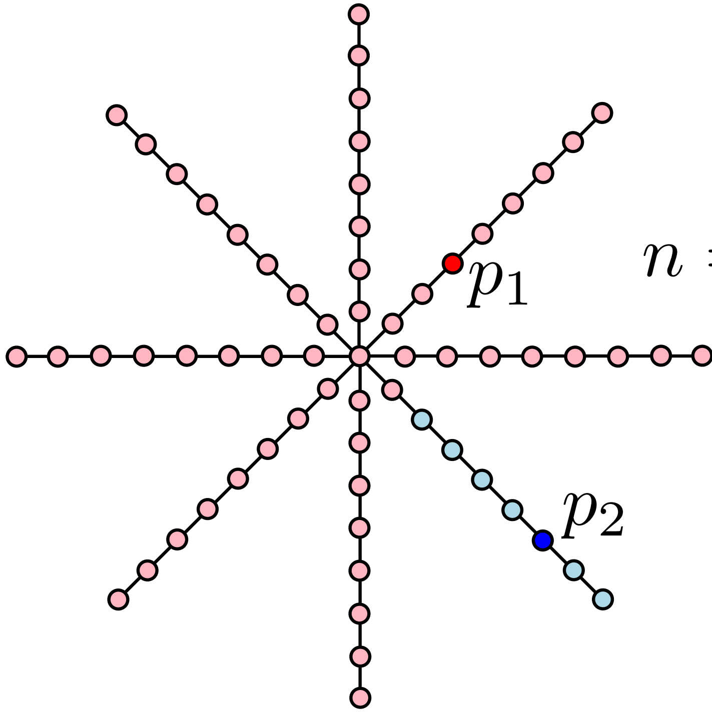
Examples: (for $k = 2$ and $n \to \infty$)

n-cycle



$\sim \delta_{(1/2,1/2)}$

n-star
(each branch of length n)



$n = 8$

$\sim \frac{1}{2}\delta_{(1,0)} + \frac{1}{2}\delta_{(0,1)}$
(closest to center wins all)

Voronoi vector for a random metric space

- Let $X = (E, d, \mu)$ be a **random metric space**

For $k \geq 2$ fixed, let $p_1, \ldots, p_k$ be random points of X

Consider the associated Voronoi vector $\mathrm{Vor}^{(k)} = (\mu(C_1), \ldots, \mu(C_k))$

Which distribution can we have for the (doubly) random vector $\mathrm{Vor}^{(k)}$?

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The model is called **Voronoi-uniform** if $\mathrm{Vor}^{(k)}$ is uniformly distributed on

$$\Delta_k := \{(x_1, \ldots, x_k), \quad x_i \geq 0, \ \textstyle\sum_{i=1}^k x_i = 1\}$$

(for $k = 2$ each component of $\mathrm{Vor}^{(2)}$ has uniform law on $[0, 1]$)

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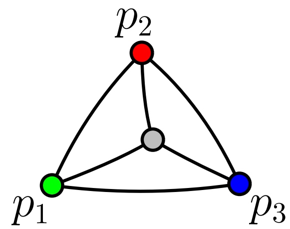
- Similarly a sequence X_n of random discrete metric spaces is said to be **Voronoi-uniform** as $n \to \infty$ if the Voronoi vector $V_n^{(k)}$

$$\text{satisfies} \quad V_n^{(k)} \xrightarrow[\text{proba}]{} \text{Uniform law on } \Delta_k$$

As it turns out, several models of random graphs have this behaviour

Example for the complete graph

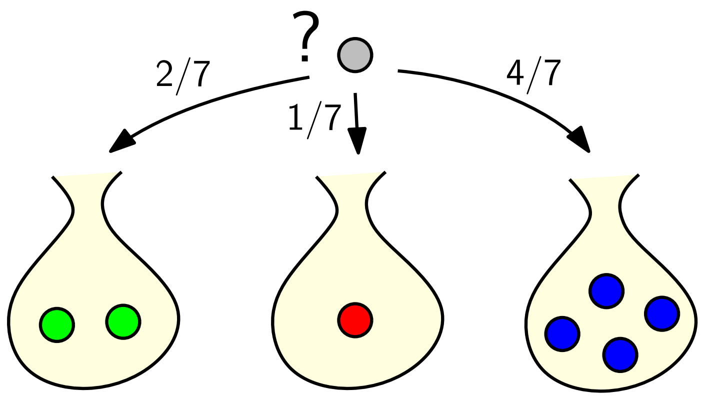
Consider the complete graph K_n with $\mathrm{Exp}(1)$ edge-lengths,



$$\forall e \in K_n, \quad \mathrm{P}(\ell(e) \geq t) = e^{-t}$$

Then this model is Voronoi-uniform as $n \to \infty$

cf Pólya urn model
starting with one ball
in each bag



- Grow the cells $C_1, \ldots, C_k$ (at unit speed) from $p_1, \ldots, p_k$
- at each time t where a new vertex v gets absorbed,

  it gets absorbed by cell C_i with probability $\dfrac{|C_i|}{|C_1| + \cdots + |C_k|}$
- convergence of urn composition (as $n \to \infty$) to uniform law on Δ_k

Results for random maps and trees

- For **random maps**:

  Conjecture: [Chapuy'16]

  For $g \geq 0$ let $Q_n^{(g)}$ be the random bipartite quadrangulation of genus g with n faces. Then $Q_n^{(g)}$ is Voronoi-uniform when $n \to \infty$

  $\Leftrightarrow$ continuum limit (Brownian map in genus g) is Voronoi-uniform

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supported by proof (using the "t_g-recurrence") that for all $g \geq 0$

$$\mathbf{E}(\mathrm{Vor}_1^{(2)} \cdot \mathrm{Vor}_2^{(2)}) = 1/6 \qquad \mathbf{E}(\mathrm{Vor}_1^{(3)} \cdot \mathrm{Vor}_2^{(3)} \cdot \mathrm{Vor}_3^{(3)}) = 1/60$$

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- For **random trees and random unicellular maps**:

Theorem: [Addario-Berry,Angel,Chapuy,F,Goldschmidt'18]

For $g \geq 0$ let $U_n^{(g)}$ be the random unicellular map of genus g with n edges

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$\Leftrightarrow$ the continuum limit is Voronoi-uniform (CRT for genus 0)

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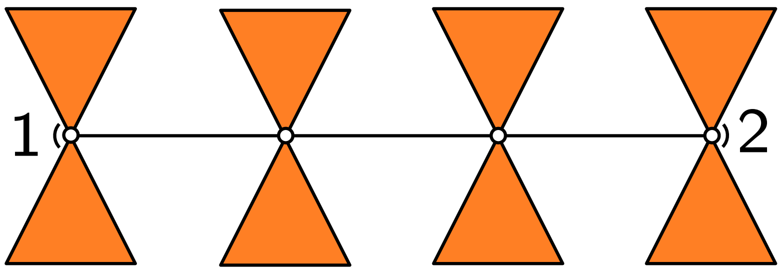
(also holds for random unicellular maps on non-orientable surfaces)

Distance between 2 random points in random trees

Consider a random plane tree on n edges with two marked corners



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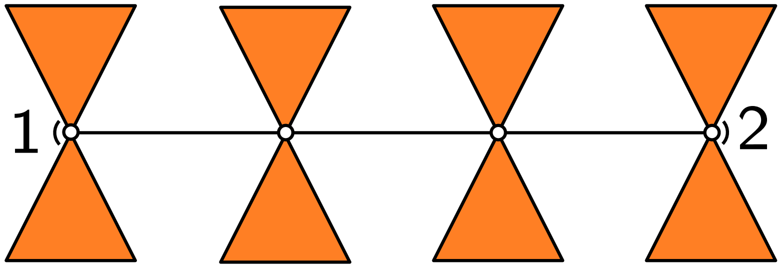

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where $a_{n,\ell} = \#$ trees on n edges with 2 marked corners at distance ℓ
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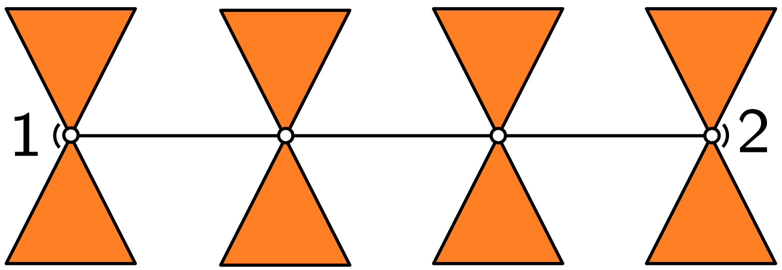

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Then $a_n = \mathrm{Cat}_n \cdot (2n-1) \sim \frac{2 \cdot 4^n}{\sqrt{\pi n}}$

And Lagrange inversion $\Rightarrow$ $a_{n,\ell} = \dfrac{\ell+1}{n} \dbinom{2n+2}{n+\ell}$

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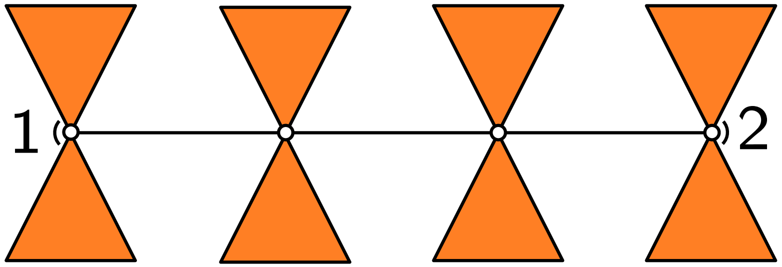

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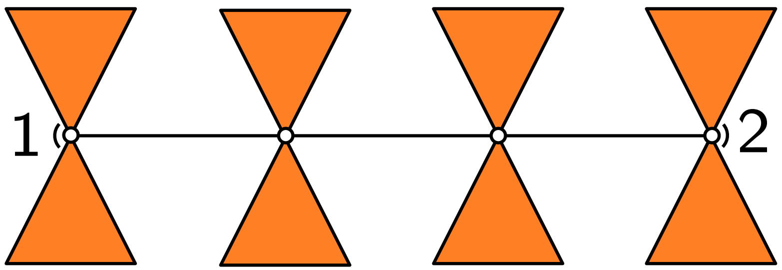

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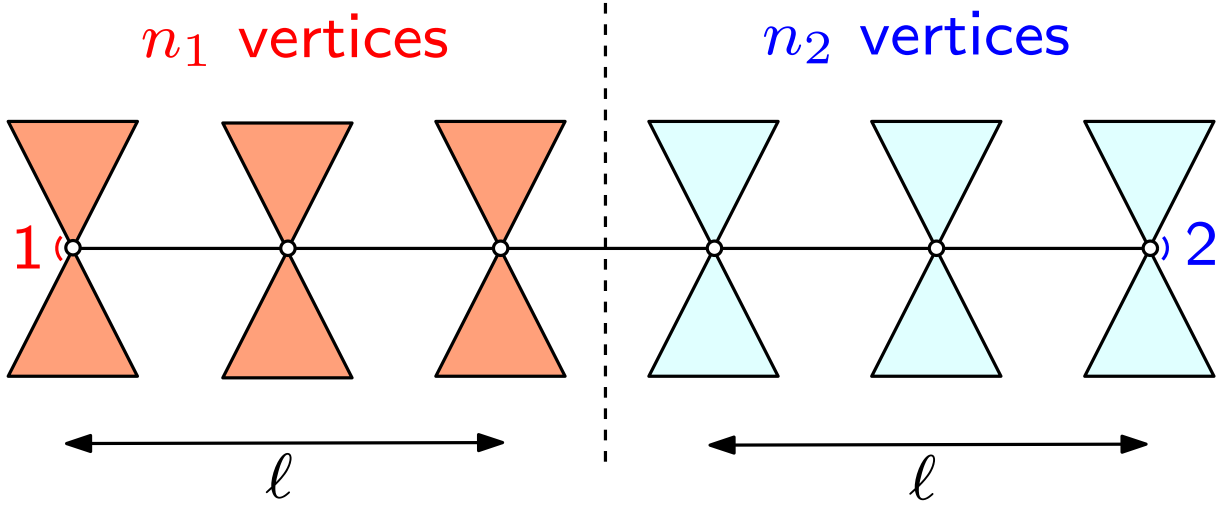
$\Rightarrow$ $\dfrac{a_{n,\ell}}{a_n} \sim \dfrac{1}{\sqrt{n}} 2xe^{-x^2}$ hence $\dfrac{D_n}{\sqrt{n}} \xrightarrow[\text{proba}]{}$ law of density $2xe^{-x^2}$ (Rayleigh law)

Joint law for the distance and Voronoi masses



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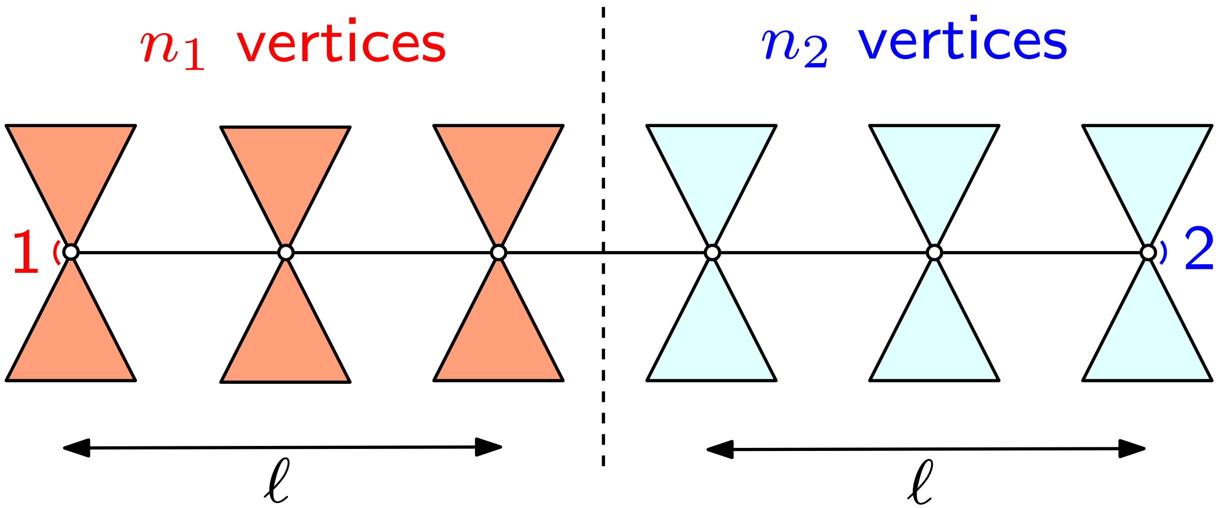

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such configurations $= a_{n_1,\ell} \cdot a_{n_2,\ell}$

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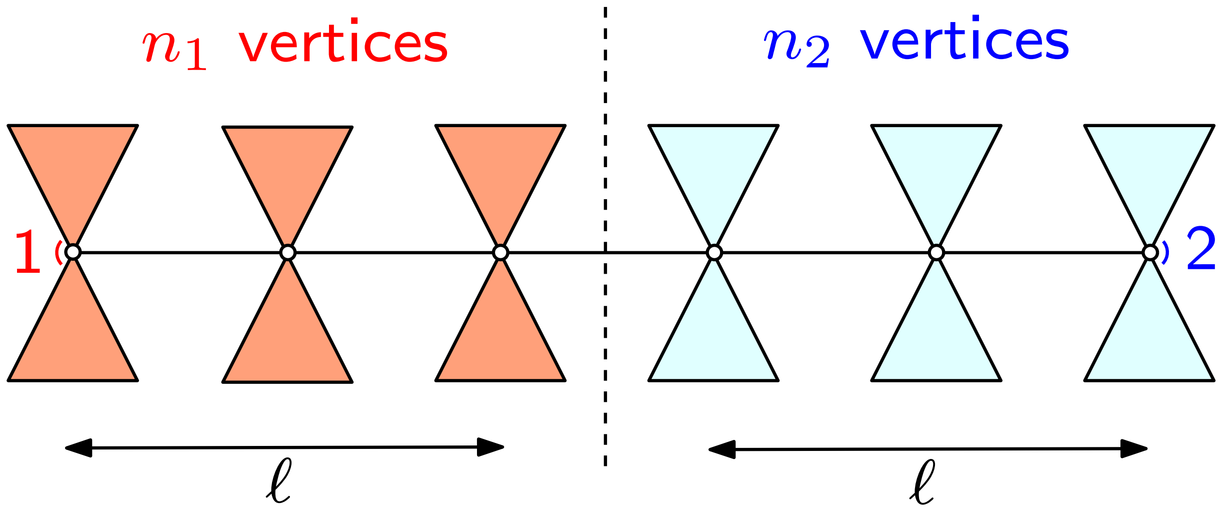

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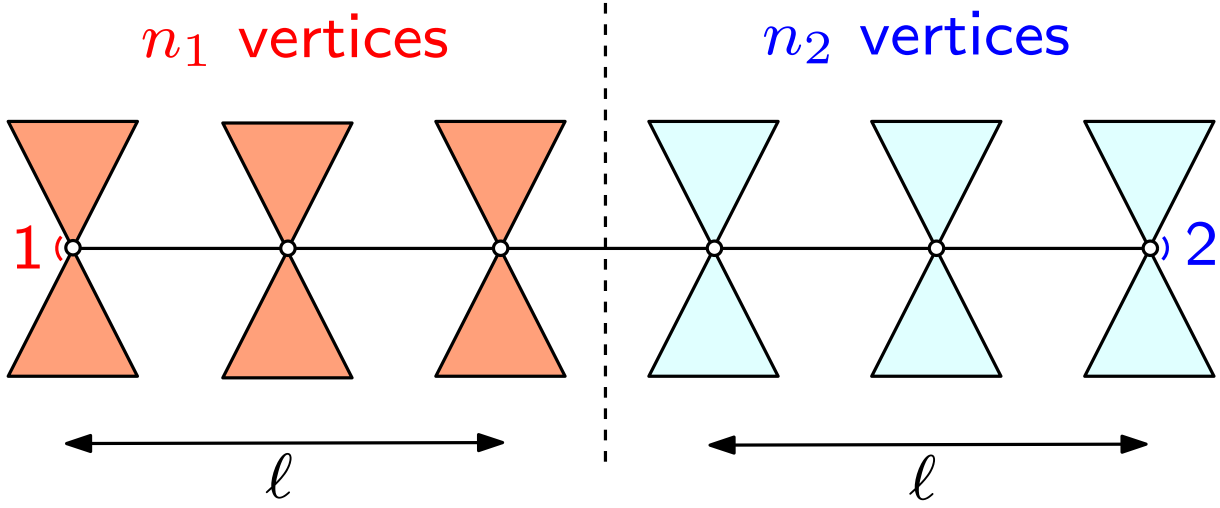

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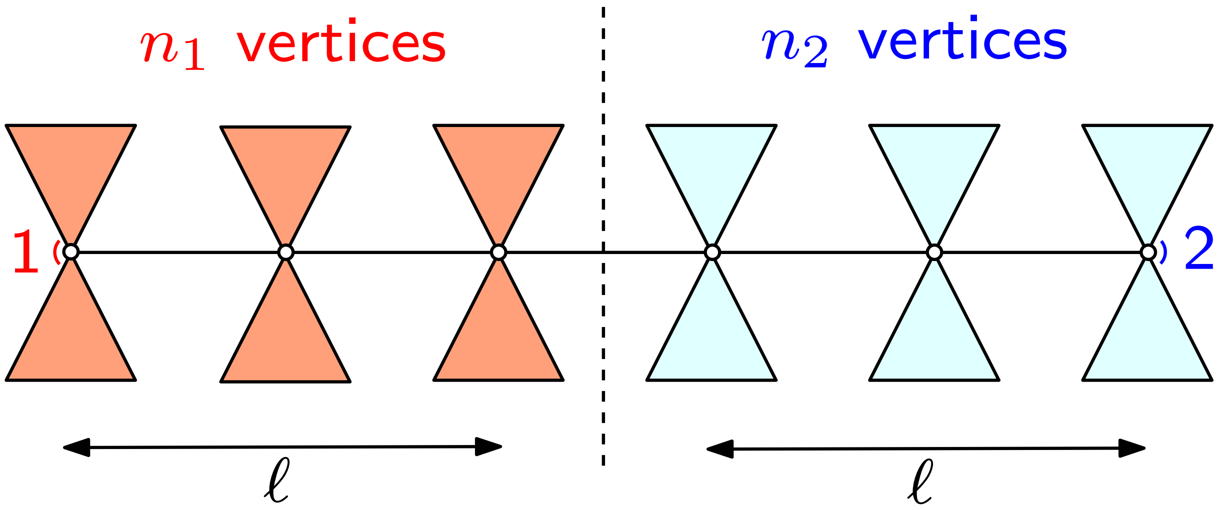

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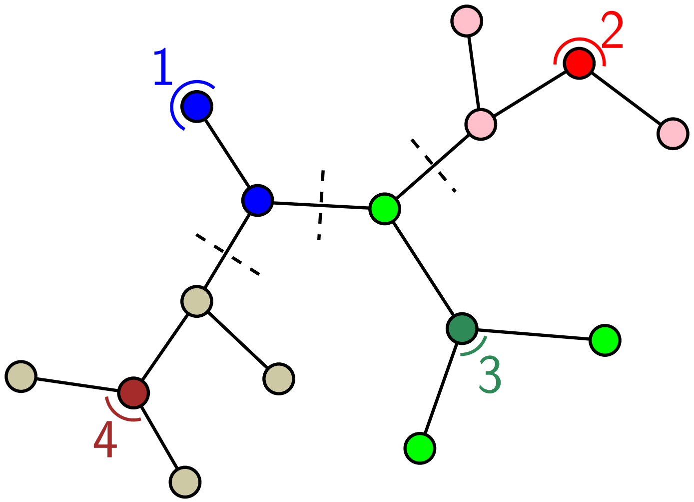
Rk: $\forall \alpha \in (0,1)$, $\int_{-\infty}^{+\infty} f(\alpha, x) \mathrm{d}x = 1$ cf change of variable $u = \frac{x}{\sqrt{\alpha(1-\alpha)}}$

$\Rightarrow$ marginal law in α is uniform on $[0,1]$ $\Rightarrow$ uniformity for random trees case $k = 2$

Bijective approach

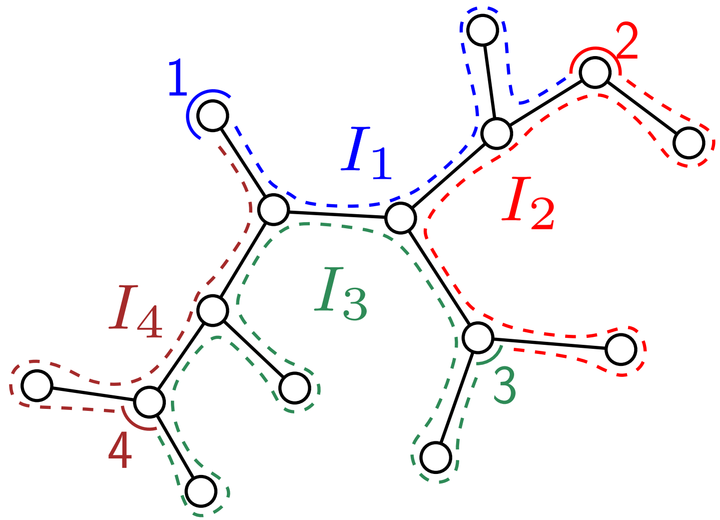
Let $\mathcal{A}_n^{(k)}$:= set of trees on n edges with k marked corners

Voronoi partition



$$\mathrm{Vor}^{(k)} = (\mu(C_1), \dots, \mu(C_k))$$

Contour partition



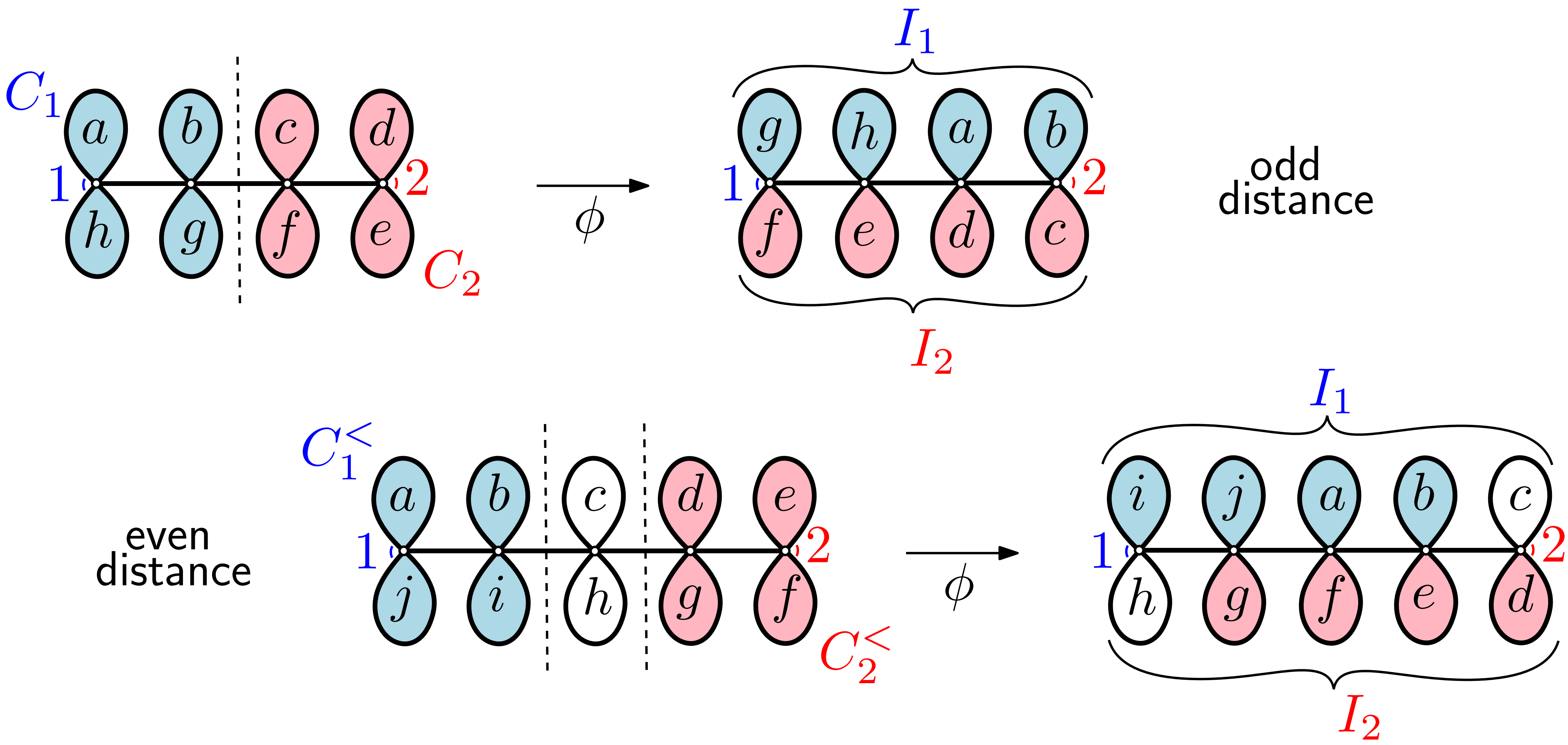
$$\mathrm{Int}^{(k)} = \frac{1}{2n}(\mathrm{length}(I_1), \dots, \mathrm{length}(I_k))$$

Idea: Find a bijection Φ from $\mathcal{A}_n^{(k)}$ to itself such that for $T' = \phi(T)$ one has $\mathrm{Int}^{(k)}(T') = \mathrm{Vor}^{(k)}(T)$ (up to $o(1)$ corrections)

This will prove uniformity, since clearly for T' taken at random in $\mathcal{A}_n^{(k)}$

$$\mathrm{Int}^{(k)}(T') \xrightarrow[\text{proba}]{} \text{Uniform law on } \Delta_k$$

Bijection for $k = 2$

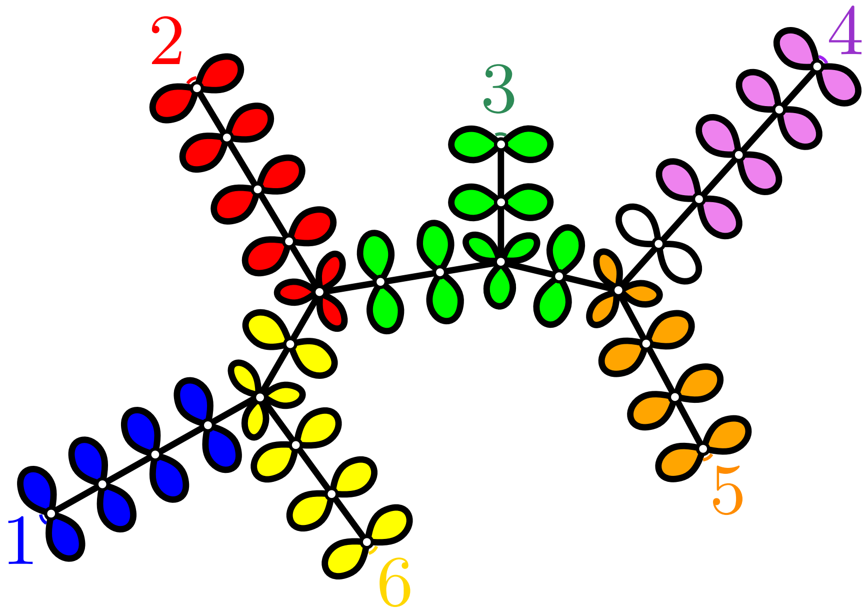


The bijection ϕ permutes the attached subtrees

For $i \in \{1, 2\}$ each attached **subtree in $C_i^<$ gets moved to I_i**

$\Rightarrow$ In proba we have $\quad \mu(C_i) \sim \mu(I_i) \sim \frac{1}{2n}\text{length}(I_i) \to \text{Unif}(0, 1)$

Bijection for $k \geq 3$ (induction on k)



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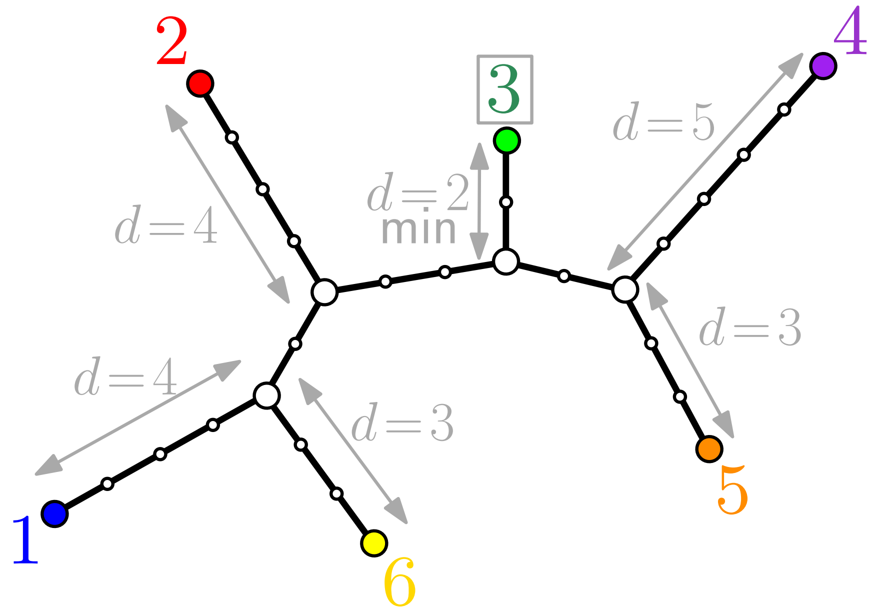

skeleton

if min=0 or min is not unique, the bijection **fails**

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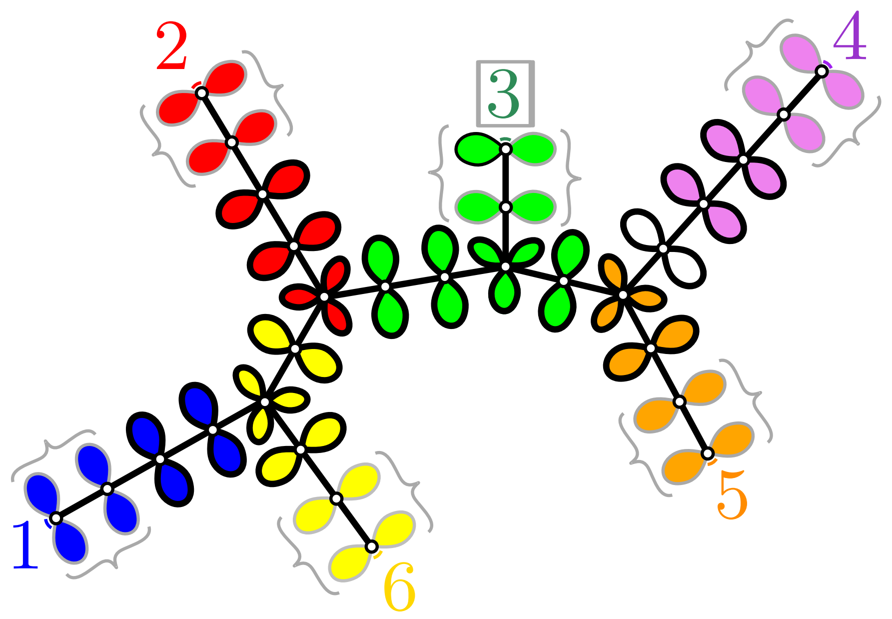

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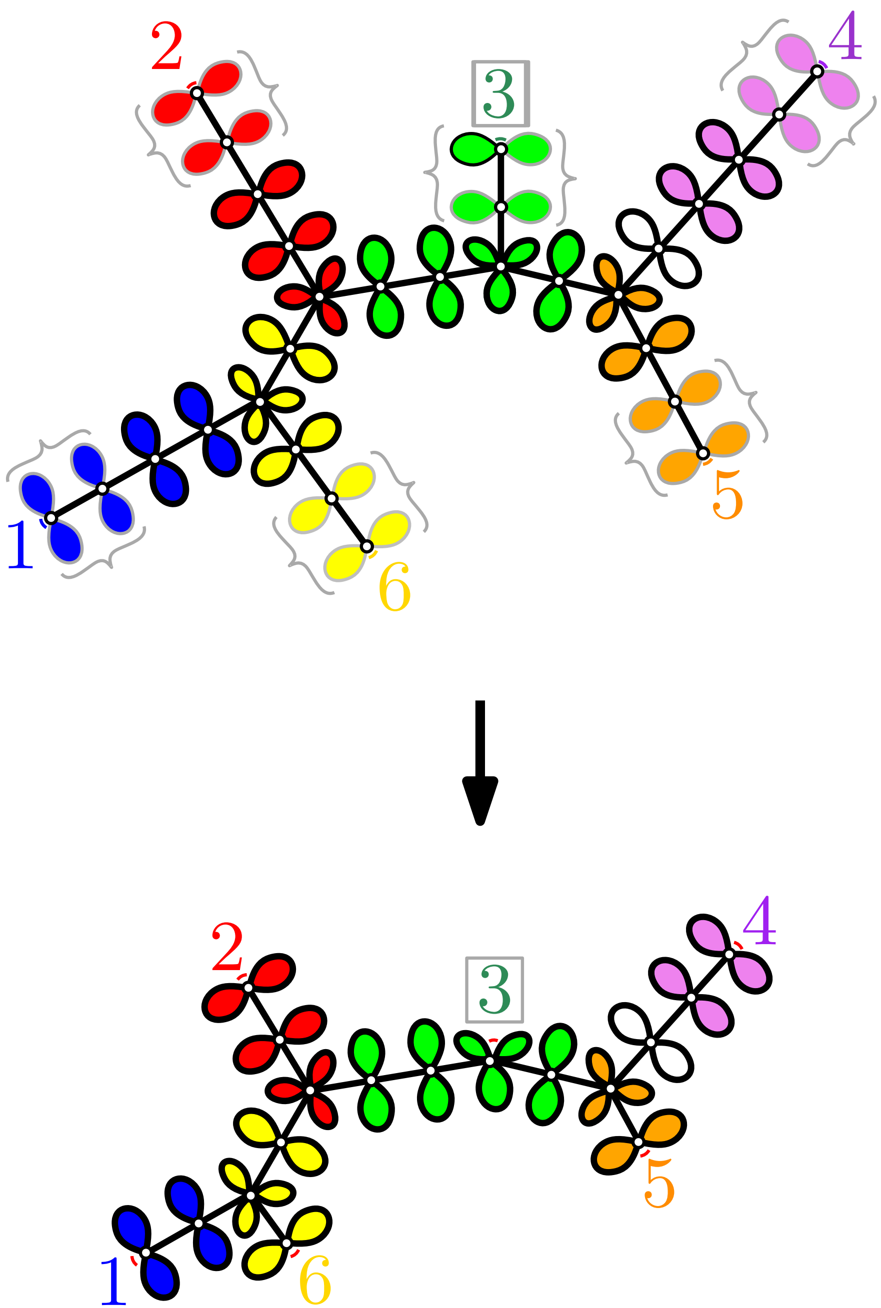

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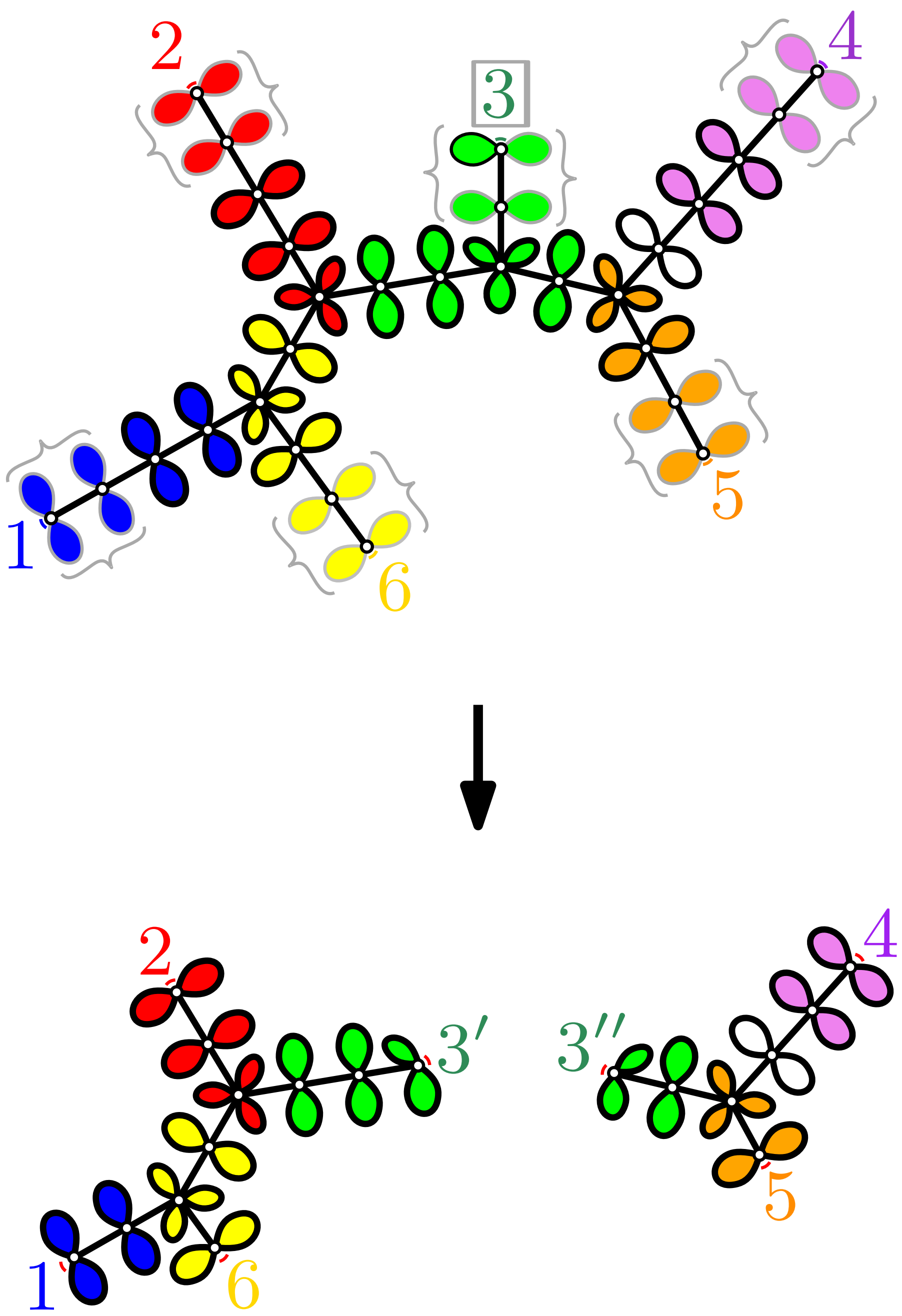

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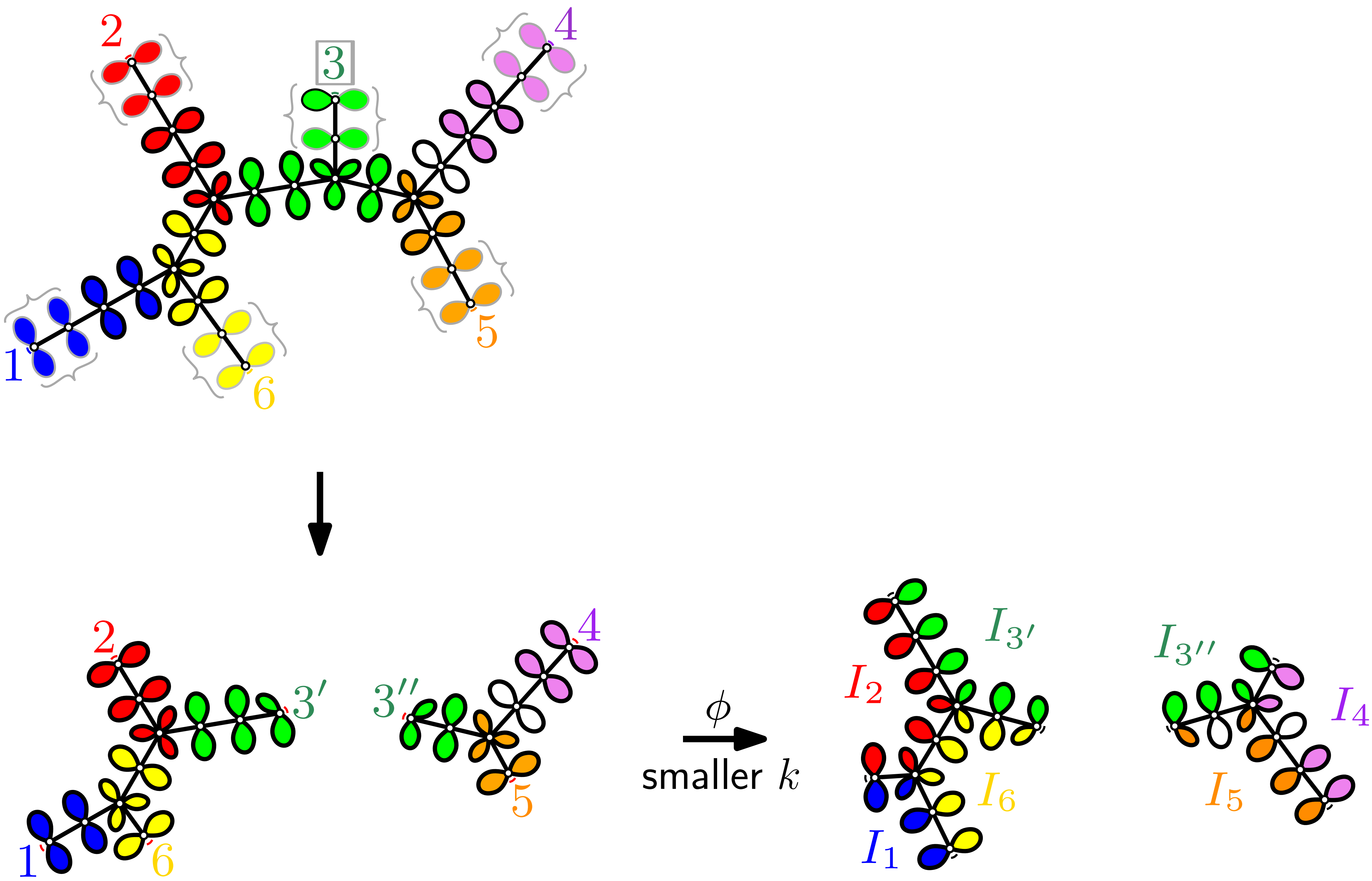

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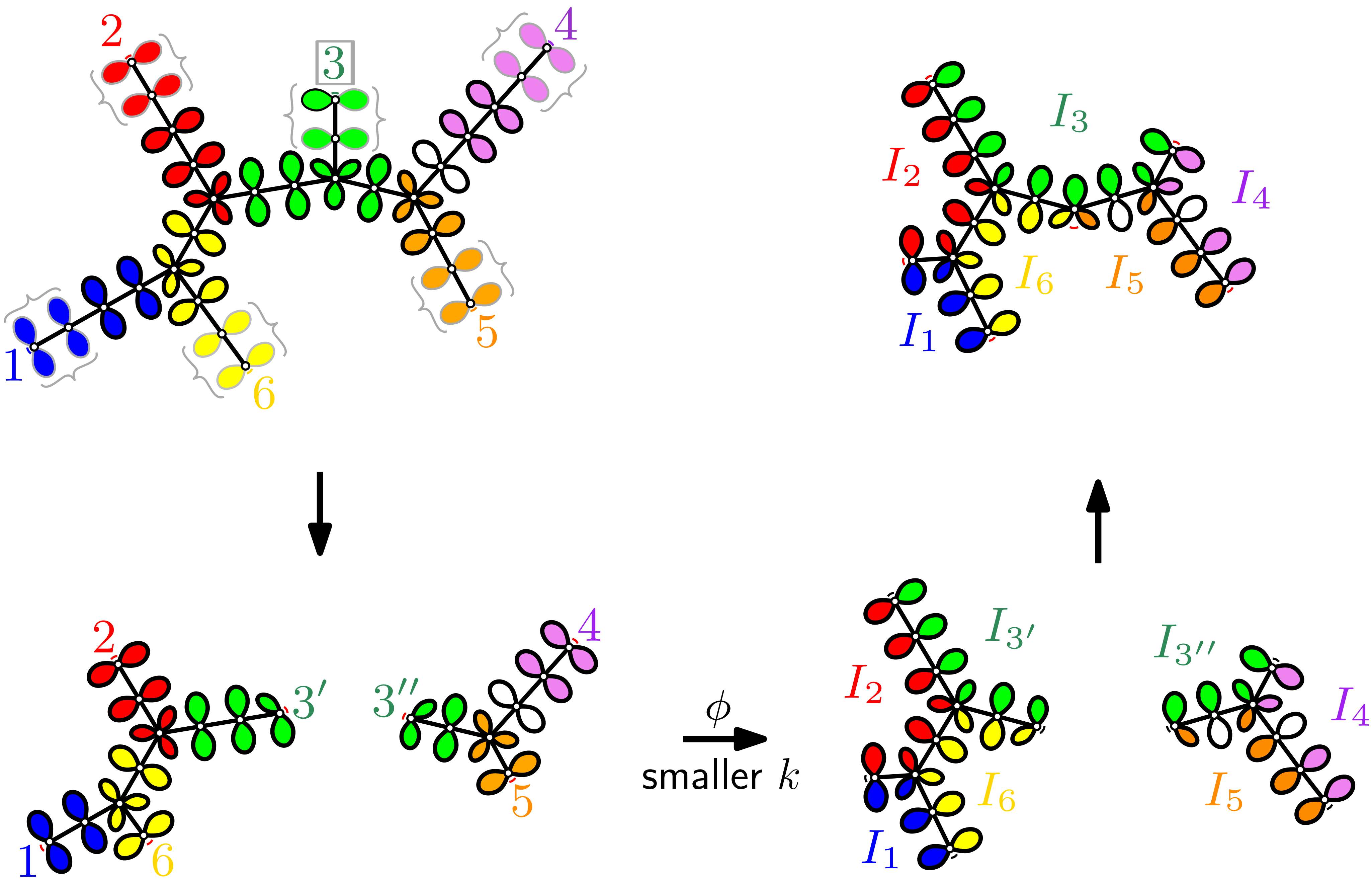

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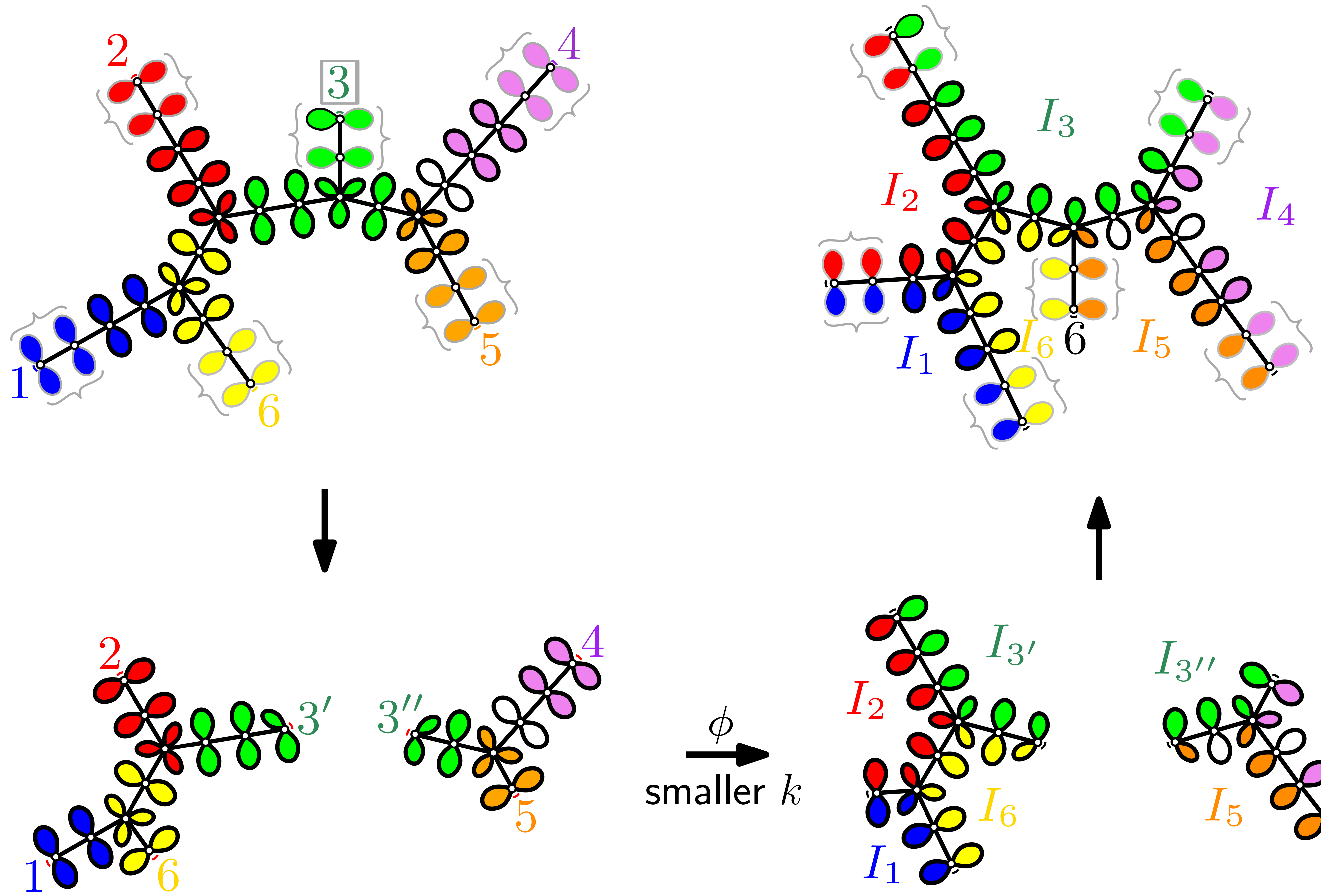


Summary of the results for trees

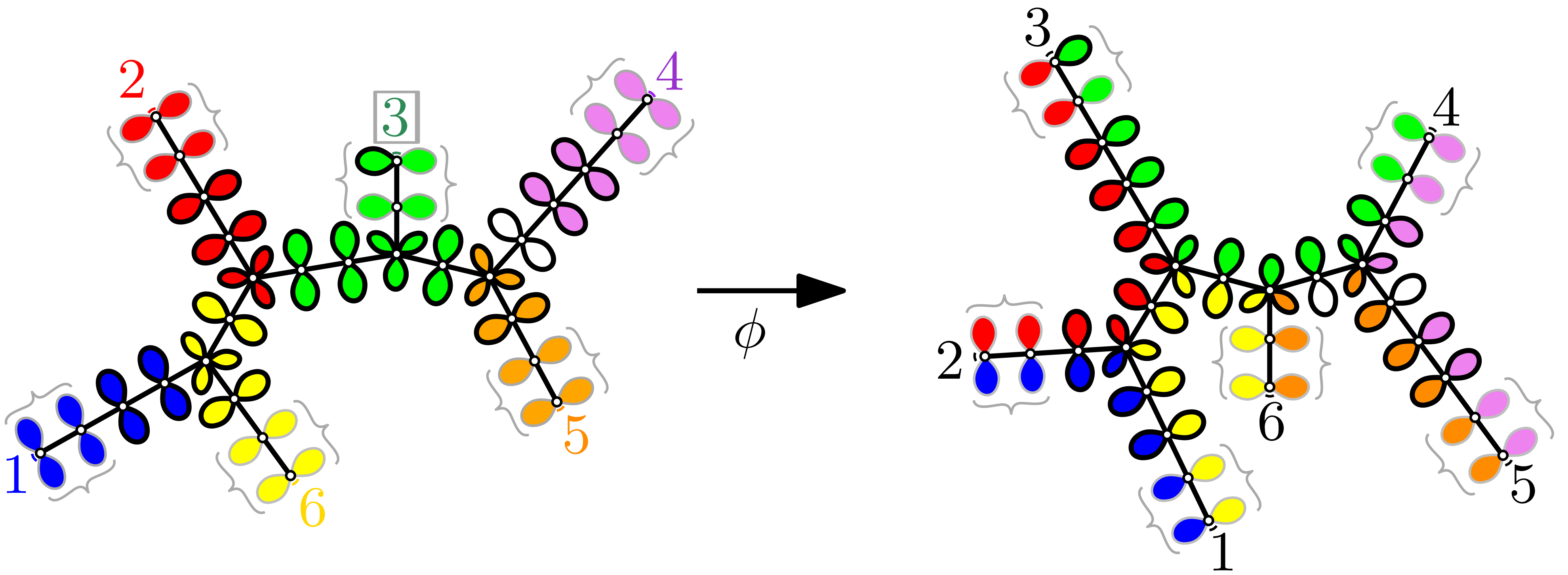
for any fixed $k \geq 2$

Let $\mathcal{A}_n^k$ be the set of trees with n edges and k marked corners

Then there is a subfamily $\mathcal{B}_n^k \subset \mathcal{A}_n^k$ (no-failure case) such that

$$|\mathcal{B}_n^k| = |\mathcal{A}_n^k| \cdot (1 - O(n^{-1/2}))$$

and a **bijection** ϕ from $\mathcal{B}_n^k$ to itself that **permutes the attached subtrees**

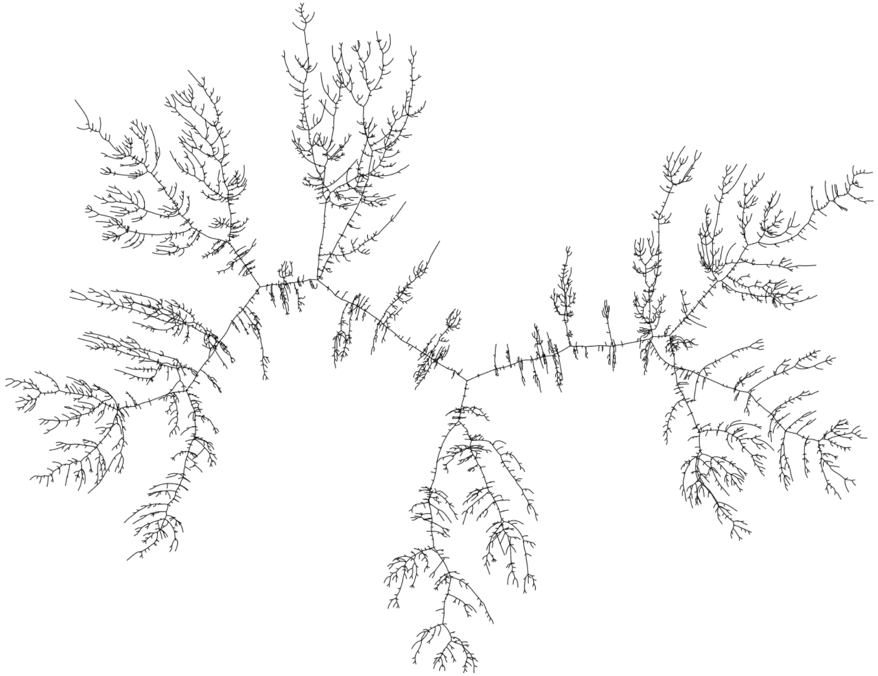


so that for $i \in [1..k]$ each attached subtree in $C_i^<$ gets moved to I_i

$\Rightarrow$ For T random in $\mathcal{A}_n^k$, $\mathrm{Vor}^{(k)}(T) \sim \mathrm{Int}^{(k)}(T) \sim$ Uniform law on Δ_k

Induced results

The CRT is the continuum limit
of random trees
(with edge lengths $/\sqrt{n}$)



So the **CRT is Voronoi-uniform**

Induced results

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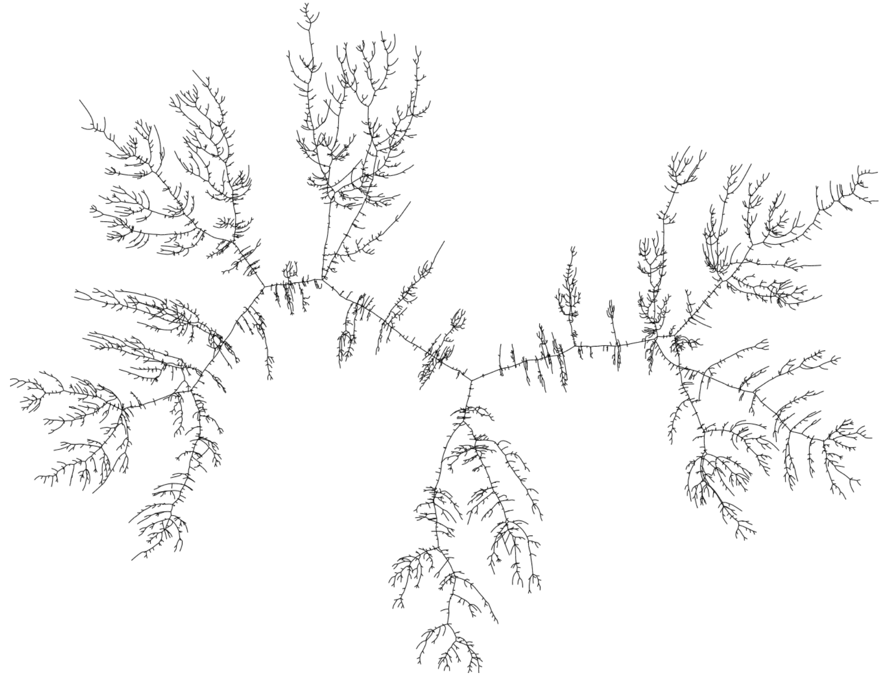

So the **CRT is Voronoi-uniform**

Gromov-Hausdorff-Prokhorov topology

$\Rightarrow$ any model of random graphs converging to the CRT is Voronoi-uniform as $n \to \infty$

This includes

- random dissections of an n-gon [Curien,Haas,Kortchemski'14] [Bettinelli'17]
- random outerplanar maps with n edges [Caraceni'16] [Stufler'17]
- random stacked triangulations of n vertices [Albenque,Marckert'08]
- random graphs of size n from a subcritical family [Panagiotou,Stufler,Weller'14] (outerplanar graphs, series-parallel graphs)

Proof of uniformity directly on the CRT

Let T be a CRT with k random points $p_1, \ldots, p_k$

To prove that T is Voronoi-uniform, we have to prove that

(i) for every $k \geq 2$, $\mathrm{Vor}^{(k)}(T)$ **and** $\mathrm{Int}^{(k)}(T)$ **are equidistributed**

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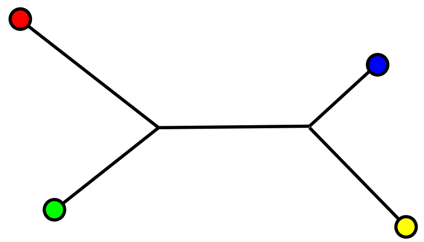
Let S be the skeleton of T (k-leaf binary tree with random edge-lengths)



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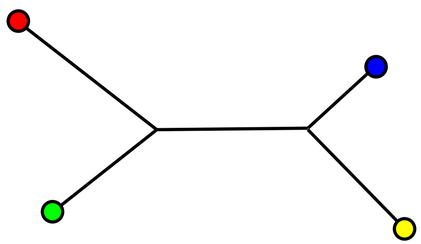
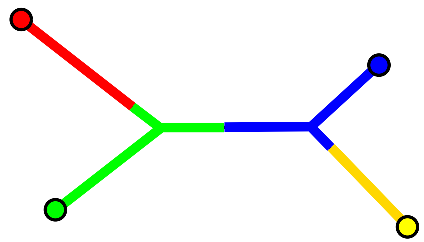
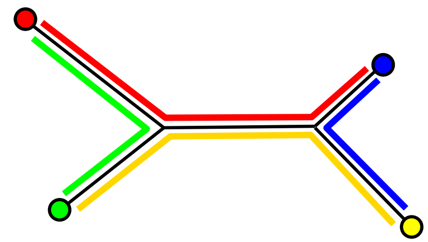




Voronoi-partition



Interval-partition

To prove (i), it is enough to prove

(ii) for every $k \geq 2$, $2\mathrm{Vor}^{(k)}(S)$ and $\mathrm{Int}^{(k)}(S)$ are equidistributed

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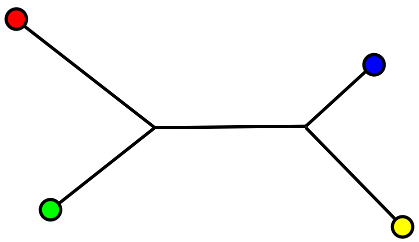

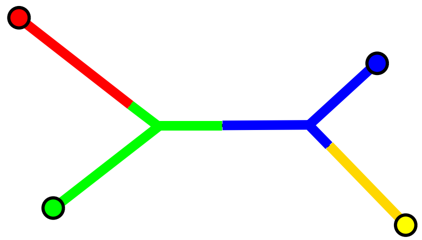

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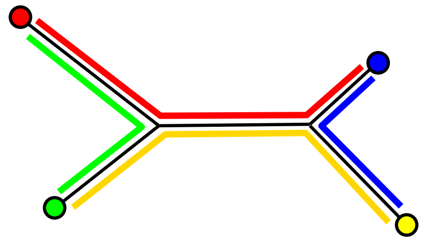

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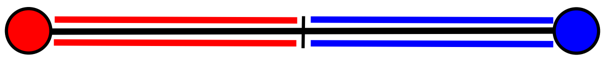
Proof for $k = 2$



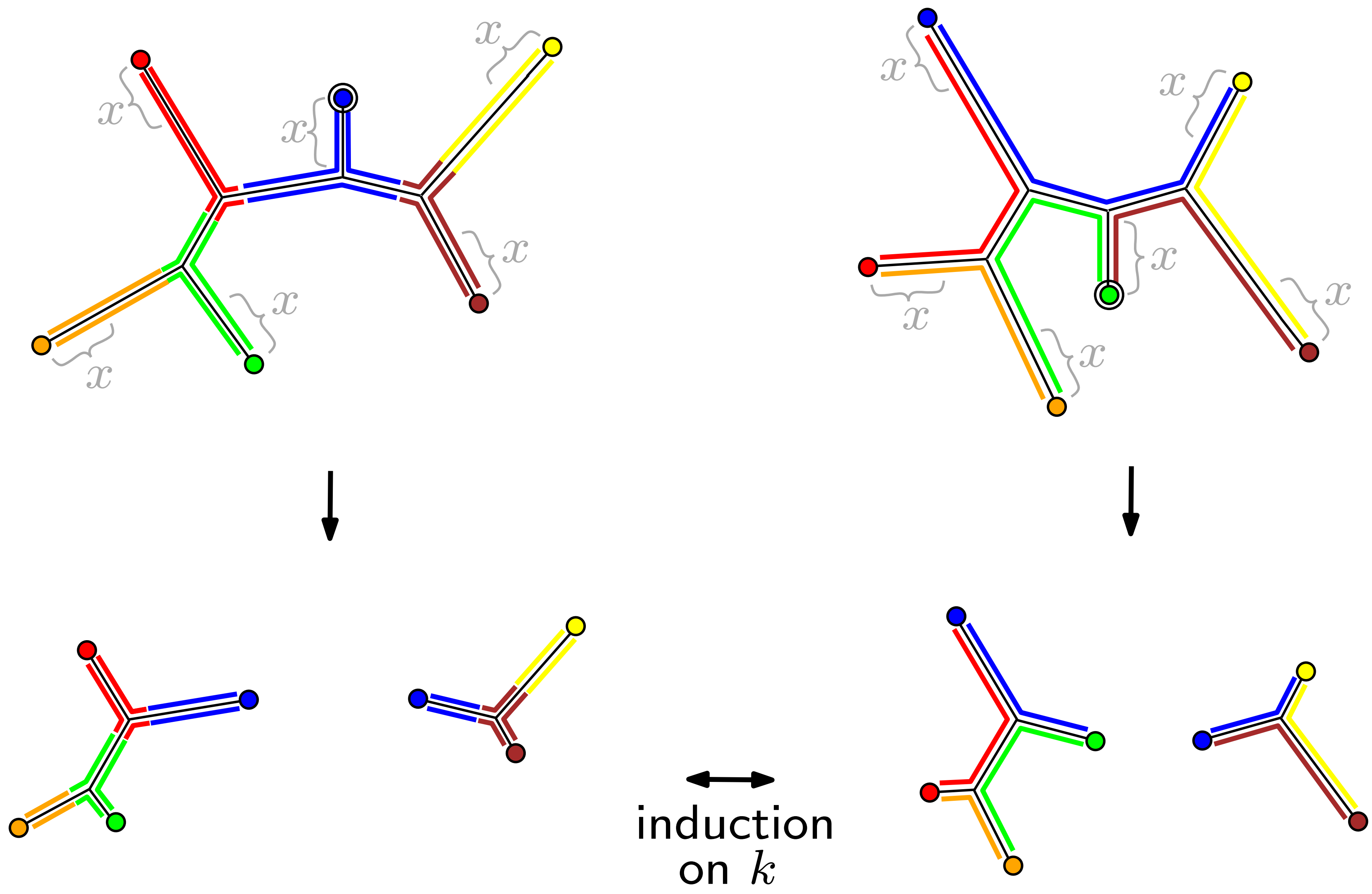
$\mathrm{Vor}^{(2)}$

$\mathrm{Int}^{(2)}$

Proof of uniformity directly on the CRT

Proof (by induction on k) that $2\,\mathrm{Vor}^{(k)}(S)$ and $\mathrm{Int}^{(k)}(S)$ are equidistributed

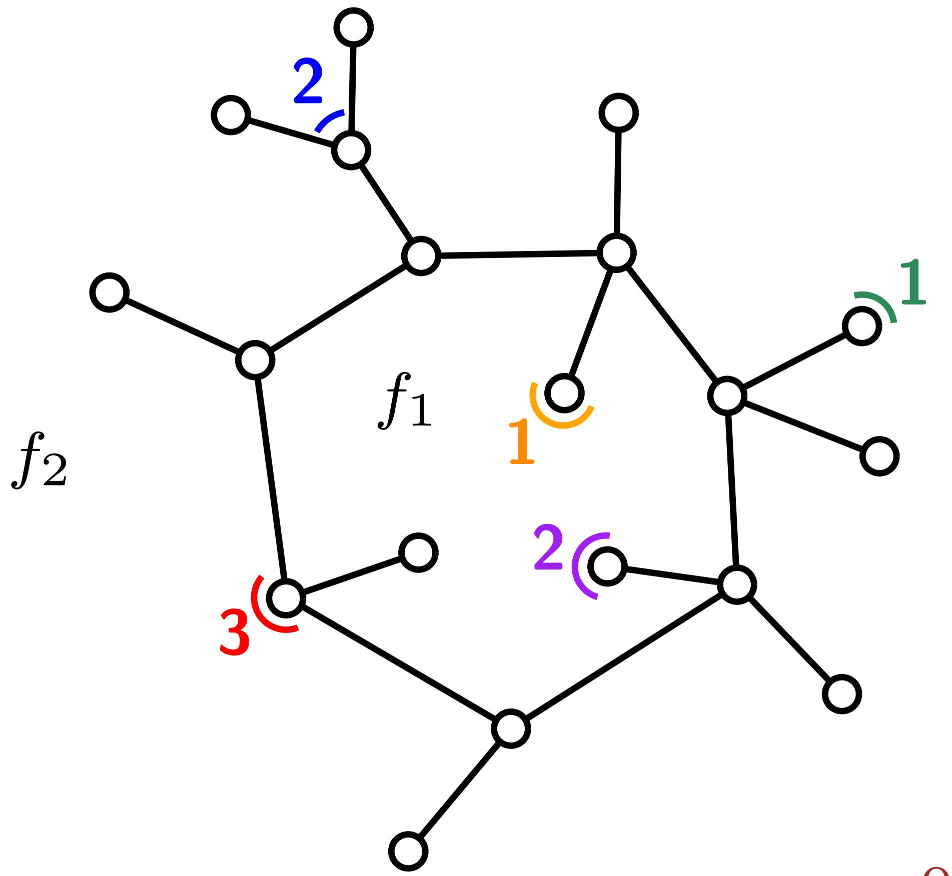
Rk: can rescale S so that the edge-lengths are independent $\mathrm{Exp}(1)$-laws



Extension to maps of finite excess

For $g \geq 0$ and $\mathbf{k} = (k_1, \ldots, k_r)$ with $k_i \geq 1$

$\mathcal{M}_n^{(\mathbf{k},g)}$:= set of maps of genus g with n edges and r faces $f_1, \ldots, f_r$ where in each face f_i there are k_i marked corners $c_{i,1}, \ldots, c_{i,k_i}$



$g = 0,\ r = 2,\ \mathbf{k} = (2, 3)$

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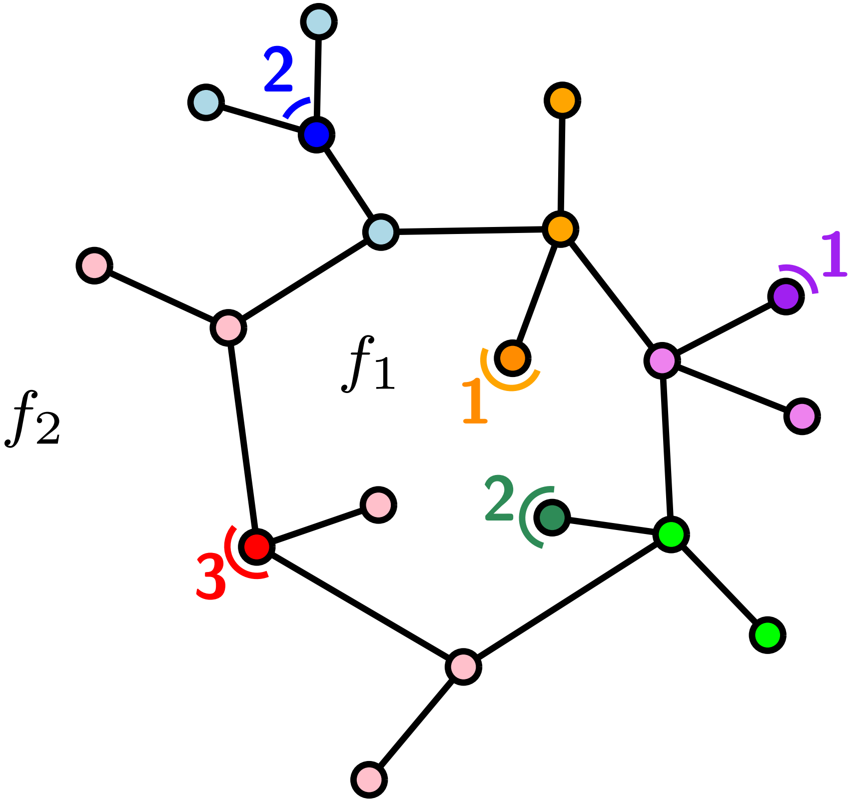
Voronoi partition

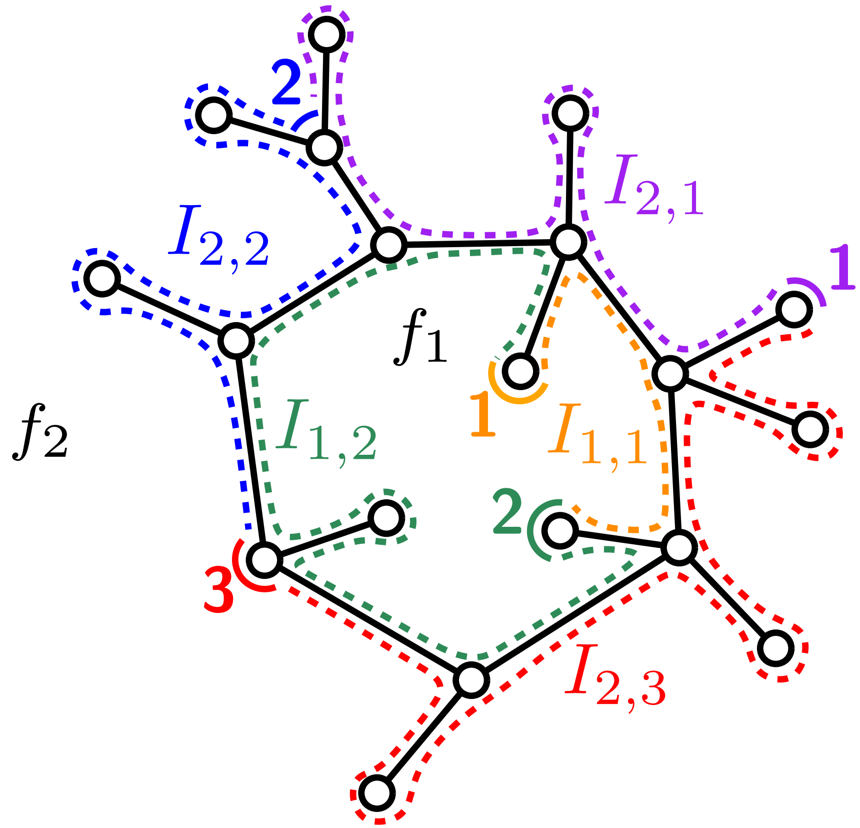


$N = k_1 + \dots + k_r$ (total number of marked corners)

$\{p_i\}_{1 \leq i \leq N} = \{$vertices at marked corners in $f_1, \dots, f_r\}$

Voronoi vector $\mathrm{Vor} := (\mu(C_1), \dots, \mu(C_N))$

Contour partition



$I_{i,j}$:= jth interval in f_i

vector $U_i := \frac{1}{2n}(\mathrm{length}(I_{i,1}), \dots, \mathrm{length}(I_{i,k_i}))$

Interval vector Int := concatenate $U_1; U_2; \dots; U_r$

Extension to maps of finite excess

Result: For $g \geq 0$ and $\mathbf{k} = (k_1, \ldots, k_r)$
there is a subfamily $\mathcal{B}_n^{(g,\mathbf{k})} \subset \mathcal{M}_n^{(g,\mathbf{k})}$ with $|\mathcal{B}_n^{(g,\mathbf{k})}| \sim |\mathcal{M}_n^{(g,\mathbf{k})}|$
and a bijection ϕ from $\mathcal{B}_n^{(g,\mathbf{k})}$ to itself such that for $M' = \phi(M)$
we have $\mathrm{Int}(M') = \mathrm{Vor}(M)$ **up to** $o(1)$ **error terms**

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- Random maps in $\mathcal{M}_n^{(g,\mathbf{k})}$ have a **scaling limit** called the $\mathrm{CRM}^{(g,\mathbf{k})}$

  The bijection implies that Vor and Int are equidistributed in the $\mathrm{CRM}^{(g,\mathbf{k})}$

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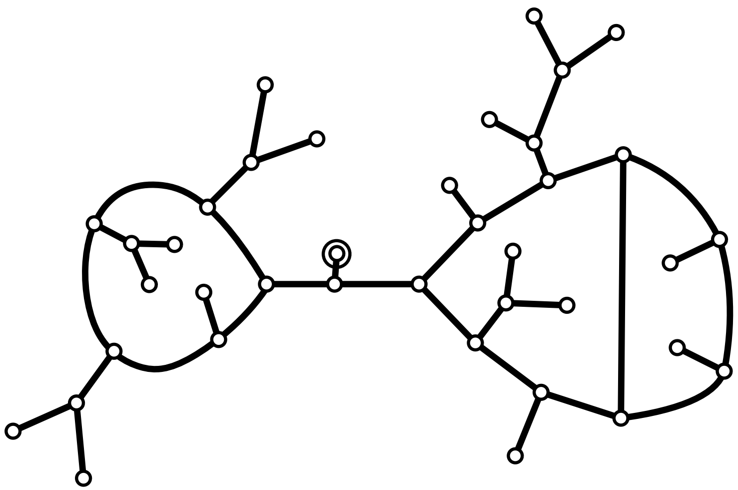
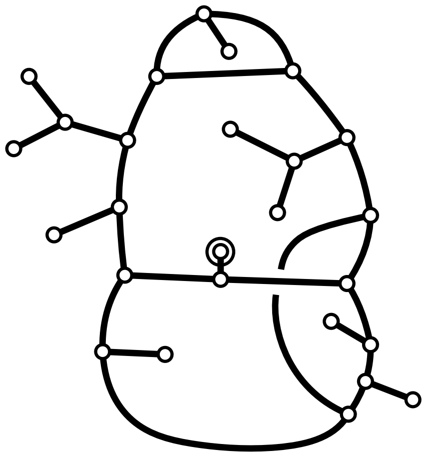
- This can be proved directly, on the associated skeleton and using induction

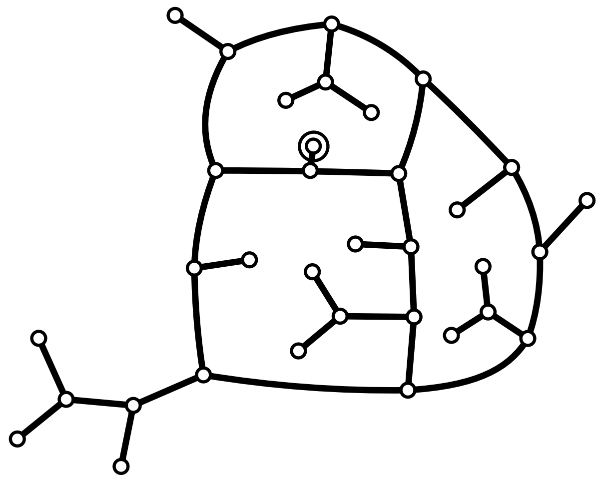
3 cases for the skeleton (surrounded the leaf with shortest incident edge)



cut



split



merge

Induced results

For $g \geq 0$ and $\mathbf{k} = (k_1, \ldots, k_r)$
the 2 vectors Vor and Int are equidistributed in the $\mathrm{CRM}^{(g,\mathbf{k})}$

- Case $r = 1$ (unicellular maps)
  Int is uniformly distributed on Δ_k, hence so is Vor
  $\Rightarrow$ the CRUM_g is Voronoi-uniform

- Case $k_1 = 1, \ldots, k_r = 1$ (one marked corner in each face)
  for a random map in $\mathcal{M}_n^{(g,\mathbf{k})}$, $\mathrm{Vor} \sim \frac{1}{2n} \cdot (\deg(f_1), \ldots, \deg(f_r))$
  $g = 0$: Tutte's slicings formula gives $\mathrm{Vor} \sim$ density $\propto x_1^{1/2} \cdots x_r^{1/2}$ on Δ_r
  (Dirichlet $(\frac{1}{2}, \ldots, \frac{1}{2})$)