Decomposition and enumeration of planar graphs

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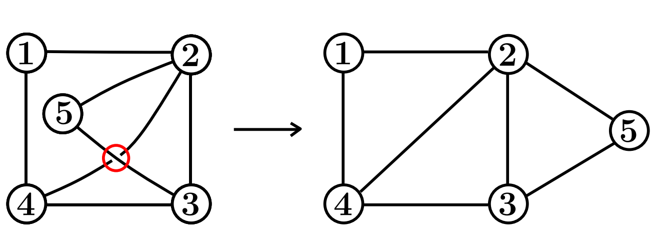
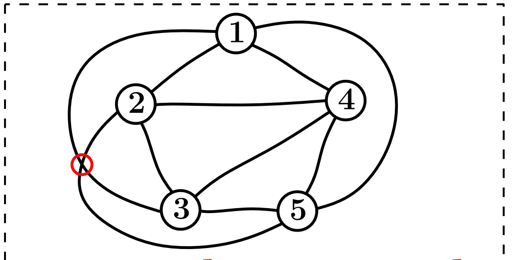
Planar graphs and planar maps

- A graph is planar iff it admits a planar drawing (no edge-crossings)



a planar graph



a non-planar graph

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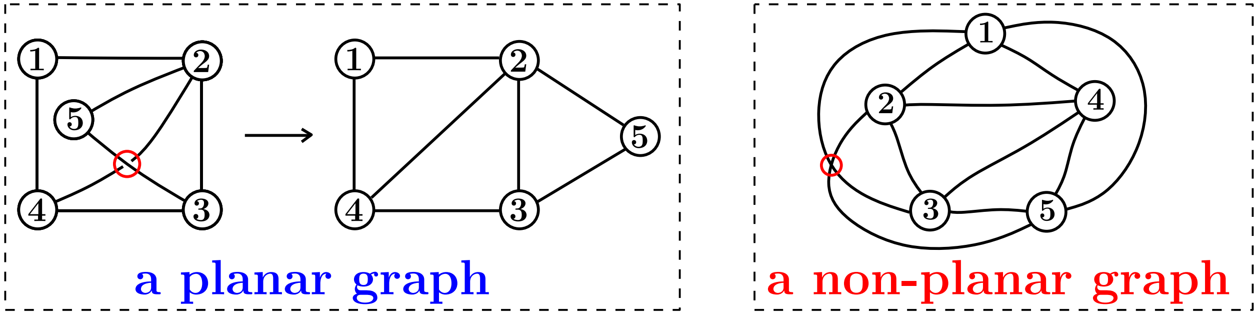
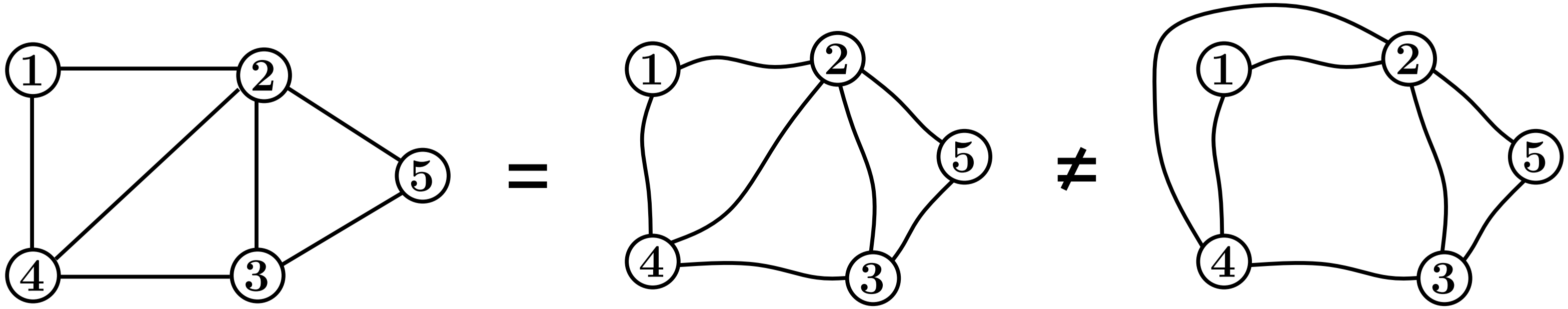


- Planar map = planar graph + planar embedding



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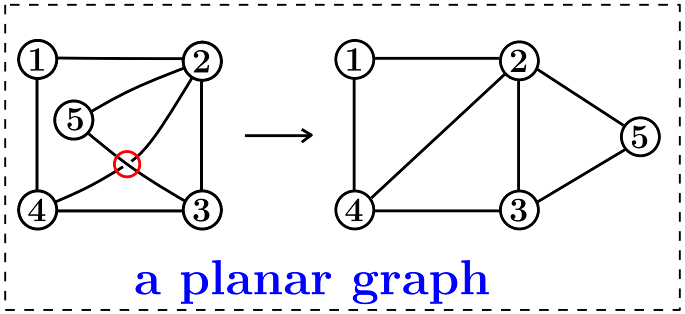

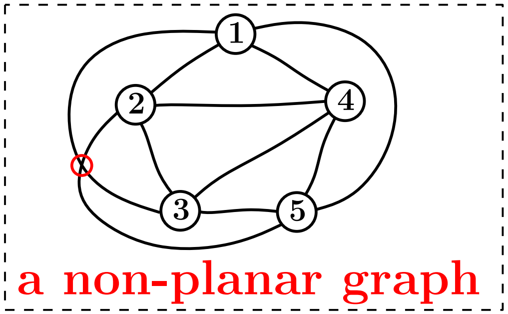

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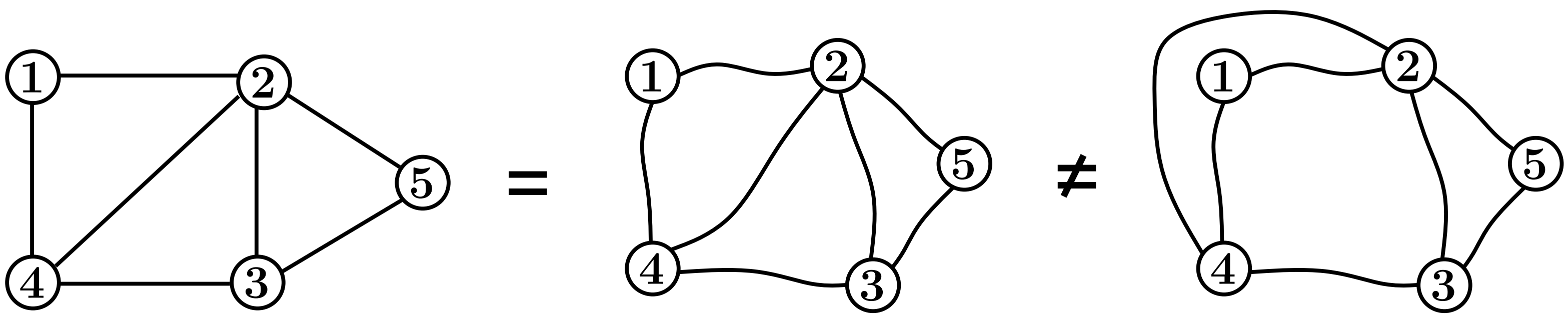

- We consider labelled graphs/maps: the n vertices carry distinct labels in $[1, \ldots, n]$.

Planar graph enumeration

Let $G[n]$ be the number of planar graphs with n vertices.

Let $G(x) = \sum_n G[n] \frac{x^n}{n!}$ be the counting series.

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- **Exact enumeration** [Bodirsky, Groepl, Kang'03]:
  The numbers $G[n]$ can be computed in polynomial time:

$$G(x) = 1x^0 + 1x + 2\frac{x^2}{2!} + 8\frac{x^3}{3!} + 64\frac{x^4}{4!} + 1023\frac{x^5}{5!} + 32071\frac{x^6}{6!} + \cdots$$

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- **Asymptotic enumeration** [Giménez and Noy'05]:
  The numbers $G[n]$ satisfies asymptotically:

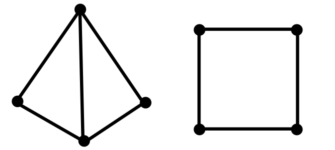
$$G[n] \sim n! g\, \gamma^n n^{-7/2}$$

  where $g \approx 4.26 \cdot 10^{-6}$ and $\gamma \approx 27.22$ are analytically computable.
  (+ limit laws for nr edges, nr connected components...)

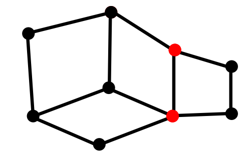
Exact enumeration of planar graphs

Families of planar graphs

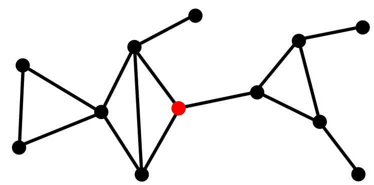
0-connected
general case



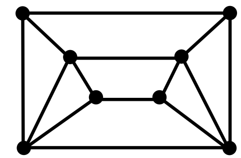
2-connected
(at least 2 vertices)



1-connected
(=connected)



3-connected
(at least 4 vertices)



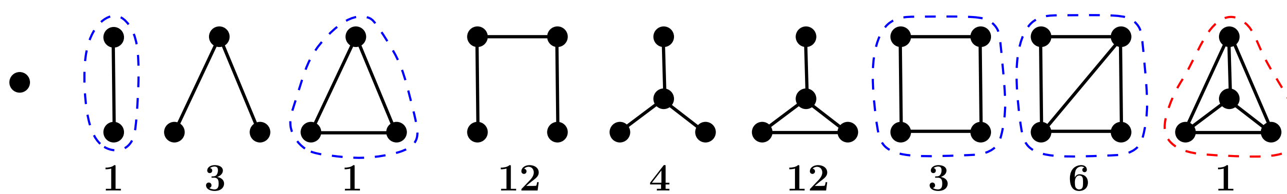
First terms:

$$G_0(x,y) = 1 + x + \frac{x^2}{2!}(1+y) + \frac{x^3}{3!}(1+3y+3y^2+y^3) + \cdots$$

$$G_1(x,y) = x + \frac{x^2}{2!}y + \frac{x^3}{3!}(3y^2+y^3) + \frac{x^4}{4!}(16y^3+15y^4+6y^5+y^6) + \cdots$$

$$G_2(x,y) = \frac{x^2}{2!}y + \frac{x^3}{3!}y^3 + \frac{x^4}{4!}(3y^4+6y^5+y^6) + \cdots$$

$$G_3(x,y) = \frac{x^4}{4!}y^6 + \frac{x^5}{5!}(15y^8+10y^9) + \cdots$$



The counting scheme

1) Equivalence with maps (Whitney):

3-connected planar maps

↕

3-connected planar graphs

2) Decomposition by increasing connectivity degree:

3-connected planar graphs

↓

2-connected planar graphs

↓

connected planar graphs

↓

planar graphs

3-connected planar graphs

- **Whitney's theorem:** Each 3-connected planar graph has two embeddings on the sphere, which differ by a reflexion (facial cycles can be read off from the graph)

$$\overrightarrow{G_3}(x, y) = \frac{1}{2}\overrightarrow{M_3}(x, y).$$

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- Enumeration of rooted 3-connected maps [Mullin, Schellenberg'68, Fusy, Poulalhon, Schaeffer'05]:

$$\overrightarrow{M_3}(x,y) = x^2y^2\Big(\frac{1}{1+xy} - \frac{y}{1+y} - \frac{(1+U)^2(1+V)^2}{(1+U+V)^3}\Big)$$

with $U = xy(1+V)^2, \quad V = y(1+U)^2.$

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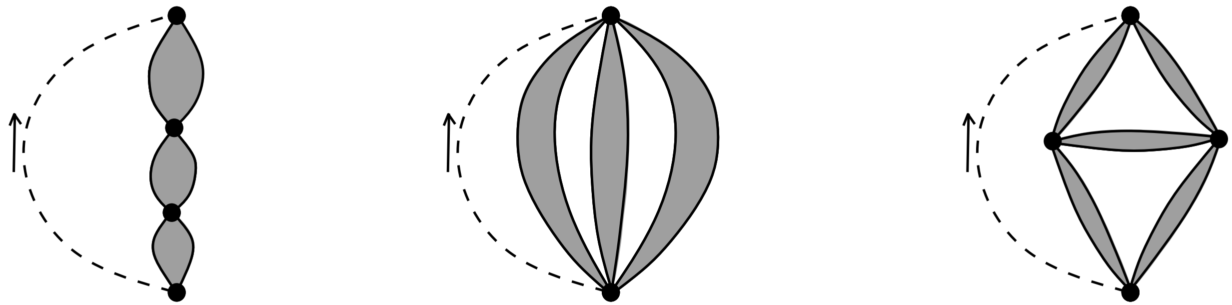
$$\Downarrow$$

$$G_3(x,y) = \frac{x^4}{4!}y^6 + \frac{x^5}{5!}(15y^8+10y^9) + \frac{x^6}{6!}(60y^9+432y^{10}+540y^{11}+195y^{12}) + \cdots$$

From 3-connected to 2-connected

Trakhtenbrot's decomposition (1958):

A rooted 2-connected planar graph is either:



$k \geq 2$ **components in series** — $k \geq 2$ **components in parallel** — **substitution at edges of a 3-connected (planar!) graph**

$\Rightarrow$ tree-like decomposition of rooted 2-connected planar graphs

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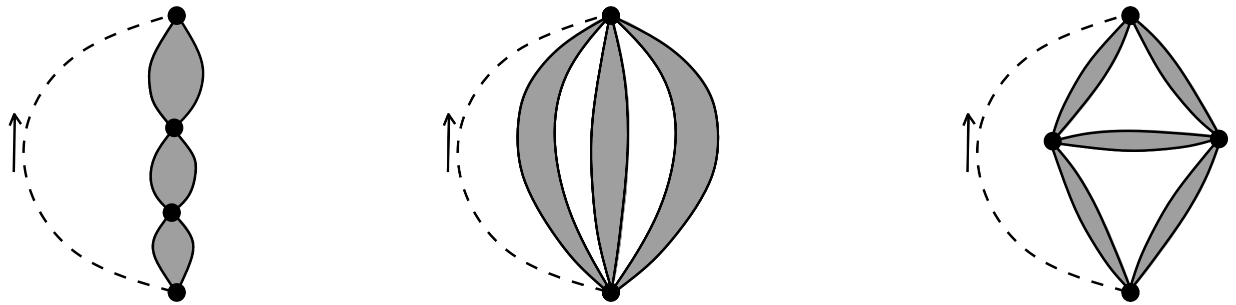

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Equation-system:

$$\left\{ \begin{array}{rcl} \color{blue}{\overrightarrow{G_2}} & = & y + S + P + \color{red}{\overrightarrow{G_3}}(x, \color{blue}{\overrightarrow{G_2}}(x,y)) \\ S & = & x(\overrightarrow{G_2} - S)/(1 - x(\overrightarrow{G_2} - S)) \\ P & = & \exp(\overrightarrow{G_2} - P) - 1 - (\overrightarrow{G_2} - P) \end{array} \right.$$

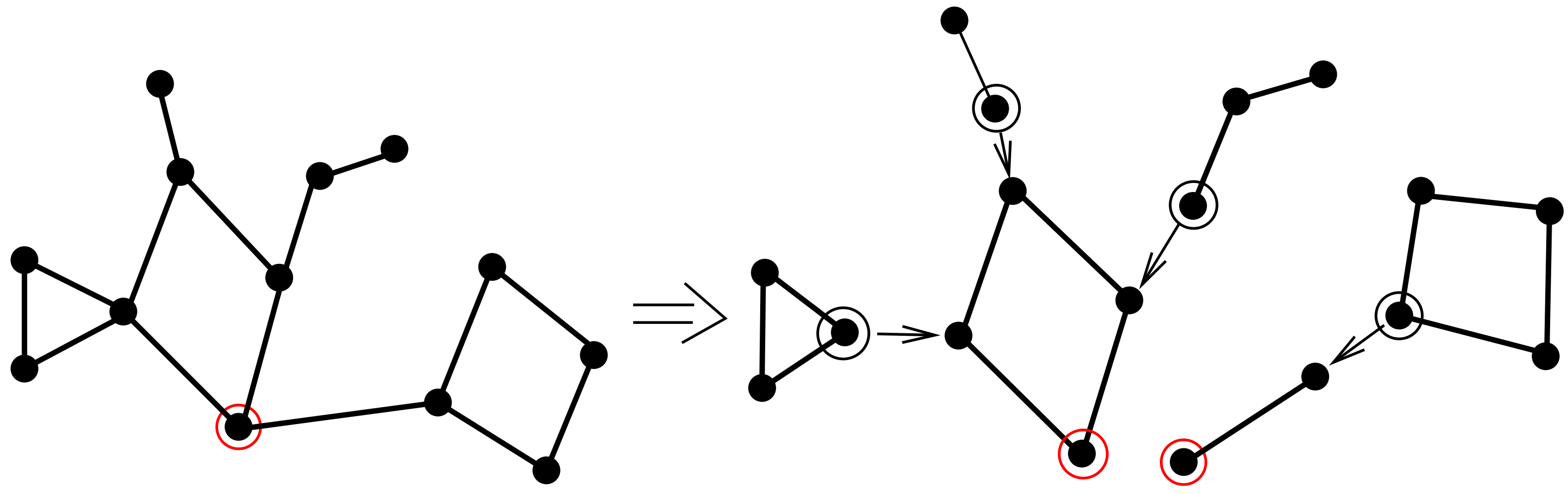
$$\Downarrow$$

$$G_2(x,y) = \frac{x^2}{2!}y + \frac{x^3}{3!}y^3 + \frac{x^4}{4!}(3y^4+6y^5+y^6) + \frac{x^5}{5!}(12y^5+70y^6+100y^7+15y^8+10y^9) + \cdots$$

From 2-connected to connected

Decomposition by vertex-substitution:
A pointed connected planar graph decomposes into a set of pointed 2-connected planar graphs where each non pointed vertex is substituted by a pointed connected planar graph.

(= first level of the decomposition in 2-connected blocks)



$$G'_1(x,y) = \exp(G'_2(xG'_1(x,y),y))$$

$$\Downarrow$$

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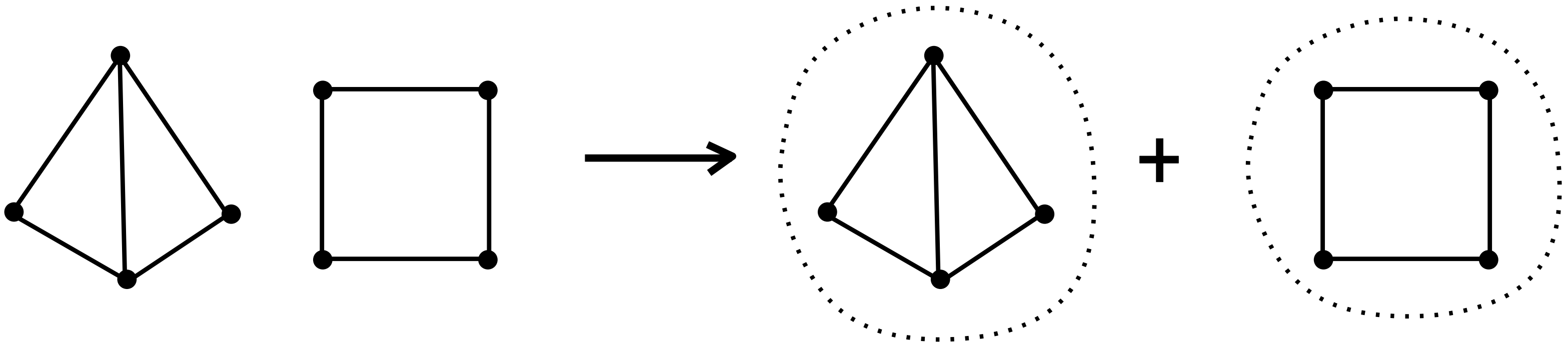
From connected to general ones

A planar graph is a set of connected planar graphs:



$$\boxed{G_0(x,y) = \exp(G_1(x,y))}$$

$$\Downarrow$$

$$G_0(x,y) = 1 + x + \frac{x^2}{2!}(1+y) + \frac{x^3}{3!}(1+3y+3y^2+y^3) + \frac{x^4}{4!}(1+6y+15y^2+20y^3+15y^4+6y^5+y^6)\cdots$$

Asymptotic enumeration of planar graphs

The approach

Analytic combinatorics (Flajolet, Sedgewick'08)

Class $\mathcal{C} = \cup_n \mathcal{C}_n$, coeff. $c_n = |\mathcal{C}_n|$, series: $C(z) = \sum_n c_n \frac{z^n}{n!}$

1. Find a combinatorial decomposition for $\mathcal{C}$
2. Translate into an equation-system satisfied by $C(z)$
3. Analyse the singularities of $C(z)$, and transfer to asymptotic formula for $|\mathcal{C}_n|$.

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Example: binary trees

1. Decomposition: tree $\rightarrow$ (left tree, node, right tree)
2. Equation-system: $\boxed{C(z) = z + C(z)^2}$
3. Analysis: square-root singularity at $z = 1/4$

$$C(z) = \frac{1 - \sqrt{1-4z}}{2} \quad \Rightarrow \quad C[n] \sim \frac{4^n}{\sqrt{\pi} n^{3/2}}$$

Asymptotics 3-connected planar graphs

Rk: asymptotics of $G_3[n]$ reduces to studying $\overrightarrow{G_3}[n,k]$, as

$$G_3[n] = \sum_k \frac{\overrightarrow{G_3}[n,k]}{2k}$$

[Bender, Richmond'84]:
The explicit expression of $\overrightarrow{G_3}(x,Y) = \frac{1}{2}\overrightarrow{M_3}(x,Y)$ yields:

$$\forall y > 0 \textbf{ fixed, } \overrightarrow{G_3}(x,Y) = \text{polynom} + c'(1 - \tfrac{x}{\rho_3(Y)})^{3/2} + \dots$$

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$$\textbf{if } k = \mu n + t\sigma\sqrt{n} \textbf{ then } \overrightarrow{G_3}[n.k] \underset{n\to\infty}{\sim} n!\, c\, \frac{\gamma^n}{n^3} e^{-t^2/2}$$

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Asymptotics 2-connected planar graphs

[Bender, Gao, Wormald'2002]:

- Singularity analysis of $\overrightarrow{G_3}(x, Y)$ from the explicit expression (similar to binary trees):

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$$\overrightarrow{G_2}(x, y) = \text{sp decomp.} + \overrightarrow{G_3}(x, \overrightarrow{G_2}(x, y)) :$$

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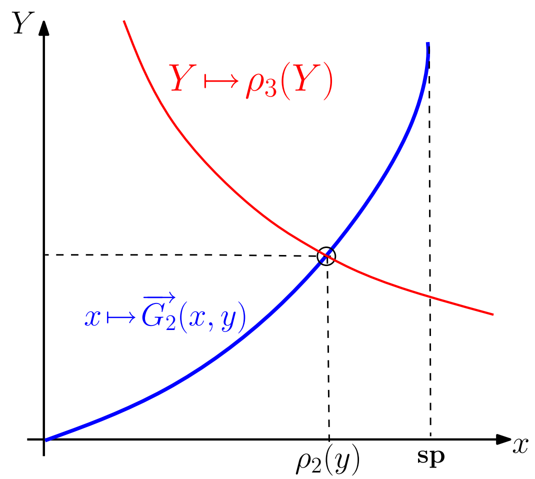

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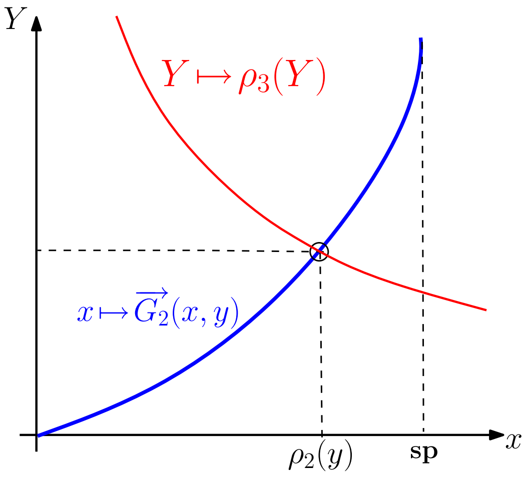

$\forall y>0$ **fixed,** $\overrightarrow{G_2}(x,y) = \text{polynom} + c'(1-\frac{x}{\rho_2(y)})^{3/2} + \ldots$

$\Downarrow$

if $k = \mu n + t\sigma\sqrt{n}$ **then** $\overrightarrow{G_2}[n.k] \underset{n\to\infty}{\sim} n!\, c\,\frac{\gamma^n}{n^3} e^{-t^2/2}$

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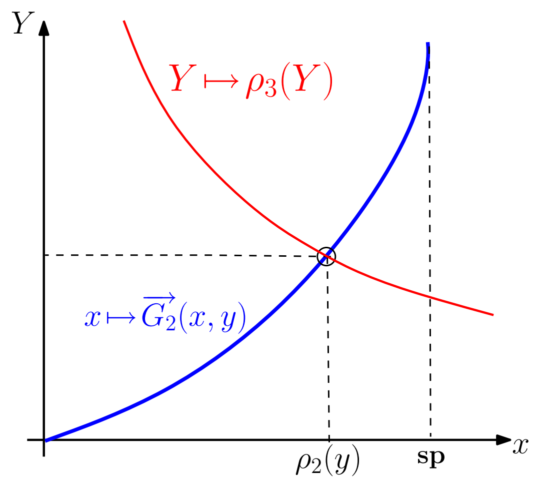

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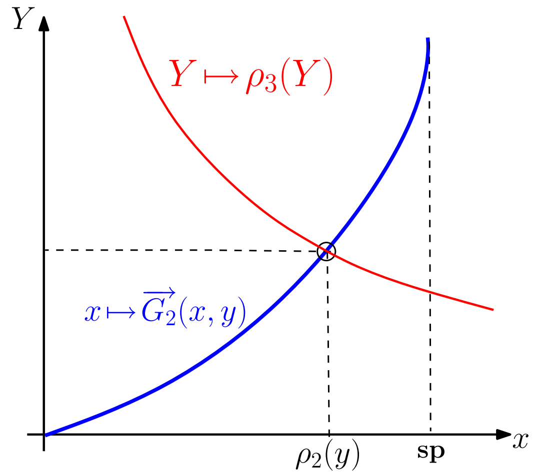

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Going to 1-connected: difficult !

- Trace the singularities from $G_2'(x,y)$ to $G_1'(x,y)$ in

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- [Giménez, Noy'05]: analytic integration of $\overrightarrow{G_2}(x,y)\mathrm{d}y$.
  [Chapuy, Fusy, Kang, Shoilekova'07, Leroux et al'07]:
  "combinatorial" integration: obtain directly $G_2(x,y)$.

Combinatorial integration on trees

- Let $T(x)$ be the series counting (unrooted) labeled trees
- Let $T^\circ(x)$ be the series counting pointed trees, specified by:

$$T^\circ(x) = x\exp(T^\circ(x))$$

$$T^\circ[n] = nT[n] \quad \Rightarrow \quad T(x) = \int_0^x \frac{T^\circ(t)}{t}\mathrm{d}t.$$

How to integrate (obtain an equation-system specifying $T(x)$) without calculations ?

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(Rq: $G_3 = \frac{1}{2}M_3 = \frac{1}{4}(\underbrace{M_3^V}_{\text{vert.}} - \underbrace{M_3^E}_{\text{edge}} + \underbrace{M_3^F}_{\text{face}})$ by Euler's relation)

and each of M_3^V, M_3^E, M_3^F is decomposable

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$$\boxed{G_1[n] \sim n!\, c'' \rho_1^{-n} n^{-7/2}}$$

Finally go to 0-connected !

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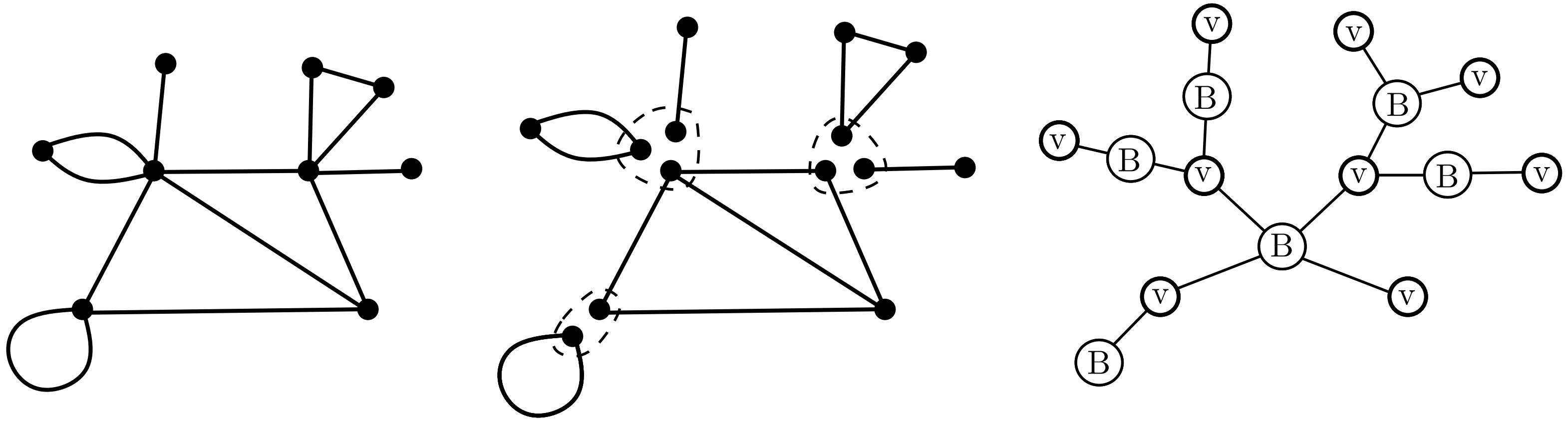
Express $G_1(x)$ directly from the (unrooted) block-tree



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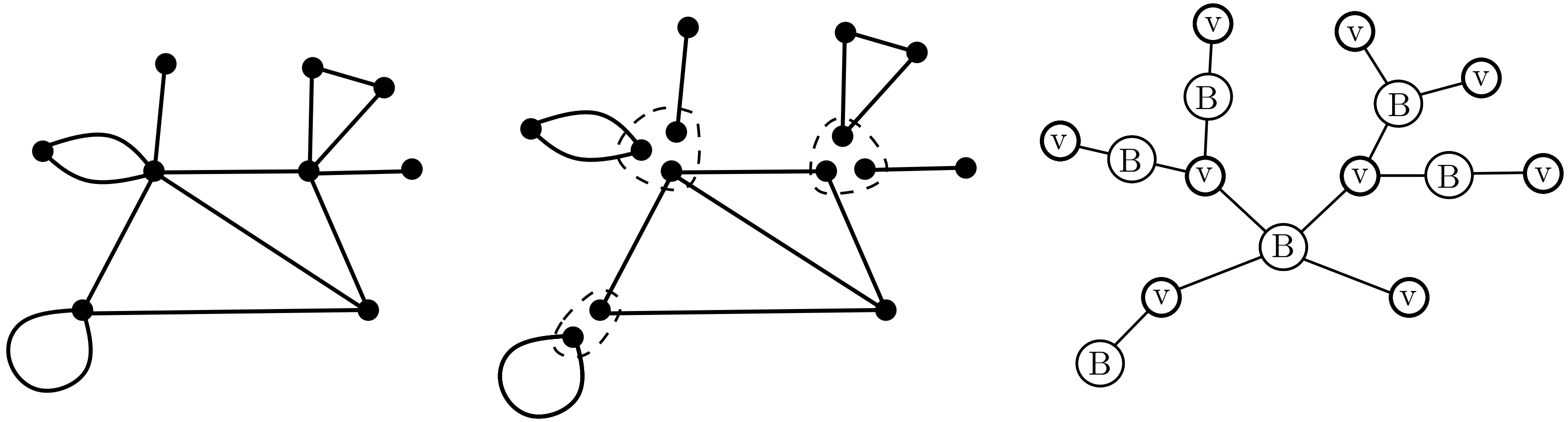

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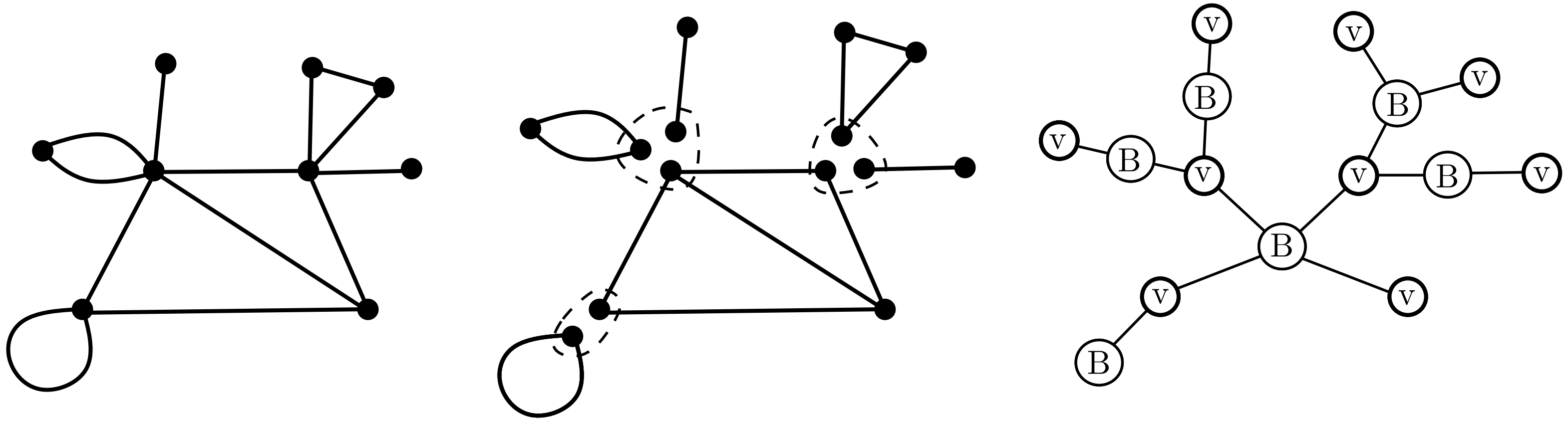

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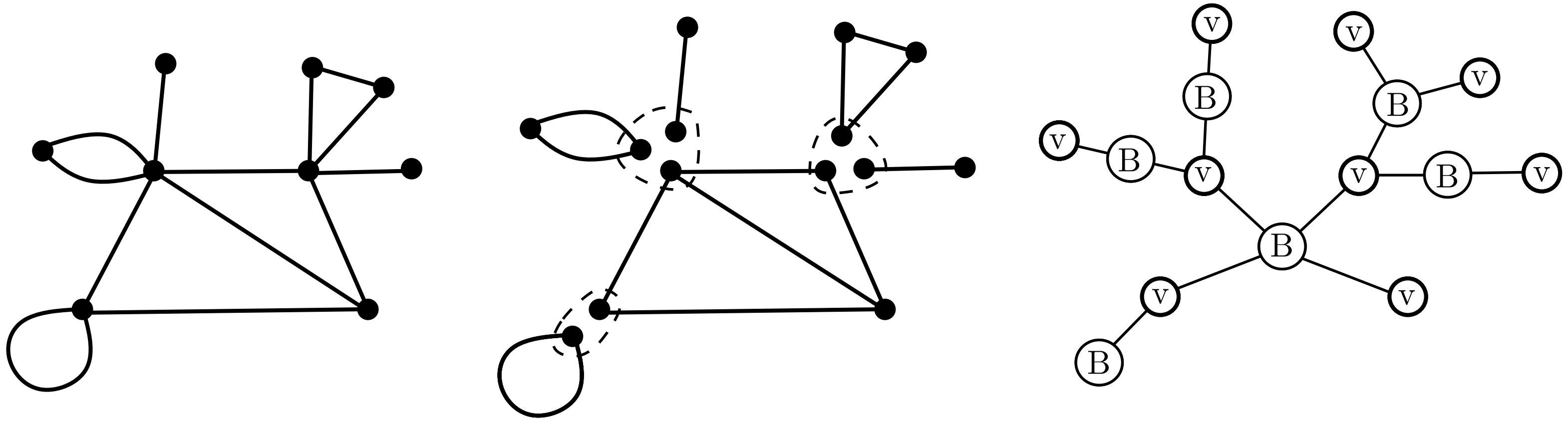

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$\Rightarrow$ singular analysis of $G_1(x)$, then of $G_0(x)$, yielding

$$\boxed{G_0[n] \sim n!\, c\gamma^n n^{-7/2}}$$

Extension to any graph family

[Giménez, Noy, Rue'07, Chapuy, Fusy, Kang, Shoilekova'07]
Theorem: For a graph family (stable under taking 3-connected components) the asymptotic study reduces to the asymptotic study for the 3-connected subfamily.

Applies to any family specified by a collection of forbidden 3-connected minors:

Typical examples:

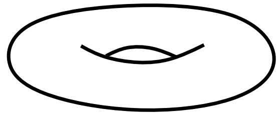
- Planar (=Forbid($K_5, K_{3,3}$)): $G[n] \sim n!\,\gamma^n n^{-7/2}$ **[Giménez, Noy'05]**
  (asymptotics determined by 3-connected maps)
- Series-parallel (=Forbid(K_4)): $G[n] \sim n!\,\gamma^n n^{-5/2}$ **[Bodirsky, Giménez, Kang, Noy'05]**
  (asymptotics determined by tree-like decomposition along 2-cuts and 1-cuts)

In full generality, only partial results for graphs with forbidden minors

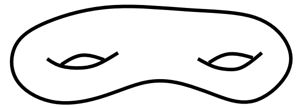
exponential growth **[Norine et al.'06]**, with refinements in **[Bernardi et al'07]**

In project: graphs on other surfaces

- Surfaces are classified according to the genus g



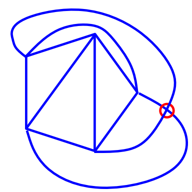
$g = 1$

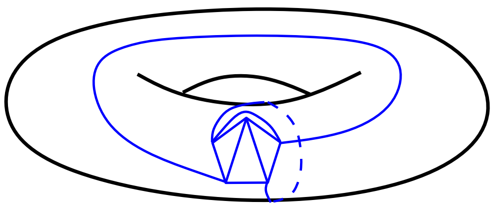


$g = 2$

- Genus of a graph = minimal genus of a surface to embedd it

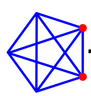
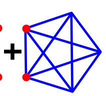
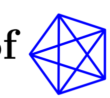
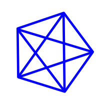
Example: genus(K_5)=1





- Let $G^{(g)}[n]$ be the number of graphs of genus g with n vertices

  Exact enumeration seems difficult (Whitney's theorem can fail, genus might not add up for decomposition along 2-cuts)

  genus of  +  = 1 ≠ genus of  + genus of 

- But asymptotic enumeration is doable! (work with B. Mohar and J. Rué
  [McDiarmid'08]: $(G^{(g)}[n]/n!)^{1/n} \to \gamma$ (same growth as in the planar case)

For "almost all" graphs of genus g, the decomposition applies nicely
(one big 3-connected map of genus g, all the other components are planar)

$\Downarrow$ implies

$$G^{(g)}[n] \sim n!\ c^{(g)} \gamma^n n^{5/2(g-1)-1}$$

(Rk: this asymptotic pattern is known for embedded graphs ([Bender et al])