Geometric representations of planar graphs and maps

Éric Fusy (CNRS/LIX)
Overview of the course

- Planar graphs and planar maps
  - structural aspects
  - combinatorial aspects

- Geometric representations

  straight-line drawings
  contact representations

+ applications & links to physical models
Structural aspects of planar graphs and maps
Graphs

A graph $G = (V, E)$ is given by two sets $V, E$ such that each $e \in E$ is an (unordered) pair of elements from $V$.

$V$ is the set of vertices, $E$ is the set of edges (links, relations).

Example:

$V = \{1, 2, 3, 4, 5, 6\}$

$E = \{\{1, 5\}, \{3, 6\}, \{1, 5\}, \{4, 5\}, \{2, 3\}, \{1, 4\}\}$
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Can also allow for loops and multiple edges

**Example:**

$V = \{a, b, c, d, e\}$

$E = \{\{a, b\}, \{b, b\}, \{b, c\}, \{c, e\}, \{b, c\}, \{a, d\}, \{d, c\}\}$
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Can also allow for loops and multiple edges

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**Def:** A graph is called **simple** if it has no loop nor multiple edges

A graph is called **connected** if it is “in one piece”
The natural abstraction for networks

- Social network
- Airline connections network
- Road network
- Electronic network
Planar graphs
A graph is called **planar** if it can be drawn **crossing-free** in the plane

$K_4$ is planar

![Non-planar drawing](image1)

![Planar drawing](image2)

$K_5$ is not planar

![Non-planar drawing with crossing](image3)

*(whatever drawing, there is always a crossing)*
Planar graphs

A graph is called **planar** if it can be drawn **crossing-free** in the plane.

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Rk: planar $\iff$ can be drawn crossing-free on the sphere
Planar maps

Def. Planar map = connected graph embedded on the sphere (up to continuous deformation)

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A map is easier to draw in the plane (implicit choice of an outer face $f_0$)
**Planar maps**

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A map has vertices, edges, and faces

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5 faces (including outer one)
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5 faces (including outer one)

Degree of a face = length of walk around $f$
Motivations for studying planar maps

- Planar networks usually **come with an explicit planar embedding**

- A natural model of **discrete surface** (formed from glued polygons)

  - Abstraction of geographic maps
  - Meshes
  - Random discrete surfaces (2D quantum gravity)

- Nice combinatorial properties!
Duality for planar maps

6 vertices, 9 edges, 5 faces

a planar map

the dual map

5 vertices, 9 edges, 6 faces

preserves \#(edges), exchanges \#(vertices) and \#(faces)
The Euler relation

Let $M = (V, E, F)$ be a planar map. Then

$$|E| = |V| + |F| - 2$$

$|V| = 6, |E| = 9, |F| = 5$
The Euler relation

Let \( M = (V, E, F) \) be a planar map. Then

\[
|E| = |V| + |F| - 2
\]

**Proof using spanning trees**

\[
|E| = (|V| - 1) + (|F| - 1)
\]

\( |V| = 6, |E| = 9, |F| = 5 \)
Kuratowski’s theorem for planar graphs
The Euler relation implies (exercise!) that $K_5$ and $K_{3,3}$ are not planar
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Hence any subdivision of $K_5$ or $K_{3,3}$ is not planar either.

A subdivision of $K_5$
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**k-connectivity in graphs**

For $k \geq 2$ a graph $G$ is called $k$-connected if $G$ is connected and remains connected when deleting any $(k - 1)$-subset of vertices.
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- not 3-connected $\iff \exists$ separating vertex-pair
**$k$-connectivity in graphs**

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- not 2-connected $\iff \exists$ separating vertex

- not 3-connected $\iff \exists$ separating vertex-pair

**Exercise:** for triangulations (faces have degree 3)

2-connected $\iff$ loopless

3-connected $\iff$ simple
The structure of the set of embeddings

For $G$ a connected planar graph, operations to change the embedding are:

- **Mirror**
- **Flip at separating vertex**
- **Flip at separating pair**
The structure of the set of embeddings
For $G$ a connected planar graph, operations to change the embedding are:

- **mirror**
- **flip at separating vertex**
- **flip at separating pair**

**Theorem** (Tutte, Whitney): any two embeddings of $G$ are related by a sequence of such operations.

Hence 3-connected planar graphs have a unique embedding (up to mirror).
A $d$-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$.
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**Rk:** A polytope $P$ induces a graph $G_P$ (vertices & edges).
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Relation with polytopes

A $d$-dimensional polytope is a bounded region $P \subset \mathbb{R}^d$ that can be obtained as $P = H_1 \cap H_2 \cap \cdots \cap H_k$ for some half-spaces $H_1, \ldots, H_k$.

Rk: A polytope $P$ induces a graph $G_P$ (vertices & edges).

Balinsky’61: If $P$ has dimension $d$, then $G_P$ is $d$-connected.

Steinitz’16: A planar graph is 3-connected iff it can be obtained as the graph of a 3D polytope.
Combinatorial aspects of planar maps
Rooted maps

A map is rooted by marking and orienting an edge

Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)
Rooted maps

A map is **rooted** by marking and orienting an edge

![A rooted map](image)

the face on the right of the root is taken as the outer face

Rooted maps are combinatorially easier than maps (no symmetry issue, root gives starting point for recursive decomposition)

<table>
<thead>
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Duality for rooted maps

same as for maps (root the dual at the dual of the root-edge)

vertices and faces play a symmetric role in rooted maps
Counting rooted maps

Let $a_n$ be the number of rooted maps with $n$ edges

<table>
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\[
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 n & 1 & 2 & 3 & 4 & 5 & 6 & 7 & \ldots \\
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**Theorem:** (Tutte’63)

\[
\frac{2 \cdot 3^n}{(n + 1)(n + 2)} \binom{2n}{n}
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Not an isolated case:

- **Triangulations** (2\(n\) faces)
  
  Loopless: \( \frac{2^n}{(n+1)(2n+1)} \binom{3n}{n} \)
  
  Simple: \( \frac{1}{n(2n-1)} \binom{4n-2}{n-1} \)

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Bijection maps $\leftrightarrow$ quadrangulations

$n$ edges
$i$ vertices
$j$ faces

$n$ faces
$i$ white vertices
$j$ black vertices
Bijection maps $\leftrightarrow$ quadrangulations

Consequence:

$\#(\text{rooted maps with } n \text{ edges}) = \#(\text{rooted quadrangulations with } n \text{ faces})$
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Consequence:

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It remains to see why this common number is $\frac{2 \cdot 3^n}{(n + 1)(n + 2)} \binom{2n}{n}$.
Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**
Counting rooted maps with one face

A rooted map with one face is called a **rooted plane tree**

Let $c_n$ be the number of rooted plane trees with $n$ edges

Let $C(z) = \sum_{n \geq 0} c_n z^n$ be the associated generating function

$C(z) = 1 + z + 2z^2 + 5z^3 + 14z^4 + \cdots$
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**Decomposition at the root:**

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\text{no edge} & \quad + \quad & \text{at least one edge}
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**recurrence:**

\[ c_0 = 1 \quad \text{and} \quad c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k} \quad \text{for} \quad n \geq 1 \]
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**Decomposition at the root:**

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- at least one edge

**recurrence:** $c_0 = 1$ and $c_n = \sum_{k=0}^{n-1} c_k c_{n-1-k}$ for $n \geq 1$

**GF equation:** $C(z) = 1 + z \cdot C(z)^2$
A rooted map with one face is called a **rooted plane tree**

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A rooted map with one face is called a rooted plane tree.

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solved as

\[ C(z) = \frac{1 - \sqrt{1 - 4z}}{2z} \]

**Taylor expansion:**

\[ C(z) = \sum_{n \geq 0} \frac{(2n)!}{n!(n+1)!} \quad \Rightarrow \quad c_n = \frac{(2n)!}{n!(n+1)!} \]

Catalan numbers
Adaptation to rooted maps

Let $m_n$ be the number of rooted maps with $n$ edges.

Let $M(z) = \sum_{n \geq 0} m_n z^n$ be the associated generating function:

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$$= 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots$$

Decomposition by deleting the root:

- no edge
- at least one edge disconnecting
- non-disconnecting
Adaptation to rooted maps

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\[ = 1 + 2z + 9z^2 + 54z^3 + 378z^4 + 2916z^5 + \cdots \]

Decomposition by deleting the root:

\[ M(z) = 1 + M(z)^2 + \text{?} \]
Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with $n$ edges and outer degree $k$.

Let $M(z, u) = \sum_{n, k \geq 0} m_{n,k} z^n u^k$ be the associated generating function

$$= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$
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Decomposition by deleting the root:

$$M(z,u) = 1 + zu^2 \cdot M(z,u)^2 + A(z,u)$$
Adding a secondary variable

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$$M(z, u) = 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$$

More generally $z^n u^k \rightarrow z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$.
Adding a secondary variable

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More generally

$$z^n u^k \rightarrow z^{n+1} \cdot (u + u^2 + \cdots + u^{k+1})$$

$$\Rightarrow A(z,u) = \sum_{n,k} m_{n,k} \ z^{n+1} \cdot (u + \cdots + u^{k+1})$$

$$= u \cdot \frac{u^{k+1} - 1}{u-1}$$
Adding a secondary variable

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Let $M(z,u) = \sum_{n,k \geq 0} m_{n,k} z^n u^k$ be the associated generating function

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More generally

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\[ \Rightarrow A(z,u) = \sum_{n,k} m_{n,k} z^{n+1} \cdot (u + \cdots + u^{k+1}) = zu \frac{uM(z,u) - M(z,1)}{u - 1} \]
Adding a secondary variable

Let \( m_{n,k} \) be the number of rooted maps with \( n \) edges and outer degree \( k \).

Let \( M(z, u) = \sum_{n,k \geq 0} m_{n,k} z^n u^k \) be the associated generating function

\[
= 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots
\]

**Decomposition by deleting the root:**

\[
M(z, u) = 1 + zu^2 \cdot M(z, u)^2 + zu \frac{uM(z, u) - M(z, 1)}{u - 1}
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Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with $n$ edges and outer degree $k$.

Let $M(z,u) = \sum_{n,k \geq 0} m_{n,k} z^n u^k$ be the associated generating function.

**Functional equation obtained:**

\[
M(z, u) = 1 + zu^2 \cdot M(z, u)^2 + zu \frac{uM(z, u) - M(z, 1)}{u - 1}
\]

of the form $P(M(z, u), M(z, 1), z, u) = 0$
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One equation, two unknown: $M(z,u)$ and $M(z,1)$

But a unique solution (2 unknown are correlated)

Equation $\Rightarrow M(z,u) = 1 + z(u+u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots$
Adding a secondary variable

Let $m_{n,k}$ be the number of rooted maps with $n$ edges and outer degree $k$.

Let $M(z,u) = \sum_{n,k \geq 0} m_{n,k} z^n u^k$ be the associated generating function.

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But a unique solution (2 unknown are correlated)

\[
M(z, u) = 1 + z(u + u^2) + z^2(2u + 2u^2 + 3u^3 + 2u^4) + \cdots
\]

Guess/and/check or explicit solution methods:

[Brown, Tutte’65, Bousquet-Mélou-Jehanne’06, Eynard’10]

\[
M(z, 1) = \frac{1}{54z^2} (-1 + 18z + (1 - 12z)^{3/2}) = \sum_{n \geq 0} \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} z^n
\]
Bijective proof: which formula to prove

Let $q_n = \#(\text{rooted quadrangulations with } n \text{ faces})$

We want to show (bijectively) that

$$q_n = \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} z^n$$
Bijective proof: which formula to prove

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Consider \( b_n = \#(\text{quad. with } n \text{ faces, a marked vertex and a marked edge}) \)
Bijective proof: which formula to prove

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\]

Consider \( b_n = \#(\text{quad. with } n \text{ faces, a marked vertex and a marked edge}) \)

Consider a diagram of a quadrangulation with a marked vertex and a marked edge.
Bijective proof: which formula to prove

Let \( q_n = \#(\text{rooted quadrangulations with } n \text{ faces}) \)

We want to show (bijectively) that

\[
q_n = \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} z^n
\]

Consider \( b_n = \#(\text{quad. with } n \text{ faces, a marked vertex and a marked edge}) \)

\[
(n + 2) \cdot q_n = 2 \cdot b_n
\]

Simple relation between \( b_n \) and \( q_n \): 

\[
\underbrace{(n + 2)}_{\#(\text{vertices})} \cdot q_n = 2 \cdot b_n
\]
Bijective proof: which formula to prove

Let $q_n = \#$(rooted quadrangulations with $n$ faces)

We want to show (bijectively) that

$$q_n = \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} z^n$$

Consider $b_n = \#$(quad. with $n$ faces, a marked vertex and a marked edge)

Simple relation between $b_n$ and $q_n$: $$q_n = 2b_n \quad \#$(vertices)

Hence showing

$$q_n = \frac{2 \cdot 3^n}{(n + 2)(n + 1)} \binom{2n}{n} z^n$$

amounts to showing

$$b_n = 3^n \frac{(2n)!}{n!(n + 1)!} = 3^n \text{Cat}_n$$
Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex \( v_0 \)

Geodesic labelling with respect to \( v_0 \): \( \ell(v) = \text{dist}(v_0, v) \)

Rk: two types of faces

- **Stretched**
- **Confluent**
Well-labelled trees

Well-labelled tree = plane tree where
- each vertex $v$ has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$
- the minimum label over all vertices is 1
The Schaeffer bijection [Schaeffer’99], also [Cori-Vauquelin’81]

Pointed quadrangulation $\Rightarrow$ well-labelled tree with min-label=1

$n$ faces

$n$ edges

Local rule in each face:

$\left[\begin{array}{c}
\begin{array}{c}
i+2 \\
i+1 \\
i+1
\end{array} \\
\begin{array}{c}
i+1 \\
i+1
\end{array} \\
i+1
\end{array}\right]$
The Schaeffer bijection [Schaeffer’99], also [Cori-Vauquelin’81]
From a well-labelled tree to a pointed quadrangulation
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From a well-labelled tree to a pointed quadrangulation

1) insert a “leg” at each corner
The Schaeffer bijection [Schaeffer’99], also [Cori-Vauquelin’81]
From a well-labelled tree to a pointed quadrangulation

1) insert a “leg” at each corner
2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree
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From a well-labelled tree to a pointed quadrangulation

1) insert a “leg” at each corner
2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree
3) create a new vertex $v_0$ outside and connect legs of label 1 to it
The Schaeffer bijection  \cite{Schaeffer'99}, also \cite{Cori-Vauquelin'81}

From a well-labelled tree to a pointed quadrangulation

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4) erase the tree-edges
The Schaeffer bijection \cite{Schaeffer'99}, also \cite{Cori-Vauquelin'81}

From a well-labelled tree to a pointed quadrangulation

1) insert a “leg” at each corner

2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree

3) create a new vertex $v_0$ outside and connect legs of label 1 to it

4) erase the tree-edges

recover the original pointed quadrangulation
The effect of marking an edge

Local rule in each face:

- marked edge
- marked half-edge
Bijective proof of counting formula

Schaeffer’s bijection \( \Rightarrow b_n = \#(\text{rooted well-labeled trees with } n \text{ edges}) \)
Bijective proof of counting formula

Schaeffer’s bijection ⇒ $b_n = \#(\text{rooted well-labelled trees with } n \text{ edges})$

$$b_n = 3^n \text{Cat}_n = 3^n \frac{(2n)!}{n!(n + 1)!}$$
Application to study distances in random maps

- **Typical distance** between (random) vertices in random maps
  the order of magnitude is $n^{1/4}$ (≠ $n^{1/2}$ in random trees)

random quadrang. \{ - [Chassaing-Schaeffer’04] probabilistic
- [Bouttier Di Francesco Guitter’03] exact GF expressions

- How does a random map (rescaled by $n^{1/4}$) “look like”?

as a (rescaled) **discrete metric space**
convergence to the “Brownian map”

[Le Gall’13, Miermont’13]

© Nicolas Curien
Extension to pointed bipartite maps

[Bouttier, Di Francesco, Guitter’04]

Local rule

Conditions:
(i) ∃ vertex of label 1
(ii) $j \leq i + 1$
Geometric representations of planar maps:
I. Straight-line drawings
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments?
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs \textit{up to continuous deformation}

\begin{align*}
\text{Question: } & \text{Does there always exist an equivalent planar drawing such that all edges are drawn as segments ?}
\end{align*}
Existence question

A planar map (with outer face) is the equivalence class of planar drawings of graphs up to continuous deformation.

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments?

(such as drawing is called a (planar) straight-line drawing)
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments? (such as drawing is called a (planar) straight-line drawing)

Remark: For such a drawing to exist, the map needs to be simple
Existence proof (reduction to triangulations)

- Any simple planar map $M$ can be completed to a simple triangulation $T$.
  (Exercise: can be done without creating new vertices, only edges)

```
```

![Diagram showing transformation from a simple planar map to a simple triangulation]
```
Existence proof (reduction to triangulations)

- Any simple planar map \( M \) can be completed to a simple triangulation \( T \) (Exercise: can be done without creating new vertices, only edges)
- A straight-line drawing of \( T \) yields a straight-line drawing of \( M \)
Existence proof (for triangulations)

**First proof:** induction on the number of vertices
Let $T$ be a triangulation with $n$ vertices
Existence proof (for triangulations)

First proof: induction on the number of vertices
Let $T$ be a triangulation with $n$ vertices.

Exercise: $T$ has at least one inner vertex $v$ of degree $\leq 5$
Existence proof (for triangulations)

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Let $T$ be a triangulation with $n$ vertices

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$T \setminus v$ has a straight-line drawing

induction

$T \setminus v$ has a straight-line drawing
Existence proof (for triangulations)

First proof: induction on the number of vertices
Let $T$ be a triangulation with $n$ vertices

Exercise: $T$ has at least one inner vertex $v$ of degree $\leq 5$

$T \setminus v$ has a straight-line drawing
Straight-line drawing algorithms

We present two famous algorithms (each with its advantages)

- Tutte’s barycentric method

- Schnyder’s face-counting algorithm
Planarity criterion for straight-line drawings

Planar

Non-planar
Planarity criterion for straight-line drawings

**Theorem:** A straight-line drawing is planar iff every inner vertex is inside the **convex hull** of its neighbours (works for triangulations and more generally for 3-connected planar graphs)
Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of $c$ in the drawing
Proof idea

• For each corner $c \in T$ let $\theta(c)$ be the angle of $c$ in the drawing.

• For each vertex $v$, let $\Theta(v) = \sum_{c \in v} \theta(c)$.
Proof idea

- For each corner \( c \in T \) let \( \theta(c) \) be the angle of \( c \) in the drawing.

- For each vertex \( v \), let \( \Theta(v) = \sum_{c \in v} \theta(c) \).

- Whatever the drawing we always have \( \sum_v \Theta(v) = 2\pi|V| \).
Proof idea

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- Whatever the drawing we always have \( \sum_v \Theta(v) = 2\pi|V| \).

- If convex hull condition holds, then \( \Theta(v) \geq 2\pi \) for each \( v \).
Proof idea

- For each corner $c \in T$ let $\theta(c)$ be the angle of $c$ in the drawing.

- For each vertex $v$, let $\Theta(v) = \sum_{c \in v} \theta(c)$

- Whatever the drawing we always have $\sum_v \Theta(v) = 2\pi |V|$.

- If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each $v$ and since $\sum_v \Theta(v) = 2\pi |V|$, must have $\Theta(v) = 2\pi$ for each $v$. 
Proof idea

• For each corner $c \in T$ let $\theta(c)$ be the angle of $c$ in the drawing

• For each vertex $v$, let $\Theta(v) = \sum_{c \in v} \theta(c)$

• Whatever the drawing we always have $\sum_v \Theta(v) = 2\pi|V|$

• If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each $v$ and since $\sum_v \Theta(v) = 2\pi|V|$, must have $\Theta(v) = 2\pi$ for each $v$

Hence locally planar at each vertex (no “folding” of triangles at a vertex) ⇒ the drawing is planar
Tutte’s barycentric method

- Outer vertices $v_1, v_2, v_3$ are fixed at fixed positions (nailed)

- Each inner vertex is at the barycenter of its neighbours

\[
x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \quad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \quad \text{for } i \geq 4
\]
Tutte’s barycentric method

- Outer vertices $v_1, v_2, v_3$ are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

\[ x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \quad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \quad \text{for } i \geq 4 \]

\[ \sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4 \]
Tutte’s barycentric method

- Outer vertices $v_1, v_2, v_3$ are fixed at fixed positions (nailed)

- Each inner vertex is at the barycenter of its neighbours

\[
\begin{align*}
  x_i &= \frac{1}{\Delta_i} \sum_{j \sim i} x_j \\
  y_i &= \frac{1}{\Delta_i} \sum_{j \sim i} y_j 
\end{align*}
\]

for $i \geq 4$

\[
\Leftrightarrow \sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} y_i - y_j = 0 \quad \text{for each } i \geq 4
\]

- This drawing exists and is unique. It minimizes the energy

\[
P = \sum_e \ell(e)^2 = \sum_{\{i, j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2
\]

under the constraint of fixed $x_1, x_2, x_3, y_1, y_2, y_3$
Tutte’s barycentric method

- Outer vertices \( v_1, v_2, v_3 \) are fixed at fixed positions (nailed)

- Each inner vertex is at the \textbf{barycenter of its neighbours}

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x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \quad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \quad \text{for } i \geq 4
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\mathcal{P} = \sum_e \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2
\]

under the constraint of fixed \( x_1, x_2, x_3, y_1, y_2, y_3 \)

- also called \textbf{spring embedding} (each edge is a spring of energy \( \ell(e)^2 \))
Tutte’s barycentric method

- Outer vertices \(v_1, v_2, v_3\) are fixed at fixed positions (nailed)

- Each inner vertex is at the **barycenter of its neighbours**

\[
x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \quad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \quad \text{ for } i \geq 4
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\sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4
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- This drawing **exists and is unique**. It minimizes the energy

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\mathcal{P} = \sum_{e} \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2
\]

under the constraint of fixed \(x_1, x_2, x_3, y_1, y_2, y_3\)

- also called **spring embedding** (each edge is a spring of energy \(\ell(e)^2\))
Advantages/disadvantages

The good!
- displays the symmetries nicely
- easy to implement (solve a linear system)
- optimal for a certain energy criterion

The less good:
- a bit expensive computationally (solve linear system of size $|V|$)
- some very dense clusters (edges of length exponentially small in $|V|$)
Schnyder woods

Schnyder wood = each inner edge is given a direction and a color (red, green, blue) so as to satisfy local rules at each vertex:

[Schyneder’89]: each (simple) triangulation admits a Schnyder wood
Fundamental property of Schnyder woods
In each color the edges form a spanning tree (rooted at the 3 outer vertex)
Shelling procedure to compute Schnyder woods at each step:
Shelling procedure to compute Schnyder woods

at each step:
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Face-counting drawing procedure
[Schnyder’90]
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Face-counting drawing procedure

[Schnyder’90]

9 inner faces

9 \times 9 \times 9 grid
Face-counting drawing procedure

[Schnyder’90]

9 inner faces

for $v$:  
- red area: 2 faces
- green area: 5 faces
- blue area: 2 faces

$9 \times 9 \times 9$ grid
Face-counting drawing procedure
[Schnyder’90]

for \( v \):
- red area: 2 faces
- green area: 5 faces
- blue area: 2 faces

draw \( v \) at the barycenter of \( \{a, b, c\} \)
with weights \( \frac{2}{9}, \frac{5}{9}, \frac{2}{9} \)
Face-counting drawing procedure
[Schnyder’90]

for $v$: red area: 2 faces
        green area: 5 faces
        blue area: 2 faces

draw $v$ at the barycenter of $\{a, b, c\}$
        with weights $\frac{2}{9}$, $\frac{5}{9}$, $\frac{2}{9}$
Face-counting drawing procedure

[Schnyder’90]

draw the other vertices according to the same rule
Face-counting drawing procedure
[Schnyder’90]

draw the edges as segments
For any triangulation $T$ with $n$ vertices, this procedure gives a planar straight-line drawing on the regular $(2n - 5) \times (2n - 5)$ grid.
Proof of planarity

at each inner vertex:

(hence inside the convex hull of neighbours)
Contact representations of planar graphs
**Contact configuration** = set of “shapes” that cannot overlap but can have contacts
Contact configuration = set of “shapes” that can not overlap but can have contacts

yields a planar map (when no...
General formulation

**Contact configuration** = set of “shapes” that can not overlap but can have contacts

yields a planar map (when no

**Problem**: given a set of allowed shapes, which planar maps can be realized as a contact configuration? Is such a representation unique?
Circle packing

[Koebe’36, Andreev’70, Thurston’85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius
Circle packing

[Koebe’36, Andreev’70, Thurston’85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius

**Exercise:** the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).
Circle packing

[Koebe’36, Andreev’70, Thurston’85]: every planar triangulation admits a contact representation by disks
The representation is unique if the 3 outer disks have prescribed radius

Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).
Hence one can lift to a circle packing on the sphere
Circle packing

[Koebe’36, Andreev’70, Thurston’85]: every planar triangulation admits a contact representation by disks

The representation is unique if the 3 outer disks have prescribed radius.

**Exercise:** the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).

Hence one can lift to a circle packing on the sphere.

There is a unique representation where the centre of the sphere is the barycenter of the contact points.
Axis-aligned rectangles in a box
• The rectangles form a tiling. The contact-map is the dual map
• This map is a triangulation of the 4-gon, where every 3-cycle is facial
Axis-aligned rectangles in a box

• The rectangles form a tiling. The contact-map is the dual map
• This map is a triangulation of the 4-gon, where every 3-cycle is facial

Is it possible to obtain a representation for any such triangulation?
Two partial dual Hasse diagrams

dual for vertical edges

dual for horizontal edges
Transversal structures
For $T$ a triangulation of the 4-gon, a transversal structure is a partition of the inner edges into 2 transversal Hasse diagrams characterized by local conditions:

$T$ admits a transversal structure iff every 3-cycle is facial
Face-labelling of the two Hasse diagrams

dual for vertical edges

a horizontal segment in each face

dual for horizontal edges

a vertical segment in each face
Face-labelling of the two Hasse diagrams

dual for vertical edges

a horizontal segment in each face

label the face by the $y$-coordinate of segment

$j > i$

a vertical segment in each face

label the face by the $x$-coordinate of segment

$k > \ell$

dual for horizontal edges
Face-labelling of the two Hasse diagrams

dual for vertical edges

Face-labelling of the two Hasse diagrams

dual for horizontal edges

a horizontal segment in each face

label the face by the y-coordinate of segment

j > i

vertex \( v \leftrightarrow \) rectangle \( R(v) \)

a vertical segment in each face

label the face by the x-coordinate of segment

\( \ell > k \)
Algorithm by reverse-engineering

For $T$ a triangulation of the 4-gon without separating 3-cycle

Each vertex $i$ is mapped to box $k$,

where $i$ and $j$ are adjacent vertices.
Square tilings

There is a unique tiling where every box is a square (needs no separating 4-cycle to be sure there is no degeneracy)

[Schramm’93]