Bijective Links on Planar Maps via Orientations

Éric Fusy

Dept. Math, University of British Columbia
- **Planar map** = graph drawn in the plane without edge-crossing, taken up to isotopy (continuous structure-preserving transformation)

- **Rooted map** = map + root edge

- **Motivations**: mesh compression, graph drawing + nice combinatorial properties
Families of planar maps

- Planar map
- Nonseparable map (no separating vertex)
- Triangulation
- Loopless map (no separating triangle)
- Irreducible triangulation (no separating triangle)
Families of planar maps

Planar map

Nonseparable map
(no separating vertex)

Triangulation

Irreducible triangulation
(no separating triangle)

Loopless map

separating vertex
Families of planar maps

Planar map
Nonseparable map
Triangulation
Irreducible triangulation
Loopless map

separating vertex
separating vertex
separating vertex

Triangulation
Irreducible triangulation
Enumeration of planar maps

Symbolic approach: Tutte, Brown

Bijective approach: Cori, Schaeffer, Bouttier-Di Francesco-Guitter

Planar maps

\[ \text{#(n edges)} = \frac{2 \cdot 3^n (2n)!}{(n+2)!n!} \]

Eulerian

\[ \text{#(n edges)} = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)!n!} \]

Loopless

\[ \text{#(n edges)} = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \]

Nonseparable

\[ \text{#(n edges)} = \frac{4(3n-3)!}{(n-1)!(2n)!} \]

4-regular

\[ \text{#(n vert.)} = \frac{2 \cdot 3^n (2n)!}{(n+2)!n!} \]

Bicubic

\[ \text{#(2n vert.)} = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)!n!} \]

Triangulations

\[ \text{#(n+3 vert.)} = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \]

Irreducible

\[ \text{#(n+3 vert.)} = \frac{4(3n-3)!}{(n-1)!(2n)!} \]
Enumeration of planar maps

- **Symbolic approach**: Tutte, Brown
- **Bijective approach**: Cori, Schaeffer, Bouttier-Di Francesco-Guitter

\[
\text{Planar maps} \quad \#(n \text{ edges}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}
\]

\[
\text{Eulerian} \quad \#(n \text{ edges}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}
\]

\[
\text{Loopless} \quad \#(n \text{ edges}) = \frac{2(4n+1)!}{(n+1)(3n+2)!}
\]

\[
\text{Nonseparable} \quad \#(n \text{ edges}) = \frac{4(3n-3)!}{(n-1)(2n)!}
\]

\[
\text{4-regular} \quad \#(n \text{ vert.}) = \frac{2 \cdot 3^n (2n)!}{(n+2)! n!}
\]

\[
\text{Bicubic} \quad \#(2n \text{ vert.}) = \frac{3 \cdot 2^{n-1} (2n)!}{(n+2)! n!}
\]

\[
\text{Triangulations} \quad \#(n+3 \text{ vert.}) = \frac{2(4n+1)!}{(n+1)(3n+2)!}
\]

\[
\text{Irreducible} \quad \#(n+3 \text{ vert.}) = \frac{4(3n-3)!}{(n-1)(2n)!}
\]
Enumeration of planar maps

- **Symbolic approach**: Tutte, Brown
- **Bijective approach**: Cori, Schaeffer, Bouthier-Di Francesco-Guitter

![Planar maps](image)

- 1. well known bijections (Tutte)
- 2. recursive bijection (Wormald)

**This talk:**
- New bijective construction for 3
- First bijective construction for 4
Well known bijections

1. Radial mapping (Tutte)

Planar maps $n$ edges $\rightarrow$ 4-regular $n$ vertices

2. Trinity mapping (Tutte)

Eulerian maps $n$ edges $\rightarrow$ Bicubic maps $2n$ vertices
Overview of the talk

1) Bijection nonseparable maps $\simeq$ irreducible triangulations
   + new duality relation for bipolar orientations

2) Bijection loopless maps $\simeq$ triangulations

3) Applications to random generation and encoding

-[Image of diagrams]
Bijection between nonseparable maps and irreducible triangulations
Bipolar orientations

Bipolar orientation = acyclic orientation with a unique source and a unique sink
Bipolar orientations

Bipolar orientation = acyclic orientation with a unique source and a unique sink

A map admits a bipolar orientation iff there is no separating vertex (nonseparable)
Transversal structures

Transversal structure = partition of inner edges into a red and a blue bipolar orientations that are transversal (introduced by Xin He’93)
Transversal structure = partition of inner edges into a red and a blue bipolar orientations that are transversal (introduced by Xin He'93)

A triangulation of the 4-gon admits a transversal structure iff there is no separating triangle (irreducible)
Reformulating the bijection

Nonseparable

Irreducible

?
Reformulating the bijection

Nonseparable

Irreducible

Ossona de Mendez’94

Fusy’05

No pattern

No patterns
Reformulating the bijection

Nonseparable

Ossona de Mendez’94

Irreducible

Fusy’05

No pattern

Bipolar orientations

Transversal structures

No patterns
How the bijection works

Bipolar

\( n \) edges

Bipolar intransitive

\( n + 1 \) vertices

without \( \mathbb{Z} \)

Transversal structures

\( n + 3 \) vertices

without \( \mathbb{Z} \)
How the bijection works

Bipolar $n$ edges

Bipolar intransitive
$n + 1$ vertices
without $Z$

Transversal structures
$n + 3$ vertices
without $Z$

No pattern

No pattern

No patterns
Bipolar→Bipolar intransitive

Start with a plane bipolar orientation
Bipolar→Bipolar intransitive

Double the root edge
Bipolar $\rightarrow$ Bipolar intransitive

Insert a white vertex in each edge
Bipolar $\rightarrow$ Bipolar intransitive

Triangulate the faces by red edges
Bipolar→Bipolar intransitive

face
edge
additions
Bipolar → Bipolar intransitive
A first bijection:

1) place a black vertex in each face
2) insert one black edge for each white vertex

Bipolar→Bipolar intransitive

Bipolar
$n$ edges

Bipolar intransitive
$n + 3$ vertices
without

Bipolar

– p.12/26
Bipolar→Bipolar intransitive

A first bijection:

1) place a black vertex in each face
2) insert one black edge for each white vertex

Remark: These are counted by the Baxter number:

\[ B_n = \frac{2}{n(n+1)^2} \sum_{k=0}^{n-1} \binom{n+1}{k} \binom{n+1}{k+1} \binom{n+1}{k+2} \]
Bipolar intransitive → Trans. struct.

Start with an intransitive bipolar orientation
Bipolar intransitive $\rightarrow$ Trans. struct.

Triangulate the faces by blue edges
Bipolar intransitive \rightarrow Trans. struct.
Bipolar intransitive $\rightarrow$ Trans. struct.

A second bijection:

Bipolar intransitive
$n + 1$ vertices
without

Transversal structures
$n + 3$ vertices
without
The bijection

Bipolar intransitive
\( n + 1 \) vertices
without \( \prec \)

Bipolar Transversal structures
\( n \) edges \( n + 3 \) vertices

Transversal structures
\( n + 3 \) vertices
without \( \prec \)
The bijection

- Bipolar structures with $n$ edges
- Bipolar intransitive structures with $n + 1$ vertices
- Transversal structures with $n + 3$ vertices

No pattern

- Bipolar structures without pattern
- Bipolar intransitive structures without pattern
- Transversal structures without pattern
The bijection

Bipolar structures with $n$ edges

Bipolar intransitive structures with $n + 1$ vertices

Transversal structures with $n + 3$ vertices

No pattern

No pattern

No patterns
The bijection

Bipolar $n$ edges

Bipolar intransitive $n + 1$ vertices without

Transversal structures $n + 3$ vertices without

Nonseparable $n$ edges

Irreducible $n + 3$ vertices

Bijection
Bijection between loopless maps and triangulations
Decomposing a loopless map

- Block decomposition
Decomposing a loopless map

- Block decomposition

- For rooted loopless maps:
  Nonseparable core where each corner is possibly occupied by a loopless map
Decomposing a triangulation

- Classical decomposition at separating triangles
Decomposing a triangulation

- Classical decomposition at separating triangles

4-connected triangulation
where each face is possibly occupied by a triangulation
Decomposing a triangulation

4-connected triangulation where each face is possibly occupied by a triangulation
Decomposing a triangulation

- Classical decomposition at separating triangles

- Here: the same after deleting an outer edge

4-connected triangulation where each face is possibly occupied by a triangulation

Irreducible triangulation where each face is possibly occupied by a triangulation
The decompositions are parallel

A loopless map $M$ ($|M| = \# \text{ edges}$) is
i) the vertex-map $\bullet$

or

ii) $|M| \geq 1$

Nonseparable core $C$

$+$$2|C|$ loopless maps $M_1, \ldots, M_{2|C|}$

$|M| = |C| + \sum_{i=1}^{2|C|} |M_i|$

A triangulation $T$ ($|T| = \# \text{ vert.}$ -3) is
i) the triangle-map $\triangle$

or

ii) $|T| \geq 1$

Irreducible core $I$

$+$$2|I|$ triangulations $T_1, \ldots, T_{2|I|}$

$|T| = |I| + \sum_{i=1}^{2|I|} |T_i|$
The decompositions are parallel

A loopless map $M$ ($|M| = \# \text{ edges}$) is:
  i) the vertex-map $\bullet$
  or
  ii) $|M| \geq 1$

Nonseparable core $C$

$2|C|$ loopless maps $M_1, \ldots, M_{2|C|}$

$|M| = |C| + \sum_{i=1}^{2|C|} |M_i|$

A triangulation $T$ ($|T| = \# \text{ vert.} -3$) is:
  i) the triangle-map $\triangle$
  or
  ii) $|T| \geq 1$

Irreducible core $I$

$2|I|$ triangulations $T_1, \ldots, T_{2|I|}$

$|T| = |I| + \sum_{i=1}^{2|I|} |T_i|$

Nonseparable maps $n$ edges \(\leftrightarrow\) bijection \(\rightarrow\) Irreducible triang. $n + 3$ vertices

Loopless maps $n$ edges \(\leftrightarrow\) bijection \(\rightarrow\) Triangulations $n + 3$ vertices
Results obtained so far

- Rooted nonseparable maps with $n$ edges

- Bipolar orientations

- Rooted irreducible triangulations with $n+3$ vertices

$\implies$

- Rooted loopless maps with $n$ edges

- Parallel decompositions

- Rooted triangulations with $n+3$ vertices
Results obtained so far

rooted nonseparable maps with \( n \) edges

\[ |N_n| = |I_n| \]

bipolar orientations

(rooted irreducible triangulations with \( n+3 \) vertices)

\[ |L_n| = |T_n| \]

\( \Rightarrow \)

rooted loopless maps with \( n \) edges

parallel decompositions

(rooted triangulations with \( n+3 \) vertices)
Results obtained so far

- Rooted nonseparable maps with $n$ edges
  - Bipolar orientations
  - Rooted irreducible triangulations with $n+3$ vertices
    \[ |N_n| = |I_n| = ? \]

$\implies$

- Rooted loopless maps with $n$ edges
  - Parallel decompositions
  - Rooted triangulations with $n+3$ vertices
    \[ |L_n| = |T_n| = ? \]
Counting the families
Irreducible triangulations \leftrightarrow ternary trees

Fusy’05: irreducible triangulations
Irreducible triang. ↔ ternary trees

Fusy’05: irreducible triangulations are in bijection with ternary trees
Irreducible triang. ↔ ternary trees

Fusy’05: irreducible triangulations are in bijection with ternary trees
Irreducible triang. $\leftrightarrow$ ternary trees

Fusy’05: irreducible triangulations are in bijection with ternary trees.
Triangulations $\leftrightarrow$ quaternary trees

Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05: triangulations are in bijection with quaternary trees
Triangulations $\leftrightarrow$ quaternary trees

Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05: triangulations are in bijection with quaternary trees.
Triangulations ↔ quaternary trees

Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05: triangulations are in bijection with quaternary trees
Triangulations ↔ quaternary trees

Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05: triangulations are in bijection with quaternary trees.
Triangulations $\leftrightarrow$ quaternary trees

Poulalhon-Schaeffer’03, Fusy-Poulalhon-Schaeffer’05: triangulations are in bijection with quaternary trees
Triangulations ↔ quaternary trees

Poulalhon-Schaeffer'03, Fusy-Poulalhon-Schaeffer'05: triangulations are in bijection with quaternary trees.
Enumerative Results

rooted nonseparable maps with \( n \) edges

\[ \iff \]

rooted irreducible triangulations with \( n+3 \) vertices

\[ \iff \]

rooted ternary trees with \( n-1 \) nodes

\[ \iff \]

rooted loopless maps with \( n \) edges

\[ \iff \]

rooted triangulations with \( n+3 \) vertices

\[ \iff \]

rooted quaternary trees with \( n \) nodes
Enumerative Results

- Rooted nonseparable maps with $n$ edges
  \[ |\mathcal{N}_n| = |\mathcal{T}_n| = \frac{4(3n-3)!}{(n-1)!(2n)!} \]
- Rooted irreducible triangulations with $n+3$ vertices
- Rooted ternary trees with $n-1$ nodes

\[ |\mathcal{L}_n| = |\mathcal{T}_n| = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \]
Enumerative Results

\[ |\mathcal{N}_n| = |\mathcal{T}_n| = \frac{4(3n-3)!}{(n-1)!(2n)!} \]

\[ |\mathcal{L}_n| = |\mathcal{T}_n| = \frac{2(4n+1)!}{(n+1)!(3n+2)!} \]
Applications
Algorithmic applications

- General scheme: (Schaeffer’99, Poulalhon-Schaeffer’03)

  maps ↔ trees ↔ “Dyck words”
Algorithmic applications

- **General scheme:** (Schaeffer’99, Poulalhon-Schaeffer’03)

  ![Diagram](image)

  - **Algorithmic applications**
  - **maps** ← → **trees** ← → **“Dyck words”**

  **optimal encoding**
  (mesh compression)
Algorithm
Algorithmic applications

- General scheme: (Schaeffer’99, Poulalhon-Schaeffer’03)

  - maps ⇄ trees ⇄ “Dyck words”

    - optimal encoding (mesh compression)
    - random generation (random discrete surfaces)

- Applies here to:
  - irreducible triangulations
  - triangulations
Algorithmic applications

- **General scheme:** (Schaeffer’99, Poulalhon-Schaeffer’03)

  \[\text{maps} \leftrightarrow \text{trees} \leftrightarrow \text{“Dyck words”}\]

- **optimal encoding**
  (mesh compression)

- **random generation**
  (random discrete surfaces)

- **Applies here to:**
  - irreducible triangulations
  - triangulations
  - as well as:
  - loopless maps
  - nonseparable maps