Duality relations for constrained walks

Éric Fusy (CNRS/LIX)

joint work with Mireille Bousquet-Mélou, Julien Courtiel, Mathias Lepoutre, Marni Mishna, and Kilian Raschel

TU Berlin, Jan 13, 2020
Duality phenomenon for paths

We say two path families $\mathcal{A}$ and $\mathcal{B}$ are dual if

- both families use the same steps, such that $\mathcal{A}$ has stronger endpoint constraint, $\mathcal{B}$ has stronger domain constraint

- there is a length-preserving bijection between $\mathcal{A}$ and $\mathcal{B}$

Example in 2D: $(\text{step-set } \{\uparrow, \leftarrow, \downarrow, \rightarrow\})$
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**Example in 2D:** (step-set $\{\uparrow, \leftarrow, \downarrow, \rightarrow\}$)

Motivations:

- mapping $\mathcal{A} \rightarrow \mathcal{B}$ for **counting** ($\mathcal{A}$ easier)
- mapping $\mathcal{B} \rightarrow \mathcal{A}$ for **random generation** (early-abort rejection)
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- mapping $\mathcal{B} \rightarrow \mathcal{A}$ for random generation (early-abort rejection)

<table>
<thead>
<tr>
<th>gen$\mathcal{B}$:</th>
<th>while not fails</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>generate random walk step by step</td>
</tr>
<tr>
<td></td>
<td>reject as soon as walk leaves domain for $\mathcal{B}$ (if not, success!)</td>
</tr>
</tbody>
</table>
Classical 1D example

\[ a_{2n} = \binom{2n}{n} \]

\( A \leftrightarrow B \)
Classical 1D example

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1st bijection:

\[ \gamma_1 \rightarrow \gamma_2 \]

\[ \text{mir}(\gamma_1) \rightarrow \gamma_2 \]

\[ k \rightarrow 2k \]
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Rk: implies

\[ k \quad 2k \]

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Classical 1D example

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\[ \mathcal{A} \quad \leftrightarrow \quad \mathcal{B} \]

1st bijection:

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**Rk:** implies

**Rk:** extends to \( r \geq 1 \) paths

\[ r = 2 \]

[Proctor'83, Elizalde'15, Hanaker et al.'17]
Classical 1D example

2nd bijection:

via Dyck paths with marked down-steps ending on $x$-axis

\[ A \]

$k = 3$ excursions below $x$-axis

flip excursions of marked steps

\[ \text{intermediate} \]

$k = 3$ marked steps

flip marked steps

\[ B \]

ends at height $2k = 6$
Outline of the talk

Duality relations for 2D walks using bijections to oriented maps

• Simple walks: \{↑, ←, ↓, →\}

• Tandem walks: \{←, ↑, ↘\} (and extension)

using Bernardi-Bonichon bijection for Schnyder woods

using Kenyon et al. bijection for bipolar orientations
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Simple walks
2D simple walk $\leftrightarrow$ pair of directed walks

$x(t) + y(t)$

$x(t) - y(t)$

$2y$
2D simple walk $\leftrightarrow$ pair of directed walks

$x(t) + y(t)$

$x(t) - y(t)$

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$x(t) + y(t)$

$x(t) - y(t)$

$2y$
2D simple walk $\leftrightarrow$ pair of directed walks

$\text{Rk: } x(t) + y(t)$

is the same as $x(t) - y(t)$
2D simple walk $\leftrightarrow$ pair of directed walks

$x(t) + y(t)$

$x(t) - y(t)$

$2y$

$x$

$y$

$\uparrow \downarrow$

$\leftrightarrow$

Rk: $\leftrightarrow$

is the same as

[Elizalde’15] path manipulations

or Schnyder woods

easy
Schnyder wood on triangulations

Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:
- at each inner vertex
- at the outer vertices

yields a spanning tree in each color

[Schnyder’89]
Bijection for Schnyder woods

Some information is redundant:

just need the blue tree and positions of the ingoing red edges
Bijection for Schnyder woods

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Bottom Dyck path:
contour of blue tree

[Bernardi, Bonichon’07]
Bijection for Schnyder woods

Some information is redundant:
just need the blue tree and positions of the ingoing red edges

Upper Dyck path:
red indegrees

4th up-step
\text{in}(u_4) = 2

Bottom Dyck path:
contour of blue tree

\[ \text{Bernardi, Bonichon'07} \]
Bijection for Schnyder woods

The mapping is a bijection from Schnyder woods with $n + 3$ vertices to non-crossing pairs of Dyck paths of lengths $2n$. 

$deg(B) = 3$

$deg(R) = 2$
Proof of
Proof of $\leftrightarrow$ Rk: $\leftrightarrow$

proof via arc diagrams also holds

[Courtiel,F,Lepoutre,Mishna’18]
Extension to prove

Bijection extended to $a = 2$
Tandem walks
Tandem walks

A **tandem-walk** is a walk in $\mathbb{Z}^2$ with step-set $\{N, W, SE\}$

in the plane $\mathbb{Z}^2$

in the half-plane $\{y \geq 0\}$

in the quarter plane $\mathbb{N}^2$

cf 2 queues in series
Duality relation for tandem walks

There is a bijection between:

- tandem walks of length $n$
  staying in the quarter plane $\mathbb{N}^2$

- tandem walks of length $n$
  staying in the half-plane $\{y \geq 0\}$
  and ending at $y = 0$

**Rk:** The bijection preserves the number of SE steps
There is a bijection between:

tandem walks of length \( n \) staying in the quadrant \( \mathbb{N}^2 \), ending at \((i, j)\)

\[ \downarrow \]

Young tableaux of size \( n \) and height \( \leq 3 \), of shape

\[ \text{tableau} \]

(after \( s \) steps, current \( y = \#N - \#SE \), current \( x = \#SE - \#W \))
Bijection with Motzkin walks

[

1  2  5  8  9  11
3  6  7  10 13
4  12

Young tableau
of height $\leq 3$

Gouyou-Beauchamps’89]
Bijection with Motzkin walks

Robinson Schensted involution with no tandem walk in $\mathbb{N}^2$

Young tableau of height $\leq 3$

[Gouyou-Beauchamps’89]
Bijection with Motzkin walks

Robinson Schensted involution with no matching

with no nesting

tandem walk in $\mathbb{N}^2$

Young tableau of height $\leq 3$

[Robinson, Schensted, Gouyou-Beauchamps'89]
Bijection with Motzkin walks

Young tableau of height $\leq 3$

Robinson Schensted

no nesting

FIFO

Motzkin walk

tandem walk in $\mathbb{N}^2$

[Gouyou-Beauchamps’89]
Bijection with Motzkin walks

Young tableau of height ≤ 3

no nesting

FIFO

Motzkin walk

LIFO

no crossing

matching with no nesting

Robinson Schensted

involution with no nesting

Gouyou-Beauchamps’89

Robinson

Schensted

[1 2 5 8 9 11]
[3 6 7 10 13]
[4 12]
An extension of the walk model

General model:

- **Step-set:**
  - the SE step
  - every step $(-i, j)$ (with $i, j \geq 0$)

  \[ \text{level} := i + j \]

Example:

![Graph showing the walk model with levels and steps labeled](image-url)
An extension of the walk model

**General model:**

- **step-set:**
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- **level:** \(i + j\)

**Example:**

- The bijection (using bipolar orientations) preserves the number of SE-steps and the number of steps in each level \(p \geq 1\)

We still have

- different bijection using automata rules [Chyzak-Yeats’18]

The bijection (using bipolar orientations) **preserves** the number of SE-steps and the number of steps in each level \(p \geq 1\)
Bipolar and marked bipolar orientations

**bipolar orientation:**

(on planar maps)

= acyclic orientation

with a unique source $S$

and a unique sink $N$

with $S, N$ incident to the outer face

inner vertex

inner face
Bipolar and marked bipolar orientations

**bipolar orientation:**
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= acyclic orientation
with a unique source $S$
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**marked bipolar orientation:**

a marked vertex $W \neq N$ on left boundary
a marked vertex $E \neq S$ on right boundary

with indegree=1

outdegree=1

inner vertex

inner face
The Kenyon et al. bijection

start with \( \begin{array}{c}
N \\
E \\
W \\
S
\end{array} \) and read the walk step by step

- **SE steps** create a new black vertex

- **steps** \((-i, j)\) create a new inner face (of degree \(i + j + 2\))
The Kenyon et al. bijection

general tandem-walk (in $\mathbb{Z}^2$) \(\xrightarrow{bijection}\) marked bipolar orientation

SE step \(\xrightarrow{\text{black vertex}}\) inner face of degree \(i+j+2\)

step \((-i, j)\) \(\xrightarrow{\text{inner face of degree } i+j+2}\)
Parameter-correspondence in the bijection

\[
\begin{align*}
\# \text{ “face-steps” of level } p & \longleftrightarrow \# \text{ inner faces of degree } p + 2 \\
\# \text{ SE-steps} & \longleftrightarrow \# \text{ black vertices} \\
1 + \# \text{ steps} & \longleftrightarrow \# \text{ plain edges (not dashed)}
\end{align*}
\]
An involution on marked bipolar orientations

\begin{align*}
N & \quad d \\
W & \quad a \\
S & \quad c+1 \\
E & \quad b+1
\end{align*}
An involution on marked bipolar orientations

Mirror

$a \leftrightarrow d$
Effect of the involution on walks

$$b+1 \quad d$$

$$W \quad a$$

$$S$$

$$N$$

$$E$$

$$\cdots$$

$$c+1$$

$$\text{involution} \quad a \leftrightarrow d$$

$$\text{start} \quad \text{end}$$

$$b \quad d$$

$$c$$

$$a$$

$$b$$

$$d$$
Proof of

\[ \leftrightarrow \]

\[ \text{involution} \]

[Bousquet-Mélou, F, Raschel’19]
Proof of

• Specialize the involution at $b = 0$

& specialize further at $d = 0$

[Bousquet-Mélou,F,Raschel’19]
General situation in duality bijections

Two families $\mathcal{A}, \mathcal{B}$ of walks

$$A(t) = \sum_n a_n t^n \quad B(t) = \sum_n b_n t^n$$

want to prove bijectively that $A(t) = B(t)$
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There is a superfamily $\mathcal{C} \supset \mathcal{A}, \mathcal{B}$ and an involution on $\mathcal{C}$ exchanging two parameters $i, j$ such that, with $C(t; u, v) = \sum c_{n, i, j} t^n u^i v^j$, we have

$$A(t) = C(t; 1, 0) \quad B(t) = C(t; 0, 1)$$
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**Ex:** for tandem walks

Mirror-involution via bipolar orientations

**Ex:** for 1D walks of even length

Exchange involution

Extension for $r \geq 1$ walks: involutivity of jeu de taquin

[Hanaker et al.'17]
Conjecture for double-tandem walks

Step-set

Known: [Yeats’14, Chyzak-Yeats’18]

Conjecture: There is an involution that realizes and preserves the length and the number of steps in \{→, ↓, ↖\}

and preserves the length and the number of steps in \{→, ↓, ↖\}