A bijection between binary trees and some dissections of the hexagon

Eric FUSY
joint work with Dominique Poulalhon and Gilles Schaeffer
Binary trees

- A binary tree is a tree embedded on the plane with:
  - vertices of degree 3: inner nodes
  - vertices of degree 1: leaves
- An edge connecting two inner nodes is called an *inner edge*
- An edge incident to a leaf is called a *stem*
Rooted binary trees

- A binary tree is rooted by marking one of its leaves
- Rooted binary trees are counted by the Catalan numbers:

\[
\frac{(2n)!}{(n + 1)!n!}
\]
Irreducible dissections

- Dissection of the hexagon in the plane with quadrangular inner faces
- Each 4-cycle delimits a face
Rooted irreducible dissections

An irreducible dissection can be rooted by choosing and orienting an edge of the hexagon.
Local closure

- Choose a stem followed by three inner edges in a ccw traversal of the tree
- Merge the extremity of the stem with the extremity of the third edge
Partial closure

Perform all local closures greedily until no local closure is possible
Complete closure

- Draw an hexagon outside of the figure
- Merge all remaining stems with vertices of the hexagon so as to create quadrangular faces
Results

• The closure application is a bijection between unrooted binary trees and unrooted irreducible dissections.

• The closure application is a 6-to-$(n + 2)$ application between rooted binary trees and rooted irreducible dissections.

• $|D'_n| = \frac{6}{n+2} |B'_n| = \frac{6(2n)!}{n!(n+2)!}$ already found by Mullin and Schellenberg.
Links with 3-connected maps

- A 3-connected map is a map that cannot be disconnected by the removal of two of its vertices.
- An irreducible quadrangulation is a quadrangulation such that each 4-cycle delimits a face
- 3-connected rooted planar maps with $n$ edges are in bijection with irreducible rooted quadrangulations with $n$ faces
Irreducible quadrangulations (all faces are squares) and irreducible dissections (outer face hexagon)

- Removing the root defines an injection between irreducible quadrangulations and irreducible dissections.

- The reciprocal fails when there is an internal path of length 3 between $s$ and $t$.

![Injection Diagram](image)

![Reciprocally Diagram](image)
Summary of the links

3–connected maps

\[ \uparrow \quad \text{bijection} \quad \downarrow \]

irreducible quadrangulations

\[ \uparrow \quad \text{injection:} \quad \downarrow \]

remove root edge \quad add edge in the outer face

irreducible dissections
Application: sampling 3-connected maps

Algorithm:

1. Sample a rooted binary tree $T \in \mathcal{B}_n'$ using a parenthesis word

2. Close $T$ to obtain an irreducible dissection $D \in \mathcal{D}_n'$

3. If no internal path of length 3 between $s$ and $t$ add an edge between $s$ and $t$ to obtain an irreducible quadrangulation $Q$. Otherwise reject and return to 1.

4. Return the 3-connected map with $n$ edges whose quadrangulation is $Q$

Complexity:

- linear time for each try

- number of rejects: geometric law with parameter
  \[ p_n = \frac{|\mathcal{Q}_n'|}{|\mathcal{D}_n'|} \rightarrow_{n \to \infty} 2^6 / 3^5. \]
  Mean number rejects: $1/p_n \rightarrow 3^5 / 2^6$
Application: counting 3-connected maps

$M(x), Q(x)$ and $D(x)$ generating functions of 3-connected maps, irreducible quadrangulations, and irreducible quadrangulations. $r(x) = x(1 + r(x))^2$ generating function of rooted binary trees

- Bijection: $M(x) = Q(x)$

- Rejection (decomposition along paths of length 3 between $s$ and $t$): $D(x) = \left(\frac{2x}{1-x} + 1\right) \frac{1}{1 - Q(x)\left(\frac{2x}{1-x} + 1\right)} - 2x - 1$

- Closure application: $D(x) = r(x)(2 - r(x))$

Finally

$$M(x) = -\frac{1}{1 + 2x + x^2r(x)(2 - r(x))} + \frac{1 - x}{1 + x}$$
Idea of the reciprocal application

- A binary tree can be oriented with outdegree 3 for each inner node
- The closure induces a certain orientation of edges of the irreducible dissection with outdegree 3 for each inner vertex
- Reciprocal: find this particular orientation and use it to open the dissection in a binary tree