Planar maps: bijections and applications

Éric Fusy (CNRS/LIX)
Rooted maps

A map is rooted by marking and orienting an edge.

Rooted maps are combinatorially easier than maps
(no symmetry issue, root gives starting point for recursive decomposition)

The 2 rooted maps with one edge

The 9 rooted maps with two edges
Counting rooted maps

Let \( a_n \) be the number of rooted maps with \( n \) edges.

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\[
\frac{2 \cdot 3^n}{(n + 1)(n + 2)} \binom{2n}{n}
\]

**Theorem:** (Tutte’63)
Counting rooted maps

Let $a_n$ be the number of rooted maps with $n$ edges

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Theorem: (Tutte’63)

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\frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n}
$$

Not an isolated case:

- Triangulations ($2n$ faces)
  
  Loopless: \( \frac{2^n}{(n+1)(2n+1)} \binom{3n}{n} \)
  
  Simple: \( \frac{1}{n(2n-1)} \binom{4n-2}{n-1} \)

- Quadrangulations ($n$ faces)
  
  General: \( \frac{2 \cdot 3^n}{(n+1)(n+2)} \binom{2n}{n} \)
  
  Simple: \( \frac{2}{n(n+1)} \binom{3n}{n-1} \)
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Not an isolated case:

- **Triangulations** (\( 2n \) faces)
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  \text{Loopless: } \frac{2^n}{(n + 1)(2n + 1)} \binom{3n}{n}
  \]
  \[
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  \]

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  \]
  \[
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  \]
Bijective aspects of planar maps
Motivations for bijections

• efficient manipulation of maps (random generation algo.)

• key ingredient to study distances (diameter,...) in random maps
  - typical distances of order $n^{1/4}$ ($\neq n^{1/2}$ in random trees)
  - random map $M$ with $n$ edges = random discrete metric space $(M, d)$

Theo: [Le Gall, Miermont’13]

$(M, \frac{1}{n^{1/4}} d)$ converges to a continuum random metric space
called the Brownian map

(analog for maps of the Continuous Random Tree)
Pointed quadrangulations, geodesic labelling

Pointed quadrangulation = quadrangulation with a marked vertex $v_0$

Geodesic labelling with respect to $v_0$: $\ell(v) = \text{dist}(v_0, v)$

Rk: two types of faces
Well-labelled trees

Well-labelled tree $= \text{plane tree where}$
- each vertex $v$ has a label $\ell(v) \in \mathbb{Z}$
- each edge $e = \{u, v\}$ satisfies $|\ell(u) - \ell(v)| \leq 1$
The Schaeffer bijection [Schaeffer’99], also [Cori-Vauquelin’81]

Pointed quadrangulation $\Rightarrow$ well-labelled tree with min-label=1

$n$ faces

$n$ edges

Local rule in each face:
Proof that it gives a tree

\[ Q \]

\[ n \text{ faces} \]
\[ n + 2 \text{ vertices} \]
Proof that it gives a tree

$Q$

$n$ faces
$n + 2$ vertices

$T$

$n$ edges
$n + 1$ vertices
Proof that it gives a tree

\[ Q \]

- \( n \) faces
- \( n + 2 \) vertices

\[ T \]

- \( n \) edges
- \( n + 1 \) vertices

Assume that \( T \) has a cycle \( C \)
Proof that it gives a tree

Assume that $T$ has a cycle $C$ where $C$ is the smallest label on $C$.

$n$ faces
$n + 2$ vertices

$n$ edges
$n + 1$ vertices
Proof that it gives a tree

Assume that $T$ has a cycle $C$

$n$ faces
$n + 2$ vertices
$n + 1$ vertices

$n$ edges

smallest label on $C$
Proof that it gives a tree

Assume that $T$ has a cycle $C$

$n$ faces
$n + 2$ vertices

$n$ edges
$n + 1$ vertices

smallest label on $C$
Proof that it gives a tree

Assume that $T$ has a cycle $C$.

The smallest label on $C$ is less than $i$.

Contradiction.
Rightmost geodesic paths

situation at a corner of the tree
Rightmost geodesic paths

situation at a corner of the tree
Rightmost geodesic paths

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implies property

\[ \geq i \]
The inverse construction [Schaeffer’99], also [Cori-Vauquelin’81]

From a well-labelled tree to a pointed quadrangulation
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2) connect each leg of label $i \geq 2$ to the next corner of label $i-1$ in ccw order around the tree.
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From a well-labelled tree to a pointed quadrangulation

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4) erase the tree-edges
The Schaeffer bijection \[\text{[Schaeffer'99], also [Cori-Vauquelin'81]}\]

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4) erase the tree-edges

recover the original pointed quadrangulation
The effect of marking an edge

Local rule in each face:

marked edge

marked half-edge
Bijective proof of counting formula

Let \( q_n = \#(\text{rooted quadrangulations with } n \text{ faces}) \)

We want to show (bijectively) that

\[
q_n = \frac{2 \cdot 3^n}{(n+2)(n+1)} \binom{2n}{n} z^n
\]

**Rk:** \( q_n \times (n+2) = \# \text{ rooted quadrangulations with } n \text{ faces} + \text{ marked vertex} \)

Hence if \( b_n := \# \text{ quadrangulations with } n \text{ faces} + \text{ marked edge} + \text{ marked vertex} \)

then \( b_n = \frac{n+2}{2} q_n \)

Hence proving formula for \( q_n \) amounts to proving \( b_n = 3^n \text{Cat}_n \)
Bijective proof of counting formula

Schaeffer’s bijection ⇒ $b_n = \#$(rooted well-labelled trees with $n$ edges)
Bijective proof of counting formula

Schaeffer’s bijection $\Rightarrow b_n = \#$(rooted well-labelled trees with $n$ edges)

$b_n = 3^n \text{Cat}_n = 3^n \frac{(2n)!}{n!(n+1)!}$
The BDG bijection for pointed bipartite maps
[Bouttier, Di Francesco, Guitter'04]
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Label vertices by distance from the marked vertex.
The BDG bijection for pointed bipartite maps

[Bouttier, Di Francesco, Guitter'04]

Construction of a labeled mobile

(i) Add a black vertex in each face
Construction of a labeled mobile

(i) Add a black vertex in each face

(ii) Each map-edge gives a mobile-edge using the local rule

\[
\begin{align*}
  &i - 1 \\
  &\quad \downarrow \\
  &\quad \quad \text{black vertex}
\end{align*}
\]

\[
\begin{align*}
  &i \\
  &\quad \downarrow \\
  &\quad \quad \text{black vertex}
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The BDG bijection for pointed bipartite maps
[Bouttier, Di Francesco, Guitter’04]

Conditions:
(i) \( \exists \) vertex of label 1
(ii) \( j \leq i+1 \)

remove the map-edges and the marked vertex \( 0 \)
The BDG bijection for pointed bipartite maps
[Bouttier, Di Francesco, Guitter’04]

Local rule

Conditions:
(i) ∃ vertex of label 1
(ii) \( j \leq i+1 \)

Theorem: The mapping is a **bijection**.

face of degree \( 2i \) ←→ black vertex of degree \( i \)
Rewriting labelled mobiles as trees with arrows

Conditions:

(i) \( \exists \) vertex of label 1

(ii) \( \delta = i - j \geq -1 \)

Condition:

each black vertex has as many buds as neighbors
Enumerative consequence

Tutte’s slicings formula (1962):

Let \( B[n_1, n_2, \ldots, n_k] \) be the number of rooted bipartite maps with \( n_i \) faces of degree \( 2i \) for \( i \in [1..k] \). Then

\[
B[n_1, \ldots, n_k] = 2 \frac{e!}{v!} \prod_{i=1}^{k} \frac{1}{n_i!} \left( \frac{2i - 1}{i - 1} \right)^{n_i}
\]

where \( e = \# \text{edges} = \sum_i i n_i \) and \( v = \# \text{vertices} = e - k + 2 \)

('contains' formula for rooted quadrangulations, \( n_2 = n, n_i = 0 \) for \( i \neq 2 \))
Reformulation of bijection using orientations

Distance-labeling

Geodesic orientation

Local rule

\[ \delta = i - j \geq -1 \]

\[ \delta + 1 \text{ buds} \]
**Definition of blossoming mobiles**

- **Blossoming mobile** = bipartite tree (black/white vertices) where each corner at a black vertex carries $i \geq 0$ buds

  \[
  \text{excess} = \text{number of edges} - \text{number of buds}
  \]

  A blossoming mobile of excess $-2$
Definition of blossoming mobiles

- **Blossoming mobile** = bipartite tree (black/white vertices)
  where each corner at a black vertex carries \( i \geq 0 \) buds

\[
\text{excess} = \text{number of edges} - \text{number of buds}
\]

- A blossoming mobile is called **balanced** iff each black vertex has as many buds as neighbors

\[
\text{Rk: implies that the excess is } 0
\]
Summary of the reformulation

Condition:
Each black vertex has as many buds as neighbors

Theorem: The mapping is a bijection between pointed bipartite maps and balanced blossoming mobiles

face of degree $2i$ $\leftrightarrow$ black vertex of degree $2i$
Summary of the reformulation

Theorem: The mapping is a bijection between pointed bipartite maps and balanced blossoming mobiles

\[
\text{face of degree } 2i \leftrightarrow \text{black vertex of degree } 2i
\]

(Other bijection by Schaeffer'97 in the dual setting of eulerian maps)

Condition:
Each black vertex has as many buds as neighbors

Local rule:

[Diagram of a local rule with a black vertex and its neighbors]
Extension for pointed orientations with no ccw cycle

- More generally, we **obtain a blossoming mobile** (of excess 0) if we start from a vertex-pointed orientation such that:
  - the marked vertex \( v_0 \) is a **“source”** (no incoming edge)
  - every vertex is **accessible** from \( v_0 \) by a directed path
  - **there is no ccw cycle** (with \( v_0 \in \text{outer face} \))
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**Theorem**: Let $\mathcal{O}_0$ be this family of orientations, then the correspondence is a bijection with mobiles of excess 0
Proof that it gives a tree

Start from an oriented map $M \in O_0$ and apply the local rule

Let $G$ be the graph of red edges and their incident vertices
Proof that it gives a tree

Start from an oriented map \( M \in \mathcal{O}_0 \) and apply the local rule

Let \( G \) be the graph of red edges and their incident vertices

\( G \) has \( |V_M| - 1 \), white vertices, \( |F_M| \) black vertices, and \( |E_M| \) edges
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Euler relation: $|E_M| = |V_M| + |F_M| - 2$

$\Rightarrow G$ has one more vertices than edges

hence $G$ is a tree iff $G$ is acyclic
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$v_0$
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Assume $G$ has a cycle:

$v_0 \circlearrowleft \quad e_1$
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\[ e_1 \quad \ldots \quad e_2 \quad v_0 \]
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Assume $G$ has a cycle:

![Diagram](image_url)
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**Euler relation:** \( |E_M| = |V_M| + |F_M| - 2 \)

\[ \Rightarrow G \text{ has one more vertices than edges} \]

hence \( G \) is a tree iff \( G \) is acyclic

Assume \( G \) has a cycle :

\[ v_0 \]

\[ e_1, e_2, e_3 \]
Proof that it gives a tree

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Proof that it gives a tree

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**Euler relation:** \(|E_M| = |V_M| + |F_M| - 2\)

\(\Rightarrow\) \( G \) has one more vertices than edges

Hence \( G \) is a tree iff \( G \) is acyclic.

Assume \( G \) has a cycle:

\[ \begin{align*}
  v_0 & \quad e_1 \quad e_2 \quad e_3 \quad e_4
\end{align*} \]
**Proof that it gives a tree**

Start from an oriented map $M \in \mathcal{O}_0$ and apply the local rule.

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hence $G$ is a tree iff $G$ is acyclic.

Assume $G$ has a cycle:

prisoner ccw cycle

$\Rightarrow$ contradiction
Extension for mobiles of excess $\leq 0$
More generally the “source” can be a $d$-gonal source as the outer face for any $d \geq 0$

Example for $d = 3$

For $d > 0$, we take the $d$-gonal source as the outer face

More generally the “source” can be a $d$-gon, for any $d \geq 0$
Extension for mobiles of excess \( \leq 0 \)
More generally the “source” can be a \( d \)-gonal, for any \( d \geq 0 \)
Example for \( d = 3 \)

For \( d > 0 \), we take the \( d \)-gonal source as the outer face

Let \( \mathcal{O} \) be the family of these orientations, still with the conditions
- the \( d \)-gonal source has no ingoing edge
- accessibility of every vertex from the source
- no ccw cycle
Theorem [Bernardi-F’10]: $\Phi$ is a bijection between $\mathcal{O}$ and blossoming mobiles of $\leq 0$ excess. Moreover,

- degree of external face $\leftrightarrow$ excess
- degree of internal faces $\leftrightarrow$ degree of black vertices
- indegree of internal vertices $\leftrightarrow$ degree of white vertices

cf [Bernardi’07], [Bernardi-Chapuy’10]
Extension for mobiles of excess $\leq 0$

- Inverse mapping (tree $\rightarrow$ cactus $\rightarrow$ closure operations)
Scheme for a general bijective strategy

1) Map family $\mathcal{C}$ identifies with a subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees
Scheme for a general bijective strategy

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**Example:** $\mathcal{C} =$ Family of simple triangulations

$\mathcal{C} \simeq$ subfamily $\mathcal{O}_C$ of $\mathcal{O}$ with
   - Face-degree $= 3$
   - Vertex-indegree $= 3$
**Scheme for a general bijective strategy**

1) Map family $\mathcal{C}$ identifies with a **subfamily** $\mathcal{O}_C$ of $\mathcal{O}$ with conditions on:

- Face degrees
- Vertex indegrees

**Example:** $\mathcal{C} = \text{Family of simple triangulations}$

\[ \mathcal{C} \simeq \text{subfamily } \mathcal{O}_C \text{ of } \mathcal{O} \text{ with:} 
\begin{align*}
&\text{• Face-degree } = 3 \\
&\text{• Vertex-indegree } = 3
\end{align*} \]

2) **Specialize** the ‘meta bijection’ $\Phi$ to the subfamily $\mathcal{O}_C$

\[ \text{degree of internal faces } \leftrightarrow \text{degree of black vertices} \]
\[ \text{indegree of internal vertices } \leftrightarrow \text{degree of white vertices} \]
\( \alpha \)-orientations

Let \( G = (V, E) \) be a graph
Let \( \alpha \) be a function from \( V \) to \( \mathbb{N} \)

\[
\begin{array}{c}
\alpha : \\
a \rightarrow 2 \\
b \rightarrow 1 \\
c \rightarrow 2 \\
d \rightarrow 0 \\
e \rightarrow 2 \\
\end{array}
\]
\textbf{\(\alpha\)-orientations}

Let \( G = (V, E) \) be a graph

Let \( \alpha \) be a function from \( V \) to \( \mathbb{N} \)

\[
\begin{array}{cccc}
a & b & c & d & e \\
2 & 1 & 2 & 0 & 2 \\
\end{array}
\]

Def: An \( \alpha \)-orientation is an orientation of \( G \) where for each \( v \in V \)

\[
\text{indegree}(v) = \alpha(v)
\]
**α-orientations**

Let $G = (V, E)$ be a graph.
Let $\alpha$ be a function from $V$ to $\mathbb{N}$.

<table>
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<tr>
<th>$\alpha$</th>
<th>$a$</th>
<th>$b$</th>
<th>$c$</th>
<th>$d$</th>
<th>$e$</th>
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<tbody>
<tr>
<td>$\rightarrow$</td>
<td>$2$</td>
<td>$1$</td>
<td>$2$</td>
<td>$0$</td>
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Def: An $\alpha$-orientation is an orientation of $G$ where for each $v \in V$,
\[
\text{indegree}(v) = \alpha(v)
\]
\(\alpha\)-orientations: criteria for existence

- If an \(\alpha\)-orientation exists, then

\[
\begin{align*}
(i) \ & \sum_{v \in V} \alpha(v) = |E| \\
(ii) \ & \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S|
\end{align*}
\]
α-orientations: criteria for existence

• If an α-orientation exists, then

\begin{align*}
(\text{i}) \quad & \sum_{v \in V} \alpha(v) = |E| \\
(\text{ii}) \quad & \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S| \\
\end{align*}

• If the α-orientation is accessible from a vertex \( u \in V \) then

\( (\text{iii}) \quad \sum_{v \in S} \alpha(v) > |E_S| \) whenever \( u \not\in S \) and \( S \neq \emptyset \)

\( \bar{S} \) and \( S \) are subsets of the vertex set \( V \).
**α-orientations: criteria for existence**

- If an α-orientation **exists**, then

\[
∀ S ⊆ V, \sum_{v \in S} α(v) ≥ |E_S|
\]

(i) \( \sum_{v \in V} α(v) = |E| \)

(ii) \( ∀ S ⊆ V, \sum_{v \in S} α(v) ≥ |E_S| \)

- If the α-orientation is **accessible** from a vertex \( u \in V \) then

\[
\sum_{v \in S} α(v) > |E_S| \quad \text{whenever} \quad u /∈ S \quad \text{and} \quad S ≠ ∅
\]

(iii) \( \sum_{v \in S} α(v) > |E_S| \quad \text{whenever} \quad u /∈ S \quad \text{and} \quad S ≠ ∅ \)

**Lemma (folklore):** The conditions are necessary and sufficient
\(\alpha\)-orientations: criteria for existence

- If an \(\alpha\)-orientation exists, then

\[
(i) \sum_{v \in V} \alpha(v) = |E| \\
(ii) \forall S \subseteq V, \sum_{v \in S} \alpha(v) \geq |E_S|
\]

- If the \(\alpha\)-orientation is accessible from a vertex \(u \in V\) then

\[
(iii) \sum_{v \in S} \alpha(v) > |E_S| \text{ whenever } u \notin S \text{ and } S \neq \emptyset
\]

**Lemma (folklore):** The conditions are necessary and sufficient

\(\Rightarrow\) accessibility from \(u \in V\) just depends on \(\alpha\) (not on which \(\alpha\)-orientation)
\(\alpha\)-orientations for plane maps

**Fundamental lemma:** If a plane map admits an \(\alpha\)-orientation, then it admits a **unique** \(\alpha\)-orientation **without ccw circuit**, called **minimal**.
**α-orientations for plane maps**

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**Uniqueness proof:** if $O_1 \neq O_2$, edges where $O_1$ and $O_2$ disagree form an eulerian suborientation of $O_1 \Rightarrow$ contains a circuit (ccw in $O_1$ or $O_2$).
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Set of $\alpha$-orientations = *distributive lattice* 

[Khueller et al’93], [Propp’93], [O. de Mendez’94], [Felsner’03]
**α-orientations for plane maps**

**Fundamental lemma:** If a plane map admits an $\alpha$-orientation, then it admits a **unique** $\alpha$-orientation **without ccw circuit**, called **minimal**.

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Set of $\alpha$-orientations = **distributive lattice**

[Khueller et al'93], [Propp'93], [O. de Mendez’94], [Felsner’03]
Fact: A triangulation with $n$ internal vertices has $3n$ internal edges.
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Natural candidate for indegree function:
$$\alpha : v \mapsto 3 \text{ for each internal vertex } v.$$ 

call 3-orientation such an $\alpha$-orientation.

Application to simple triangulations
Application to simple triangulations

Fact: A triangulation admitting a 3-orientation is simple

- $k$ internal vertices
- $3k + 1$ internal edges

$k$ internal vertices

$3k + 1$ internal edges
Application to simple triangulations

**Thm [Schnyder 89]:** A simple triangulation admits a 3-orientation. (proof by shelling procedure)

**Easier proof:** Any simple planar graph $G = (V, E)$ satisfies

$$|E| \leq 3|V| - 6 \quad \text{(Euler relation)}$$

hence the existence/accessibility conditions are satisfied. □
Application to simple triangulations

- From the lattice property (taking the min) we have
  
  family $\mathcal{F}$ of simple triangulations $\leftrightarrow$ subfamily $\mathcal{O}_T$ of $\mathcal{O}$ where:
  - faces have degree 3
  - inner vertices have indegree 3

- From the bijection $\Phi$ specialized to $\mathcal{O}_T$, we have
  
  $\mathcal{F} \leftrightarrow$ mobiles where all vertices have degree 3

[Bernardi, F’10], other bijection in [Poulalhon, Schaeffer’03]
Counting formula for simple triangulations

Let $T_n = \#$ rooted simple triangulations with $n + 3$ vertices

marked face (outer) + marked edge

$\Rightarrow T_n = \frac{2(4n + 1)!}{(n + 1)!(3n + 2)!}$
**Application to simple quadrangulations**

2-orientation = orientation where each internal vertex has indegree 2

[de Fraysseix, Ossona de Mendez’01]:
A quadrangulation $Q$ admits a 2-orientation iff $Q$ is simple
Every 2-orientation is accessible from the outer contour
(proof by shelling algorithm)

![Diagram of a quadrangulation](image)

**Proof from existence criterion:**
for every simple bipartite graph $G = (V, E)$, one has $|E| \leq 2|V| - 4$
Application to simple quadrangulations

- Specializing the meta bijection $\Phi$ we get

  - indegrees $= 2$
  - face-degrees $= 4$

  - every $\bigcirc$ has degree 2
  - every $\bullet$ has degree 4

  ($\simeq$ unrooted ternary tree)
Application to simple quadrangulations

• Specializing the meta bijection $\Phi$ we get

\[
\text{indegrees} = 2 \\
\text{face-degrees} = 4
\]

• recover a bijection in [Schaeffer’99]

• bijection $\Rightarrow$ there are \( \frac{4(3n)!}{n!(2n+2)!} \) rooted simple quadrangulations with \( n \) faces

\[
\text{every } \circ \text{ has degree 2} \\
\text{every } \bullet \text{ has degree 4} \\
(\sim \text{ unrooted ternary tree})
\]
Extension to any girth and face-degrees

girth = length shortest cycle

Rk: girth \leq \text{minimal face-degree}

Our approach works in any girth $d$, with control on the face-degrees

Other approach using slice decompositions [Bouttier,Guitter’15]