Distances in plane trees and planar maps

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LIX, Ecole Polytechnique
Overview

• Structures we study:
  - paths
  - trees
  - maps

• Distance-parameters
  - typical (depth, distance between 2 vertices)
  - extremal (height, radius, diameter)
Part 1: distances in plane trees
Plane trees

- Plane tree = tree embedded in the plane

- Rooted Plane tree = plane tree + marked corner

- Rooted plane tree <-> Dyck path
Profile of a plane tree

Overview:
- show (using cyclic lemma) that $h \approx 2 \cdot$ Typical Level
- show limit profile (Rayleigh law)
Cyclic lemma to count Dyck paths

- **Def**: quasi-bridge = walk ending at \( y = -1 \)
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Dyck path + appended down-step
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• **Def**: quasi-bridge = walk ending at \( \{y = -1\} \)

Dyck path + appended down-step + marked point

Quasi-bridge (by re-rooting)
Cyclic lemma to count Dyck paths

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Quasi-bridge (by re-rooting)
Cyclic lemma to count Dyck paths

- **Def:** quasi-bridge = walk ending at \( y = -1 \)

\[
\Rightarrow D_n \cdot (2n + 1) = \binom{2n + 1}{n} \Rightarrow D_n = \frac{(2n)!}{n!(n + 1)!}
\]
Def: vertical span := MaxOrdinate - MinOrdinate

vs = 3
Vertical span and cyclic lemma

\[ \text{vs}(D) = \begin{cases} \text{vs}(Q) & \text{if marked point before MaxOrdinate} \end{cases} \]
Vertical span and cyclic lemma

\[ vs(D) = \begin{cases} 
    vs(Q) & \text{if marked point before MaxOrdinate} \\
    vs(Q) + 1 & \text{if marked point after MaxOrdinate} 
\end{cases} \]
Vertical span and cyclic lemma

\[ \text{vs}(D) = \text{vs}(Q) + (0 \text{ or } 1) \]
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Also, \( \text{vs}(D) = h + 1 \)
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Also, \( \text{vs}(D) = h + 1 \)

\[ \text{vs}(Q) = h_{\downarrow}(Q) + h_{\uparrow}(Q) + 1 \]
Vertical span and cyclic lemma

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Also, \( \text{vs}(D) = h + 1 \)

\[ \text{vs}(Q) = h_{\downarrow}(Q) + h_{\uparrow}(Q) + 1 \]

Hence \( h(D) = h_{\downarrow}(Q) + h_{\uparrow}(Q) + (0 \text{ or } -1) \)
Combinatorial interpretation of $h_{\downarrow}(Q)$

$Q \iff D + \text{marked point} \iff T + \text{marked corner}$

$h_{\downarrow}(Q) = \text{distance } L \text{ between the } 2 \text{ marked corners}$
Combinatorial interpretation of $h_\downarrow(Q)$

$Q \iff D + \text{marked point} \iff T + \text{marked corner}$

$h_\downarrow(Q) = \text{distance } L \text{ between the 2 marked corners}$

PATHS: $h(D) = h_\downarrow(Q) + h_\uparrow(Q) + (0 \text{ or } -1)$

TREES: $h(T) = L + L' + (0 \text{ or } -1)$

extremal typical same distribution as L
Distribution of $L$ (Meir & Moon’78)

- Use generating functions (cf this morning)

$$T(z) = \frac{1}{1 - zT(z)}$$

- Two marked corners at distance $k$

$$T_k(z) = z^k T(z)^{2k+2}$$

$$P_n(L = k) = \frac{[z^n]T_k(z)}{(2n+1)[z^n]T(z)} = \frac{(2k+2)n!(n+1)!}{(n+k+2)!(n-k)!}$$

(using the Lagrange inversion formula)
Distribution of $L$ (Meir & Moon’78)

(i) $\mathbb{P}_n(L = k) = \frac{[z^n]T_k(z)}{(2n + 1)[z^n]T(z)} = \frac{(2k + 2)n!(n + 1)!}{(n + k + 2)!(n - k)!}$

L=3
Distribution of $L$ (Meir & Moon’78)

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\[\forall x > 0, \quad \mathbb{P}_n(L = x\sqrt{n}) \xrightarrow[n \to \infty]{} \frac{1}{\sqrt{n}}2x \exp(-x^2)\]
Distribution of $L$ (Meir & Moon’78)

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$\forall x > 0$, $\mathbb{P}_n(L = x\sqrt{n}) \sim \frac{1}{\sqrt{n}} 2x \exp(-x^2)$

$L/\sqrt{n} \xrightarrow[n \to \infty]{} dx \cdot 2x \exp(-x^2)$  Rayleigh law
Distribution of L (Meir & Moon’78)

(i) \( \mathbb{P}_n(L = k) = \frac{[z^n]T_k(z)}{(2n + 1)[z^n]T(z)} = \frac{(2k + 2)n!(n + 1)!}{(n + k + 2)!(n - k)!} \)

\[ \implies \forall x > 0, \quad \mathbb{P}_n(L = x\sqrt{n}) \sim \frac{1}{\sqrt{n}} 2x \exp(-x^2) \]

\[ \implies L/\sqrt{n} \to \int dx \cdot 2x \exp(-x^2) \quad \text{Rayleigh law} \]

Rq: (i) implies uniform tail \( \mathbb{P}_n(L/\sqrt{n} \geq x) \leq a e^{-cx} \forall n, x \)

\( \implies \) Moments of \( L / n^{1/2} \) converge to moments of Rayleigh law
The Rayleigh law / stable laws

cf [Banderier, Flajolet, Schaeffer, Soria’01]

Case $\lambda = 1/2$

If $\mathbb{P}_n(X_n = k) \asymp [z^n] T(u)^k$

with $T(u) = 1 - c(1 - u)^{1/2} + \ldots$

then $\frac{X_n}{n^{1/2}} \rightarrow \text{Rayleigh law}$

Rk: $T(u)^k = PGF\left(\sum_{i=1}^{k} Z_i\right)$, with $\text{Tail}(Z_i) \sim k^{-3/2}$

$$\frac{1}{k^2} \sum_{i=1}^{k} Z_i \longrightarrow \text{Stable law parameter } 1/2$$
The Rayleigh law / stable laws
cf [Banderier, Flajolet, Schaeffer, Soria’01]

**General** $\lambda \in (0, 1)$

If $\mathbb{P}_n(X_n = k) \propto [z^n] T(u)^k$

with $T(u) = 1 - c(1 - u)^\lambda + \cdots$

then $\frac{X_n}{n^\lambda} \to G_\lambda(u) \, du$

**Rk:** $T(u)^k = PGF\left(\sum_{i=1}^k Z_i\right)$, with $\text{Tail}(Z_i) \sim k^{-\lambda-1}$

$$\frac{1}{k^{1/\lambda}} \sum_{i=1}^k Z_i \to \text{Stable law parameter } \lambda$$
The Rayleigh law / stable laws

cf [Banderier, Flajolet, Schaeffer, Soria’01]

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$\frac{1}{k^{1/\lambda}} \sum_{i=1}^{k} Z_i \longrightarrow$ Stable law parameter $\lambda$

Here $\lambda = 1/2$ (for maps $\lambda = 1/4$)
Expectation/tail for the height

\[ h = L + L' + (0 \text{ or } -1) \]
Expectation/tail for the height

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Expectation:

\[
\mathbb{E}_n(h) = 2 \mathbb{E}_n(L) + \epsilon, \quad \text{with } \epsilon \in [-1, 0]
\]

\[
\mathbb{E}_n(L) \sim \sqrt{\pi/2} \cdot \sqrt{n}
\]

\[
\mathbb{E} \text{(Rayleigh)}
\]

\[
\implies \mathbb{E}_n(h) \sim \sqrt{\pi} \sqrt{n}
\]

[De Bruijn, Knuth, Rice’72]
Expectation/tail for the height

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**Expectation:**
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\mathbb{E}_n(h) = 2 \mathbb{E}_n(L) + \epsilon, \quad \text{with } \epsilon \in [-1, 0]
\]
\[
\mathbb{E}_n(L) \sim \sqrt{\pi / 2} \cdot \sqrt{n}
\]
\[
\mathbb{E}(\text{Rayleigh})
\]

\[ \Rightarrow \quad \mathbb{E}_n(h) \sim \sqrt{\pi} \sqrt{n} \]
[De Bruijn, Knuth, Rice’72]

**Exponential tail:**
\[
P_n(h \geq k) \leq 2 P_n(L \geq k/2)
\]
\[
P_n(L/\sqrt{n} \geq x) \leq a e^{-cx} \quad \forall n, x
\]

\[ \Rightarrow \quad P_n(h/\sqrt{n} \geq x) \leq 2a e^{-cx} \]
Limit distribution for the height

Two possible approaches:

• Singularity analysis [Flajolet, Odlyzko’82], [Flajolet et al.’93]

\[
\text{System } y_h(z) = \frac{1}{1 - y_{h-1}(z)} \quad \text{[height } \leq h]\n\]

Singular expansion of \( y_h - y_{h-1} \) for \( h = \lfloor x \sqrt{n} \rfloor \)

\[
\implies P\left( \frac{\text{height}}{\sqrt{n}} \leq x \right) \longrightarrow \sum_{k \in \mathbb{Z}} (2k^2x^2 - 1)e^{-k^2x^2}
\]

• Continuous limit [Aldous]

If functional \( F : C[0, 1] \rightarrow \mathbb{R} \) is continuous for \( \| . \|_{\infty} \), then

\[
F(D_n/\sqrt{n}) \longrightarrow F(\text{brownian excursion})
\]

Image credit
J.F. Marckert
Part 2: distances in planar quadrangulations
Planar maps

- **Planar map** = planar graph embedded on the sphere
  
  ![Planar map](image1)
  
  ![Embedded in the plane](image2)

- **Quadrangulation** = planar map with faces of degree 4
  
  ![Quadrangulation](image3)
Profile of a pointed quadrangulation

Profile for vertices: (4,4,4,2)  Profile for edges: (4,8,8,6)
Well-labelled trees

- A well-labelled tree is a plane tree where:
  - each vertex v has a non-negative label
  - the labels at each edge \((v,v')\) differ by at most 1
  - at least one vertex has label 1
Well-labelled trees

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• Rooted well-labelled tree = well-labelled tree + marked corner

(there are $3^n \frac{(2n)!}{n!(n+1)!}$ such trees with $n$ edges)
Well-labelled tree -> pointed quadrangulation

[Schaeffer’98], also [Cori&Vauquelin’81]
Well-labelled tree -> pointed quadrangulation

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1) Place a red leg in each corner
Well-labelled tree -> pointed quadrangulation

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Well-labelled tree -> pointed quadrangulation

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2) Repeat: - choose a leg of label i>1
    - “throw” it to next corner of label i-1
Well-labelled tree -> pointed quadrangulation

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Well-labelled tree $\rightarrow$ pointed quadrangulation

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Well-labelled tree -> pointed quadrangulation

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3) Create a new vertex labelled 0 in the outer face
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Well-labelled tree -> pointed quadrangulation

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4) Connect all remaining legs (label 1) to the new vertex
Well-labelled tree -> pointed quadrangulation

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4) Connect all remaining legs (label 1) to the new vertex
Well-labelled tree $\rightarrow$ pointed quadrangulation

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5) Delete the black edges
Well-labelled tree -> pointed quadrangulation

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Well-labelled tree -> pointed quadrangulation

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- faces are of degree 4
Well-labelled tree $\rightarrow$ pointed quadrangulation

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- faces are of degree 4
Well-labelled tree -> pointed quadrangulation

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- faces are of degree 4
- labels = distances from pointed vertex
The mapping is a bijection

**Theorem** [Schaeffer’98]: The mapping is a bijection from well-labelled trees to pointed quadrangulations.

- vertex label $i$ ↔ vertex at distance $i$
- corner label $i$ ↔ edge at level $i$
- edge ↔ face
The mapping is a bijection

**Theorem** [Schaeffer’98]: The mapping is a bijection from well-labelled trees to pointed quadrangulations

Corollary: there are \( \frac{3^n (2n)!}{n!(n + 1)!} \) quadrangulations with \( n \) faces, a marked vertex, and a marked edge.
Relative levels

$T + 2$ marked corners $c_1, c_2 \leftrightarrow (Q, v) + 2$ marked edges $e_1, e_2$

\[ \ell(c_2) - \ell(c_1) = \text{level}(e_2) - \text{level}(e_1) \]
Relative levels are in the scale $n^{1/4}$

$L$ is of order $n^{1/2}$

$\Delta := \ell(c_2) - \ell(c_1)$ is of order $\sqrt{L}$, i.e., $n^{1/4}$
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Precisely

$$\frac{\Delta}{n^{1/4}} \xrightarrow{n \to \infty} \int dt \, g(t)$$

where

$$g(t) := 2\sqrt{\frac{3}{\pi}} \int_0^{+\infty} e^{-3t^2/4x} \sqrt{x} e^{-x^2} dx$$

$$g(t) = \Theta(t^{1/3} e^{-ct^{4/3}})$$

$$c := \frac{3^{2/3} 5}{8}$$
Relation typical level / radius
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$h \downarrow +1 = \text{Level(random edge)}$  \quad L := h_\downarrow +1/2 = \text{Level} - 1/2
Relation typical level / radius

\[ h \downarrow + 1 = \text{Level(random edge)} \]
\[ L := h \downarrow + 1/2 = \text{Level - 1/2} \]

\[ r = L + L' \]

extremal  typical  same distribution as L
Illustration

- For pointed quadrangulations with 2 faces.

\[
E(r) = \frac{2+2+1}{3} = \frac{5}{3}
\]

\[
E(L) = \frac{7/2 + 5/2 + 3/2}{9} = \frac{5}{6}
\]

\[E(r) = 2 \times E(L)\]

in each fixed size.
Consequence on the profile

typical level-difference $n^{1/4}$

typical level

radius
Consequence on the profile

Typical level (& radius) also of order $n^{1/4}$:
- Chassaing-Schaeffer’04: continuous limit (brownian snake)
- Bouttier-Di Francesco-Guitter’03: exact GF expressions
Exact GF expression

[Bouttier, Di Francesco, Guitter’03]

\[ R_k(z) := \text{GF well-labelled trees with root-label } \leq k \]
Exact GF expression

[Boultier, Di Francesco, Guitter’03]

\[ R_k(z) := \text{GF well-labelled trees with root-label } \leq k \]

Equation: \[ R_k(z) = \frac{1}{1 - z(R_{k-1}(z) + R_k(z) + R_{k+1}(z))} \]

\[ R = \lim_{k \to \infty} R_k \quad \text{satisfies} \quad R = \frac{1}{1 - 3zR} \]
Exact GF expression
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\[
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\]

**Exact solution:** \[
R_k = R \frac{(1 - x^k)(1 - x^{k+3})}{(1 - x^{k+1})(1 - x^{k+2})}
\]

where \[ x + \frac{1}{x} + 1 = \frac{1}{zR^2} \]
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where \( x + \frac{1}{x} + 1 = \frac{1}{zR^2} \)

**Rk:** \( x = 1 - c(1 - z/\rho)^{1/4} + \cdots \)

\[
\Rightarrow \frac{\text{Level}}{n^{1/4}} \rightarrow du \quad g(u)
\]

related to Stable\(_{1/4}\)