Planar maps: bijections and applications

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Geometric representation of planar maps

Various methods can be used to draw a map on the plane/sphere
Existence question

planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation
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(such as drawing is called a (planar) straight-line drawing)
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planar map (with outer face) = equivalence class of planar drawings of graphs up to continuous deformation

Question: Does there always exist an equivalent planar drawing such that all edges are drawn as segments?
(such as drawing is called a (planar) straight-line drawing)

Remark: For such a drawing to exist, the map needs to be simple
Existence proof (reduction to triangulations)

• Any simple planar map $M$ can be completed to a simple triangulation $T$
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• Any simple planar map $M$ can be completed to a simple triangulation $T$
• A straight-line drawing of $T$ yields a straight-line drawing of $M$
Existence proof (for triangulations)

First proof: induction on the number of vertices
Let $T$ be a triangulation with $n$ vertices
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$T \backslash v$ has a straight-line drawing

$\Rightarrow$

induction

$T \backslash v$ has a straight-line drawing
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Straight-line drawing algorithms

We present two classical algorithms

- Tutte’s barycentric method

- Schnyder’s face-counting algorithm
Planarity criterion for straight-line drawings

Planar

Non-planar
Planarity criterion for straight-line drawings

Theorem: a straight-line drawing is planar iff every inner vertex is inside the convex hull of its neighbours

(works for triangulations and more generally for 3-connected planar graphs)
Proof idea

- For each corner \( c \in T \) let \( \theta(c) \) be the angle of \( c \) in the drawing.
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• If convex hull condition holds, then $\Theta(v) \geq 2\pi$ for each $v$.
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Hence locally planar at each vertex (no “folding” of triangles at a vertex) ⇒ the drawing is planar.
Tutte’s barycentric method

- Outer vertices $v_1, \ldots, v_d$ are fixed at fixed positions (nailed)
- Each inner vertex is at the barycenter of its neighbours

$$x_i = \frac{1}{\Delta_i} \sum_{j \sim i} x_j \quad y_i = \frac{1}{\Delta_i} \sum_{j \sim i} y_j \quad \text{for } i \geq 4$$
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\[\Leftrightarrow \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{and} \quad \sum_{j \sim i} x_i - x_j = 0 \quad \text{for each } i \geq 4\]
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for each $i \geq 4$

- This drawing exists and is unique. It minimizes the energy

\[
\mathcal{P} = \sum_e \ell(e)^2 = \sum_{\{i,j\} \in T} (x_i - x_j)^2 + (y_i - y_j)^2
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under the constraint of fixed $x_1, \ldots, x_d, y_1, \ldots, y_d$
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- Also called spring embedding (each edge is a spring of energy $\ell(e)^2$)
Schnyder woods on triangulations

Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:
- At each inner vertex
- At the outer vertices

Yields a spanning tree in each color

[Schnyder’89]
Schnyder’s face-counting algorithm

Outer vertices: equilateral triangle
Inner vertices: barycentric placement

place $A$ at $\frac{4}{9}a_1 + \frac{2}{9}a_2 + \frac{3}{9}a_3$
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2 faces in blue area
3 faces in blue area
4 faces in red area

place \( A \) at \( \frac{4}{9}a_1 + \frac{2}{9}a_2 + \frac{3}{9}a_3 \)

straight-line drawing

sheer
grid \( (2n-5) \times (2n-5) \)

[Schnyder’90]
Proof of planarity

at each inner vertex:

(hence inside the convex hull of neighbours)
Transversal structures

For $T$ a triangulation of the 4-gon, a transversal structure is a partition of the inner edges into 2 transversal Hasse diagrams characterized by local conditions:

$T$ admits a transversal structure iff every 3-cycle is facial.
Rectangle tilings and dual triangulation
The dual map is a triangulation of the 4-gon, where every 3-cycle is facial.
Rectangle tilings and dual triangulation

The dual is naturally endowed with a transversal structure

dual for vertical edges

dual for horizontal edges
Face-labelling of the two Hasse diagrams

dual for vertical edges

da dual for horizontal edges

a horizontal segment in each face

a vertical segment in each face
Face-labelling of the two Hasse diagrams

dual for vertical edges

dual for horizontal edges

a horizontal segment in each face

a vertical segment in each face

label the face by the $y$-coordinate of segment

label the face by the $x$-coordinate of segment

$j > i$

$\ell > k$
Face-labelling of the two Hasse diagrams

dual for vertical edges

dual for horizontal edges

a horizontal segment in each face

label the face by the $y$-coordinate of segment

vertex $v \leftrightarrow$ rectangle $R(v)$

a vertical segment in each face

label the face by the $x$-coordinate of segment

bounding $x, y$-coordinates given by labels

$j > i$

$\ell > k$
Algorithm by reverse-engineering

For $T$ a triangulation of the 4-gon without separating 3-cycle

[Kant, He’92]
Algorithm by reverse-engineering

For $T$ a triangulation of the 4-gon without separating 3-cycle

Each vertex $\rightarrow$ box

Where

[Kant, He’92]
Rectangle tilings and electrical networks

other way of associating a planar map to a rectangle tiling

nice way to visualize Kirchhoff’s laws

\[ R_k: \text{aspect ratio of a rectangle} \leftrightarrow \text{resistance of corresponding link in the network} \]
Rectangle tilings and electrical networks

other way of associating a planar map to a rectangle tiling

nice way to visualize Kirchhoff’s laws

Rk: aspect ratio of a rectangle $\leftrightarrow$ resistance of corresponding link in the network

Given a network with resistances $= 1$

one gets a square tiling representation

by solving the Kirchhoff’s laws

$\text{cf ‘squaring the square’}$
Square tilings dual to triangulations

Question: Given $T$ a triangulation of the 4-gon, does there always exist a square tiling whose dual is $T$?
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Yes! up to allowing for degeneracies (empty squares)
solution via computing the ‘optimal metric’ of $T$
(no known algorithm by solving linear equation systems)
Circle packing

[Koebe’36, Andreev’70, Thurston’85]: every planar triangulation admits a contact representation by disks. The representation is unique if the 3 outer disks have prescribed radius.
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Exercise: the stereographic projection maps circles to circles (considering lines as circle of radius $+\infty$).
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Hence one can lift to a circle packing on the sphere.
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Hence one can lift to a circle packing on the sphere

There is a unique representation where the centre of the sphere is the barycenter of the contact points
Contact representations with prescribed shapes

Generalized statement:

for any triangulation $T$ and a prescribed convex shape for each vertex there exists a contact representation of $T$

(possibility of degeneracies if shapes are not smooth)

Example (Eppstein’s blog post)

isocahedron