Planar maps: bijections and applications

Éric Fusy (CNRS/LIX)

AEC summer school, Hagenberg, 2018
Universality phenomena for maps

For ‘any’ standard family $\mathcal{M} = \bigcup_n \mathcal{M}_n$ of rooted maps
($p$-angulations, loopless, 2-connected, 3-connected, etc.)

- $m_n = \text{Card}(\mathcal{M}_n)$ satisfies $m_n \sim c \gamma^n n^{-5/2}$ for some constants $c, \gamma$
Universality phenomena for maps

For ‘any’ standard family $\mathcal{M} = \bigcup_n \mathcal{M}_n$ of rooted maps

($p$-angulations, loopless, 2-connected, 3-connected, etc.)

- $m_n = \text{Card}(\mathcal{M}_n)$ satisfies $m_n \sim c \gamma^n n^{-5/2}$ for some constants $c, \gamma$

- scaling limit point of view:
  for $M_n$ a random map in $\mathcal{M}_n$ and $v_1, v_2$ two random vertices in $M_n$
  let $X_n = \text{distance}(v_1, v_2)$

Then $X_n / n^{1/4} \to \text{universal proba. dist.}$ & $(M_n, d_{n^{1/4}}) \to \text{Brownian map}$
Universality phenomena for maps
For ‘any’ standard family $\mathcal{M} = \bigcup_n \mathcal{M}_n$ of rooted maps
($p$-angulations, loopless, 2-connected, 3-connected, etc.)

- $m_n = \text{Card}(\mathcal{M}_n)$ satisfies $m_n \sim c\gamma^n n^{-5/2}$ for some constants $c, \gamma$

- scaling limit point of view:
  for $M_n$ a random map in $\mathcal{M}_n$ and $v_1, v_2$ two random vertices in $M_n$
  let $X_n = \text{distance}(v_1, v_2)$

  Then $\frac{X_n}{n^{1/4}} \to$ universal proba. dist. & $(M_n, \frac{d}{n^{1/4}}) \to$ Brownian map

- local limit point of view
  let $Y_n^{(r)} = \#(\text{vertices at distance } \leq r \text{ from root-vertex in } M_n)$
  let $B^{(r)} := \lim_{n \to \infty} \mathbb{E}(Y_n^{(r)})$

  Then $B^{(r)} \sim \kappa \cdot r^4$ as $r \to \infty$
Looking for other universality classes

Structured planar map = pair \((M, X)\), with \(M\) a rooted map and \(X\) a combinatorial structure on \(M\).

We can consider some natural families \(\mathcal{S} = \bigcup_n S_n\) of structured maps:

- spanning tree
- eulerian orientation
- Schnyder wood
- bipolar orientation
Watabiki predictions

If a model of maps gives asymptotic behaviours of the form \( \kappa \gamma^n n^{-\alpha} \) then the central charge of the model is \( c = -\frac{(3\alpha - 5)(2\alpha - 5)}{\alpha - 1} \)

prediction: \( B(r) \sim \text{constant} \times r^\beta \) with \( \beta = 2\frac{\sqrt{25 - c} + \sqrt{49 - c}}{\sqrt{25 - c} + \sqrt{1 - c}} \)

<table>
<thead>
<tr>
<th></th>
<th>( \alpha )</th>
<th>( c )</th>
<th>( \beta )</th>
<th>( 1/\beta )</th>
</tr>
</thead>
<tbody>
<tr>
<td>no structure</td>
<td>5/2</td>
<td>0</td>
<td>4</td>
<td>0.25</td>
</tr>
<tr>
<td>spanning tree</td>
<td>3</td>
<td>-2</td>
<td>( \frac{3+\sqrt{17}}{2} )</td>
<td>( \approx 0.28 )</td>
</tr>
<tr>
<td>Bipolar ori.</td>
<td>4</td>
<td>-7</td>
<td>( \frac{4+2\sqrt{7}}{3} )</td>
<td>( \approx 0.32 )</td>
</tr>
<tr>
<td>Schnyder wood</td>
<td>5 ( \frac{25}{2} )</td>
<td>( \frac{5+\sqrt{41}}{4} )</td>
<td>( \approx 0.35 )</td>
<td></td>
</tr>
</tbody>
</table>

upper/lower bounds for \( \beta \) (consistent with prediction) [Gwynne, Holden, Sun’17]
Plan for today

- Review of bijective links (and discuss some connections/applications)
  - Structured maps
  - Lattice walks in quadrant (or in a 2d cone)

Explains asymptotic behaviour, cf [Denisov-Wachtel’2015]

\[ a_n = \# \text{ walks of length } n \text{ in } K \text{ with fixed endpoints} \]

Then \( a_n \sim \kappa \gamma^n n^{-p-1} \), with \( p = \frac{\pi}{\theta} \)

\( \theta = \pi/2 \) for spanning trees, \( \pi/3 \) for bipolar orientations, \( \pi/4 \) for Schnyder woods
Tree-rooted maps

(map + spanning tree)
Contour encoding of a tree-rooted map [Mullin’67]
Contour encoding of a tree-rooted map \cite{Mullin'67}

contour encoding of the tree $T$:

$a a a a a a a a a a a$

Dyck word
Contour encoding of a tree-rooted map [Mullin’67]

Contour encoding of the tree $T$:

$$\overline{a\ a\ a\ a\ a\ a\ a\ a\ a\ a}$$

Dyck word

Enriched contour encoding:

$$\overline{a\ b\ b\ a\ a\ b\ b\ a\ a\ a\ a\ b\ b\ b\ a\ a\ b\ a}$$

shuffle of two Dyck words
Contour encoding of a tree-rooted map [Mullin’67]

Contour encoding of the tree $T$:

```
a a a a a a a a a a
```

Dyck word

Enriched contour encoding:

```
a b b a a b b a a a a b b b a a b a
```

shuffle of two Dyck words

Rk: red word is the contour word for the dual spanning tree
Contour encoding of a tree-rooted map

countour encoding of the tree $T$:

\[
\text{Dyck word}
\]

enriched contour encoding:

\[
\text{shuffle of two Dyck words}
\]

\[
\textbf{Rk: red word is the contour word for the dual spanning tree}
\]

excursion in quadrant, with steps
Contour encoding of a tree-rooted map [Mullin’67]

Contour encoding of the tree $T$:

```
  a a a a a a a a a a
```

Dyck word

Enriched contour encoding:

```
  a b b a a b b a a a a a b b b a a b a a b a
```

shuffle of two Dyck words

Rk: red word is the contour word for the dual spanning tree

$t_n = \# \text{ tree-rooted maps with } n \text{ edges satisfies}$

$$t_n = \sum_{k=0}^{n} \binom{2n}{2k} \text{Cat}_k \text{Cat}_{n-k}$$
Contour encoding of a tree-rooted map [Mullin’67]

Contour encoding of the tree \( T \):

\[
\text{enriched contour encoding:}
\]

\[
\text{shuffle of two Dyck words}
\]

\[
\text{Rk: red word is the contour word}
\]

\[
t_n = \# \text{ tree-rooted maps with } n \text{ edges satisfies}
\]

\[
t_n = \sum_{k=0}^{n} \binom{2n}{2k} \text{Cat}_k \text{Cat}_{n-k} = \text{Cat}_n \text{Cat}_{n+1}
\]

\[
\text{cf } \binom{s + t}{n} = \sum_{k=0}^{n} \binom{s}{k} \binom{t}{n-k}
\]
Contour encoding of a tree-rooted map [Mullin’67]

Contour encoding of the tree $T$:

$a$ $a$ $a$ $a$ $a$ $a$ $a$ $a$ $a$

Dyck word

enriched contour encoding:

$a$ $b$ $b$ $a$ $a$ $b$ $b$ $a$ $a$ $a$ $a$ $b$ $b$ $b$ $a$ $a$ $a$ $b$ $a$

shuffle of two Dyck words

Rk: red word is the contour word for the dual spanning tree

$t_n = \# \text{ tree-rooted maps with } n \text{ edges satisfies}$

$t_n = \sum_{k=0}^{n} \binom{2n}{2k} \text{Cat}_k \text{Cat}_{n-k} = \text{Cat}_n \text{Cat}_{n+1}$

$\text{cf } \left( \begin{array}{c} s + t \\ n \end{array} \right) = \sum_{k=0}^{n} \binom{s}{k} \binom{t}{n-k}$

Hence $t_n \sim \frac{4}{\pi} 16^n n^{-3}$ with $n^{-3}$ ‘universal’ for tree-rooted maps (cf exercise)
Direct proof that $t_n = \text{Cat}_n \text{Cat}_{n+1}$

- First step:

  - Tree-rooted map
  - Local rule
  - Oriented rooted map (root-accessible & no ccw cycle)
Direct proof that $t_n = \text{Cat}_n \text{Cat}_{n+1}$

• First step:
  - tree-rooted map
  - local rule (root-accessible & no ccw cycle)

• Second step:
  - blue tree has $n + 1$ edges
  - red tree has $n$ edges
Direct proof that $t_n = \text{Cat}_n \text{Cat}_{n+1}$ [Bernardi’07]

- First step:
  - tree-rooted map
  - local rule
  - oriented rooted map (root-accessible & no ccw cycle)

- Second step:
  - local rule
  - blue tree has $n + 1$ edges
  - red tree has $n$ edges

(the bijection $\Phi$ used previously this week is closely related to 2nd step)
Schnyder woods
Schnyder woods on triangulations

Schnyder wood = choice of a direction and color (red, green, or blue) for each inner edge, such that:

Local conditions:
- at each inner vertex
- at the outer vertices

yields a spanning tree in each color

[Schnyder’89]
Equivalence with 3-orientations

can propagate the colors (uniquely) from any 3-orientation

Schnyder wood

3-orientation

outdegree 3 at inner vertices
outdegree 0 at outer vertices

conditions
Bijective encoding of Schnyder woods

[Bernardi, Bonichon’09]

Schnyder woods on \( n + 3 \) vertices

non-intersecting pairs of Dyck paths of lengths \( 2n \)
Let $s_n =$ total number of Schnyder woods over triangulations with $n + 3$ vertices

- **Exact formula:**
  $$s_n = \text{Cat}_n \text{Cat}_{n+2} - \text{Cat}_{n+1} \text{Cat}_{n+1} = \frac{6(2n)! (2n + 2)!}{n! (n + 1)! (n + 2)! (n + 3)!}$$

- **Asymptotic formula:**
  $$s_n \sim \frac{24}{\pi} 16^n n^{-5}$$
The Tamari lattice $\mathcal{L}_n$ is the partial order on Dyck paths of length $2n$ for the covering relation (amounts to right rotation in corresponding binary trees) for the Tamari lattice for $n = 4$. 
The Tamari lattice

The Tamari lattice $\mathcal{L}_n$ is the partial order on Dyck paths of length $2n$ for the covering relation

$\leq$ (amounts to right rotation in corresponding binary trees)

the Tamari lattice for $n = 4$ it has 68 intervals

Interval in $\mathcal{T}_n = \text{pair } (t, t')$ such that $t \leq t'$
The Tamari lattice \( \mathcal{L}_n \) is the partial order on Dyck paths of length \( 2n \) for the covering relation

\[
\text{(amounts to right rotation in corresponding binary trees)}
\]

the Tamari lattice for \( n = 4 \)

it has 68 intervals

Interval in \( \mathcal{T}_n = \text{pair } (t, t') \) such that \( t \leq t' \)

**Theorem** [Chapoton’06]: there are \( \frac{2}{n(n+1)} \binom{4n+1}{n-1} \) intervals in \( \mathcal{L}_n \)

**Rk**: This is also the number of simple triangulations with \( n + 3 \) vertices
Characterization of intervals by length-vectors
Characterization of intervals by length-vectors

Rk: if $t \leq t'$ in $\mathcal{L}_n$, then $t$ is below $t'$
Characterization of intervals by length-vectors

Rk: if $t \leq t'$ in $\mathcal{L}_n$, then $t$ is below $t'$

the converse is not true!
Characterization of intervals by length-vectors

Rk: if $t \leq t'$ in $\mathcal{L}_n$, then $t$ is below $t'$
the converse is not true!

Q: How to characterize pairs forming an interval in $\mathcal{L}_n$?
Characterization of intervals by length-vectors

Rk: if $t \leq t'$ in $\mathcal{L}_n$, then $t$ is below $t'$
the converse is not true!

Q: How to characterize pairs forming an interval in $\mathcal{L}_n$?

**Length-vector** $L_D$ of $D$:

$L_D = (4, 1, 2, 1)$
Characterization of intervals by length-vectors

**Rk:** if \( t \leq t' \) in \( \mathcal{L}_n \), then \( t \) is below \( t' \)
the converse is not true!

**Q:** How to characterize pairs forming an interval in \( \mathcal{L}_n \)?

**Length-vector** \( L_D \) of \( D \):

\[
L_D = (4, 1, 2, 1)
\]

**Lem:** \( D \leq D' \) in \( \mathcal{L}_n \) iff \( L_D \leq L_{D'} \)
Specializing the bijection for Schnyder woods

[Bernardi, Bonichon’09]

**Property:** A triangulation has a unique Schnyder wood with no cw cycle.

**Property:** A non-crossing pair of Dyck paths is an interval in $\mathcal{L}_n$ iff the corresponding Schnyder wood has no cw cycle.

---

![Diagram](image-url)

- The first diagram shows a triangulation with a cw cycle.
- The second diagram shows a triangulation without a cw cycle.
- The corresponding Dyck paths and their length-vectors are shown on the right.

- For the triangulation with a cw cycle, the length-vectors are $4 \ 1 \ 2 \ 1$.
- For the triangulation without a cw cycle, the length-vectors are $2 \ 1 \ 2 \ 1$.

---

The length-vectors for the Dyck paths are:

- $4 \ 1 \ 2 \ 1$ for the triangulation with a cw cycle.
- $2 \ 1 \ 2 \ 1$ for the triangulation without a cw cycle.
Specializing the bijection for Schnyder woods

[Bernardi, Bonichon’09]

**Property:** A triangulation has a unique Schnyder wood with no cw cycle

**Property:** A non-crossing pair of Dyck paths is an interval in $\mathcal{L}_n$ iff the corresponding Schnyder wood has no cw cycle

⇒ intervals in $\mathcal{L}_n$ are in bijection with simple triangulations with $n + 3$ vertices
Bipolar orientations
Definition

Let $M$ be a planar map with two marked outer vertices $S, N$

**Bipolar orientation** of $M = \text{acyclic orientation of } M$

with $S$ the unique source and $N$ the unique sink

---

**local conditions**

- inner vertex
- inner face
Enumeration by edges

The number $b_n$ of bipolar orientations with $n - 1$ edges is

$$b_n = \frac{2}{n^2(n+1)} \sum_{k=0}^{n-1} \binom{n+1}{r-1} \binom{n+1}{r} \binom{n+1}{r+1}$$

Baxter numbers

cf bijections

$k + 2$ vertices

$n - k$ faces

+ Gessel-Viennot lemma

$b_n$ also counts many other classes (pattern-avoiding permutations, square tilings, etc.)
Enumeration by edges

The number $b_n$ of bipolar orientations with $n - 1$ edges is

$$b_n = \frac{2}{n^2(n + 1)} \sum_{k=0}^{n-1} \binom{n + 1}{r - 1} \binom{n + 1}{r} \binom{n + 1}{r + 1}$$

Baxter numbers

cf bijections

$k + 2$ vertices
$n - k$ faces

$+$ Gessel-Viennot lemma

$b_n$ also counts many other classes (pattern-avoiding permutations, square tilings, etc.)

Asymptotics: $b_n \sim \frac{25}{\pi \sqrt{3}} 8^n n^{-4}$
**Enumeration by edges**

The number $b_n$ of bipolar orientations with $n - 1$ edges is

\[
b_n = \frac{2}{n^2(n + 1)} \sum_{k=0}^{n-1} \binom{n+1}{r-1} \binom{n+1}{r} \binom{n+1}{r+1}
\]

Baxter numbers

$\sum_{k=0}^{n-1} \binom{n+1}{r-1} \binom{n+1}{r} \binom{n+1}{r+1}$

*cf bijections*

$k + 2$ vertices

$n - k$ faces

$+$ Gessel-Viennot lemma

$b_n$ also counts many other classes (pattern-avoiding permutations, square tilings, etc.)

**Asymptotics:** $b_n \sim \frac{2^5}{\pi \sqrt{3}} 8^n n^{-4}$

We show a bijection by Kenyon, Miller, Sheffield and Wilson

with lattice walks in quadrant ($+\text{control on face degrees}$)

explains universality of $n^{-4}$ for bipolar ori. $+$ appli. to lattice walk enumeration
The Kenyon et al. bijection

Tandem walks

step set

$\begin{align*}
\text{level 1} \\
\text{level 2} \\
\text{level 3}
\end{align*}$

$y$

$x$

$\bullet \text{SE}$

Tandem walks in quadrant (start & end at 0)

bijection

bipolar orientations inside bi-gon

length $n$

$n + 1$ edges

step level $r$

inner face of degree $r + 2$

SE step

vertex $\notin \{S, N\}$
The Kenyon et al. bijection
Consequences of the bijection

- The linear mapping that sends \( \begin{vmatrix} \pi/2 \\ \pi/3 \end{vmatrix} \) to turns the covariance matrix of step-set to \( I_2 \)

\[ \Rightarrow \text{universality of the subexponential order } n^{-4} \text{ for bipolar orientations} \]
Consequences of the bijection

- The linear mapping that sends $\pi/2$ to $\pi/3$ turns the covariance matrix of step-set to $I_2$
  
  $\Rightarrow$ universality of the subexponential order $n^{-4}$ for bipolar orientations

- Let $Q(t; z_1, z_2, \ldots)$ be the GF of tandem walks in the quadrant (starting at the origin, free endpoint)
  with $t$ for the length, $z_r$ for steps of level $r$

  Then $Q(t; z_1, z_2, \ldots)$ also counts tandem walks in upper half-plane $\{y \geq 0\}$ (starting at 0, ending at $\{y = 0\}$)
Consequences of the bijection

- The linear mapping that sends $\pi/2$ to $\pi/3$

  turns the covariance matrix of step-set to $I_2$

  $\Rightarrow$ universality of the subexponential order $n^{-4}$ for bipolar orientations

- Let $Q(t; z_1, z_2, \ldots)$ be the GF of tandem walks in the quadrant
  (starting at the origin, free endpoint)
  
  with $t$ for the length, $z_r$ for steps of level $r$

Then $Q(t; z_1, z_2, \ldots)$ also counts tandem walks in upper half-plane $\{y \geq 0\}$

  (starting at 0, ending at $\{y = 0\}$)

  $\Rightarrow Y \equiv tQ(t)$ is given by

$$Y = t \cdot (1 + w_0Y + w_1Y^2 + w_2Y^3 + \cdots)$$

  where $w_i = z_i + z_{i+1} + z_{i+2} + \cdots$
Consequences of the bijection

- The linear mapping that sends $\frac{\pi}{2}$ to $\frac{\pi}{3}$ turns the covariance matrix of step-set to $I_2$

  $$\Rightarrow$$ universality of the subexponential order $n^{-4}$ for bipolar orientations

- Let $Q(t; z_1, z_2, \ldots)$ be the GF of tandem walks in the quadrant
  (starting at the origin, free endpoint)

  with $t$ for the length, $z_r$ for steps of level $r$

Then $Q(t; z_1, z_2, \ldots)$ also counts tandem walks in upper half-plane $\{y \geq 0\}$

  (starting at 0, ending at $\{y = 0\}$)

  $$\Rightarrow Y \equiv t Q(t) \text{ is given by } Y = t \cdot (1 + w_0 Y + w_1 Y^2 + w_2 Y^3 + \cdots)$$

  where $w_i = z_i + z_{i+1} + z_{i+2} + \cdots$

proof using the extended version of the bijection

(also possible by kernel method for walks with large steps [Bostan, Bousquet-Mélou, Melczer’18])