BIJECTIONS FOR MAPS WITH BOUNDARIES: KRIKUN'S FORMULA FOR TRIANGULATIONS, AND A QUADRANGULATION ANALOGUE

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Abstract. We present bijections for planar maps with boundaries. In particular, we obtain bijections for triangulations and quadrangulations of the sphere with boundaries of prescribed lengths. For triangulations we recover the beautiful factorized formula obtained by Krikun using a (technically involved) generating function approach. The analogous formula for quadrangulations is new. Our method is to show that these maps with boundaries can be endowed with certain "canonical" orientations, making them amenable to the master bijection approach we developed in previous articles. As an application of our formulas, we note that they provide an exact solution of the dimer model on rooted triangulations and quadrangulations.

1. Introduction

In this article, we present bijections for maps with boundaries. Recall that a *map* is a decomposition of the 2D sphere into vertices, edges, and faces, considered up to continuous deformation[1] (see precise definitions in Section 2). A *map with boundaries* is a map with a set of distinguished faces called *boundary faces* which are pairwise vertex-disjoint, and have simple-cycle contours (no pinch point). We call *boundaries* the contours of the boundary faces. We can think of the boundary faces as holes in the sphere, and maps with boundary as a decomposition of a sphere with holes into vertices, edges and faces. A *triangulation with boundaries* (resp. *quadrangulation with boundaries*) is a map with boundaries such that every non-boundary face has degree 3 (resp. 4).

The main results obtained in this article are bijections for triangulations and quadrangulations with boundaries. The bijection establishes a correspondence between these maps and certain types of plane trees. This, in turns, easily yields factorized enumeration formulas with control on the number and lengths of the boundaries. In the case of triangulations, the enumerative formula was first discovered by Krikun [11] (via a technically involved "guessing/checking" generating function approach). The case of quadrangulations is new.

The strategy we applied is to adapt to maps with boundaries the "master bijection" approach we developed in [2, 3] for maps without boundaries. Roughly speaking, this strategy reduces the problem of finding bijections, to the problem of exhibiting canonical orientations characterizing these classes of maps.

Let us now state the main enumerative results. We call a map with boundary *multi-rooted* if the r boundary faces are labeled with distinct numbers in $[r] = \{1, \ldots, r\}$, and each one has a marked corner; see Figure 1. For $m \geq 0$ and $a_1, \ldots, a_r$ positive integers, we denote $\mathcal{T}(m; a_1, \ldots, a_r)$ (resp. $\mathcal{Q}(m; a_1, \ldots, a_r)$) the set of multi-rooted triangulations (resp. quadrangulations) with r boundary faces, and m internal vertices (vertices not on the

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[1] We deal exclusively with planar maps in this article and call them simply *maps*.

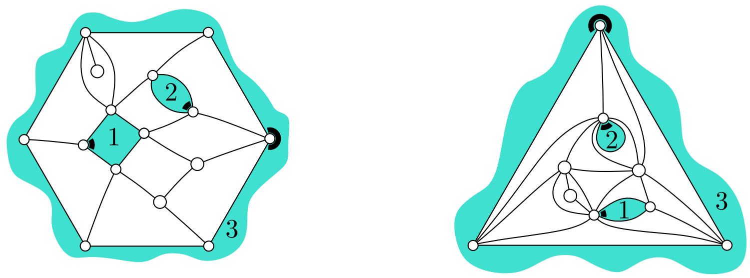


FIGURE 1. Left: a quadrangulation in $\mathcal{Q}[3;4,2,6]$. Right: a triangulation in $\mathcal{T}[3;2,1,3]$.

boundaries), such that the boundary labeled i has length a_i for all $i \in [r]$. In 2007 Krikun proved the following result:

Theorem 1 (Krikun [11]). *For $m \geq 0$ and $a_1, \ldots, a_r$ positive integers,*

$$|\mathcal{T}[m;a_1,\ldots,a_r]| = \frac{4^k(e-2)!!}{m!(2b+k)!!}\prod_{i=1}^{r} a_i\binom{2a_i}{a_i}, \tag{1}$$

where $b := \sum_{i=1}^r a_i$ is the total boundary length, $k := r+m-2$, and $e = 2b+3k$ is the number of edges (and the notation $n!!$ stands for $\prod_{i=0}^{\lfloor (n-1)/2\rfloor}(n-2i)$).

We obtain a bijective proof of this result, and also prove the following analogue:

Theorem 2. *For $m \geq 0$ and $a_1, \ldots, a_r$ positive integers,*

$$|\mathcal{Q}[m;2a_1,\ldots,2a_r]| = \frac{3^k(e-1)!}{m!(3b+k)!}\prod_{i=1}^{r} 2a_i\binom{3a_i}{a_i}, \tag{2}$$

where $b := \sum_{i=1}^r a_i$ is the half-total boundary length, $k := r+m-2$, and $e = 3b+2k$ is the number of edges.

Note that (1) gives in particular $|\mathcal{T}[0;a+2]| = \frac{(2a)!}{a!(a+1)!}$, hence can be seen as a far-reaching generalization of the Catalan formula for the number of triangulations of a polygon (without internal points). Similarly, we recover from (2) the formula $|\mathcal{Q}[0;2a+2]| = \frac{(3a)!}{a!(2a+1)!}$ for the number of quadrangulations of a polygon (without internal points). Equation (2) also yields

$$|\mathcal{Q}[m;2]| = \frac{2\cdot 3^m(2m)!}{m!(m+2)!},$$

which is the well-known formula for the number of rooted quadrangulations with $m+2$ vertices (upon seeing the root-edge as blown into a boundary face of degree 2), and more generally it yields

$$|\mathcal{Q}[m;2a]| = \frac{3^{m-1}(3a+2m-3)!}{m!(3a+m-1)!}\frac{(3a)!}{(2a-1)!a!},$$

which is the formula given in [6, Eq.(2.12)] for the number of rooted quadrangulations with a simple boundary of length $2a$, and m internal vertices.

As a side remark, let us discuss the counterparts of (1) and (2) when we remove the condition for the boundaries to be simple and pairwise disjoint. Let $\widehat{\mathcal{T}}[n;a_1,\cdots,a_r]$ (resp. $\widehat{\mathcal{Q}}[n;a_1,\ldots,a_r]$) be the set of maps with r distinguished faces labeled $1,\ldots,r$ of respective degrees $a_1, \ldots, a_r$, each having a marked corner, such that the other faces have degree 3 (resp. 4). It is easy to deduce from Tutte's slicings formula [19] that

$$|\widehat{\mathcal{Q}}[n; 2a_1, \ldots, 2a_r]| = \frac{(e-1)!}{v!n!} 3^n \prod_{i=1}^{r} 2a_1 \binom{2a_i - 1}{a_i},$$

where $v = n + 2 + \sum_{i=1}^{r}(a_i - 1)$ is the total number of vertices, and $e = 2n + \sum_{i=1}^{r} a_i$ is the total number of edges. However no factorized formula should exist for $|\widehat{\mathcal{T}}[n; a_1, \ldots, a_r]|$, since the formula for $r = 1$ is already complicated [12].

Let us also mention that, for the case of triangulations with a single (simple) boundary, there are nice factorized formulas when adding a girth constraint (the girth is at most 3 due to the non-boundary faces having degree 3): in girth at least 2 (loopless) a formula is given in [13] (with a bijective proof in [15]), and in girth 3 a formula is given in [8] (with bijective proofs in [16, 2, 1]). However it is not known if these formulas extend to more than one boundary, and we have not succeeded in applying our method to deal with more than one boundary under a girth constraint.

This article is organized as follows. In Section 2 we set our definitions about maps, and adapt the *master bijection* established in [2] to maps with boundaries. In Section 3, we define canonical orientations for quadrangulations with boundaries, and obtain a bijection with a class of trees called *mobiles* (the case where one of the boundary has size 2 is simpler, while the general case requires to first cut the map into two pieces). In Section 4 we treat similarly the case of triangulations. In Section 5, we count mobiles and obtain (1) and (2); we also derive from our formulas (both for coefficients and generating functions) exact solutions of the dimer model on rooted quadrangulations and triangulations. In Section 6, we prove the existence and uniqueness of the needed canonical orientations for triangulations and quadrangulations with boundaries. Lastly, in Section 7, we discuss additional results and perspectives.

2. Maps and the master bijection

In this section we set our definitions about maps and orientations, recall the master bijection established in [2], and adapt it to maps with boundaries.

2.1. **Maps and weighted biorientations.** A *map* is a decomposition of the 2D sphere into *vertices* (points), *edges* (homeomorphic to open segments), and *faces* (homeomorphic to open disks), considered up to continuous deformation. A map can equivalently be defined as a drawing (without edge crossings) of a connected graph, considered up to continuous deformation. Each edge of a map is thought as made of two *half-edges* that meet in its middle. A *corner* is the region between two consecutive half-edges around a vertex. The *degree* of a vertex or face x, denoted $\deg(x)$, is the number of incident corners. A *rooted map* is a map with a marked corner c_0; the incident vertex v_0 is called the root vertex, and the half-edge (resp. edge) following c_0 in clockwise order around v_0 is called the *root half-edge* (resp. *root edge*). A map is said to be *bipartite* if the underlying graph is bipartite, which happens precisely when every face has even degree. A *plane map* is a map with a face distinguished as its *outer face*. We think about plane maps, as drawn in the plane, with the outer face being the infinite face. The non-outer faces are called *inner faces*; vertices and edges are called *outer* or *inner* depending on whether they are incident to the outer face or not. The degree of the outer face is called the *outer degree*.

A *biorientation* of a map M is the assignment of a direction to each half-edge of M, that is, each half-edge is either outgoing or ingoing at its incident vertex. For $i \in \{0, 1, 2\}$, an edge is called i-way if it has i ingoing half-edges. An *orientation* is a biorientation such that every edge is 1-way. If M is a plane map endowed with a biorientation, then a *ccw cycle* (resp. *cw-cycle*) of M is a simple cycle C of edges of M such that each edge of C is either 2-way or 1-way with the interior of C on its left (resp. on its right). The biorientation is called *minimal* if there is no ccw cycle, and *almost-minimal* if the only ccw cycle is the outer face contour (in which case the outer face contour must be a simple cycle). For u, v two vertices of M, v is said to be *accessible* from u if there is a path $P = u_0, u_1, \ldots, u_k$ of vertices of M such that $u_0 = u$, $u_k = v$, and for $i \in [1..k-1]$, the edge (u_i, u_{i+1}) is either 1-way from u_i to u_{i+1} or 2-way. The biorientation is said to be *accessible* from u if every vertex of M is accessible from u. A *weighted biorientation* of M is a biorientation of M where each half-edge is assigned a weight (in some additive group). A $\mathbb{Z}$-*biorientation* is a weighted biorientation such that weights at ingoing half-edges are positive integers, while weights at outgoing half-edges are non-positive integers.

2.2. **Master bijection for $\mathbb{Z}$-bioriented maps.** We first define the families of bioriented maps involved in the master bijection. Let d be a positive integer. We define $\mathcal{O}_d$ as the family of plane maps of outer degree d endowed with a $\mathbb{Z}$-biorientation which is minimal and accessible from every outer vertex, and such that every outer edge is either 2-way or is 1-way with an inner face on its right [2]. We define $\mathcal{O}_{-d}$ as the family of plane maps of outer degree d endowed with a $\mathbb{Z}$-biorientation which is almost-minimal and accessible from every outer vertex, and such that outer edges are 1-way with weights $(0, 1)$, and each inner half-edge incident to an outer vertex is outgoing.

Next, we define the families of trees involved in the master bijection. We call *mobile* an unrooted plane tree with two kinds of vertices, black vertices and white vertices (vertices of the same types can be adjacent), where each corner at a black vertex possibly carries additional dangling half-edges called *buds*; see Figure 3 (right) for an example. The *excess* of a mobile is defined as the number of half-edges incident to a white vertex, minus the number of buds. A *weighted mobile* is a mobile where each half-edge, except for buds, is assigned a weight. A $\mathbb{Z}$-mobile is a weighted mobile such that weights of half-edges incident to white vertices are positive integers, while weights at half-edges incident to black vertices are non-positive integers. For $d \in \mathbb{Z}$, we denote by $\mathcal{B}_d$ the family of $\mathbb{Z}$-mobiles of excess d.

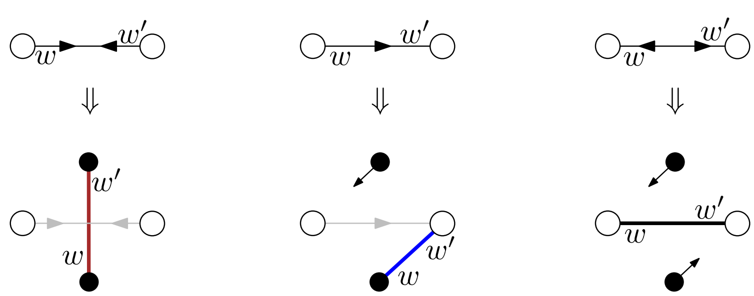


FIGURE 2. The local rule performed at each edge (0-way, 1-way or 2-way) in the master bijection Φ.

Let $d \in \mathbb{Z}\backslash\{0\}$. We now recall the master bijection Φ introduced in [2] between $\mathcal{O}_d$ and $\mathcal{B}_d$. For $O \in \mathcal{O}_d$, we obtain a mobile $T \in \mathcal{B}_d$ by the following steps (see Figure 3 for examples):

[2]By a slight abuse of notation, we will often use the terminology "biorientation" to denote a map endowed with a biorientation.

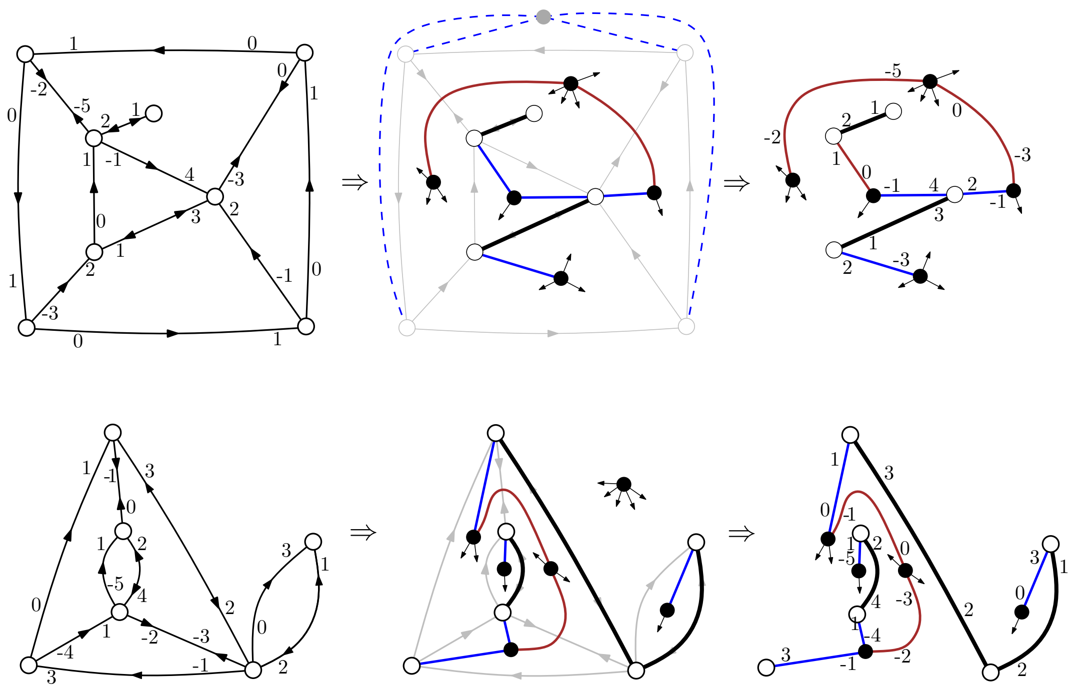


FIGURE 3. The master bijection from a plane $\mathbb{Z}$-bioriented plane map in $\mathcal{O}_d$ to a $\mathbb{Z}$-mobile of excess d (the top example has $d = -4$, the bottom-example has $d = 5$).

(1) insert a black vertex in each face (including the outer face) of O;
(2) apply the local rule of Figure 2 (which involves a transfer of weights) to each edge of O;
(3) erase the original edges of O and the black vertex b inserted in the outer face of O; if $d > 0$ erase also the d buds at b, if $d < 0$ erase also the $|d|$ outer vertices of O and the $|d|$ edges from b to each of the outer vertices.

Theorem 3 ([2])**.** *For $d \in \mathbb{Z}\backslash\{0\}$, the mapping Φ is a bijection between $\mathcal{O}_d$ and $\mathcal{B}_d$.*

The master bijection has the nice property that several parameters of a $\mathbb{Z}$-biorientation $O \in \mathcal{O}_d$ can be read on the associated $\mathbb{Z}$-mobile $T = \Phi(O)$. We define the *weight* (resp. the *indegree*) of a vertex $v \in O$ as the total weight (resp. total number) of ingoing half-edges at v, and we define the *weight* of a face $f \in O$ as the total weight of the outgoing half-edges having f on their right. For a vertex $v \in T$, we define the *degree* of v as the number of half-edges incident to v (including buds if v is black), and we define the *weight* of v as the total weight of the half-edges (excluding buds) incident to v. It is easy to see that if $O \in \widehat{\mathcal{O}}_d$ and $T = \Phi(O)$, then

- every inner face of O corresponds to a black vertex in T of same degree and same weight,
- for $d > 0$ (resp. $d < 0$), every vertex (resp. every inner vertex) $v \in O$ corresponds to a white vertex $v' \in T$ of the same weight and such that the indegree of v equals the degree of v'.

2.3. **Adaptation to maps with boundaries.** A face f of a map is said to be *simple* if the number of vertices incident to f is equal to the degree of f (in other words there is no pair of corners of f incident to the same vertex). A *map with boundaries* is a map M where the set of faces is partitioned into two subsets: *boundary faces* and *internal faces*, with the constraint that the boundary faces are simple, and the contours of any two boundary faces are vertex-disjoint; these contour-cycles are called the *boundaries* of M. Edges (and similarly half-edges and vertices) are called boundary edges or internal edges depending on whether they are on a boundary or not. If M is a *plane* map with boundaries, whose outer face is a boundary face, then the contour of the outer face is called the *outer boundary* and the contours of the other boundary faces are called *inner boundaries*.

For M a map with boundaries, a $\mathbb{Z}$-*biorientation* of M is called *consistent* if the boundary edges are all 1-way with weights $(0,1)$ and have the incident boundary face on their right. For $d \in \mathbb{Z}\backslash\{0\}$, denote by $\widehat{\mathcal{O}}_d$ the family of plane maps with boundaries endowed with a consistent $\mathbb{Z}$-biorientation, such that the outer face is a boundary face for $d < 0$ and an internal face for $d > 0$, and when forgetting which faces are boundary faces, the underlying $\mathbb{Z}$-bioriented plane map is in $\mathcal{O}_d$.

Define now a *boundary mobile* as a mobile where every corner at a white corner might carry additional dangling half-edges called *legs* (as buds, legs carry no weight). White vertices having at least one leg are called *boundary vertices*. The *degree* of a white vertex v is the number of non-leg half-edges incident to v. Define the *excess* of a boundary mobile as the number of half-edges incident to a white vertex (including the legs) minus the number of buds. For $d \in \mathbb{Z}$, denote by $\widehat{\mathcal{B}}_d$ the set of boundary $\mathbb{Z}$-mobiles of excess d (the constraint that non-leg half-edges at white vertices have positive weight holds also for boundary vertices).

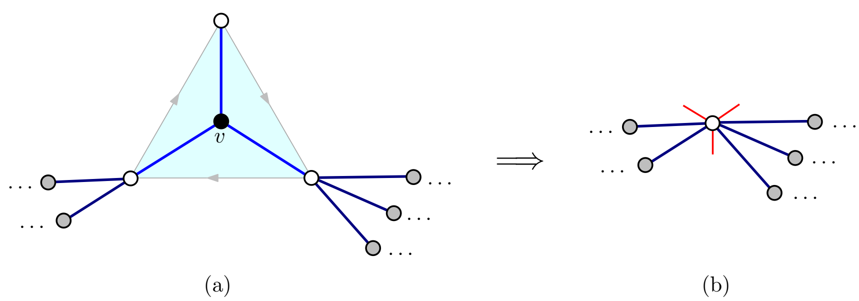


FIGURE 4. Reduction operation at the black vertex v corresponding to a boundary face in $O \in \widehat{\mathcal{O}}_d$.

We can now specialize the master bijection. For $O \in \widehat{\mathcal{O}}_d$, let $T = \Phi(O)$ be the associated $\mathbb{Z}$-mobile. Note that each inner boundary face f of O of degree k yields a black vertex v of degree k in T such that v has no bud, and the k neighbors $w_1, \ldots, w_k$ of v are the white vertices corresponding to the vertices around f. We perform the following operation represented in Figure 4: we insert one leg at each corner of v, then contract the edges incident to v, and finally recolor b as white. Doing this for each inner boundary we obtain (without loss of information) a boundary $\mathbb{Z}$-mobile T' of the same excess as T, called the *reduction* of T. We denote by $\widehat{\Phi}$ the mapping such that $\widehat{\Phi}(O) = T'$.

We now argue that $\widehat{\Phi}$ is a bijection between $\widehat{\mathcal{O}}_d$ and $\widehat{\mathcal{B}}_d$. For a boundary mobile T', the *expansion* of T' is the mobile T obtained from T' by applying to every boundary vertex the process of Figure 4 in reverse direction: a boundary vertex with k legs yields in T a distinguished black vertex of degree k with no buds, and with only white neighbors. Note that, if T' has non-zero excess d and if $O \in \mathcal{O}_d$ denotes the $\mathbb{Z}$-bioriented plane map associated to T by the master bijection, then each distinguished face $f \in O$ (i.e., a face associated to a distinguished black vertex of T) is simple; indeed if $k \geq 1$ denotes the degree of f, the corresponding black vertex $v \in T$ has k white neighbors, which thus correspond to k distinct vertices incident to f. In addition the contours of the distinguished inner faces are disjoint since the expansions of any two distinct boundary vertices of T' are vertex-disjoint in T. Lastly, for $d \in \mathbb{Z}_-^*$, the outer face is simple and disjoint from the contours of the inner distinguished faces (indeed the vertices around an inner distinguished face of O are all present in T, hence are inner vertices of O). We thus conclude that O belongs to $\widehat{\mathcal{O}}_d$, upon seeing the distinguished faces (including the outer face for $d \in \mathbb{Z}_-^*$) as boundary faces. The following statement summarizes the discussion:

Theorem 4. *The master bijection $\widehat{\Phi}$ adapted to consistent $\mathbb{Z}$-biorientations is a bijection between $\widehat{\mathcal{O}}_d$ and $\widehat{\mathcal{B}}_d$ for each $d \in \mathbb{Z}\backslash\{0\}$.*

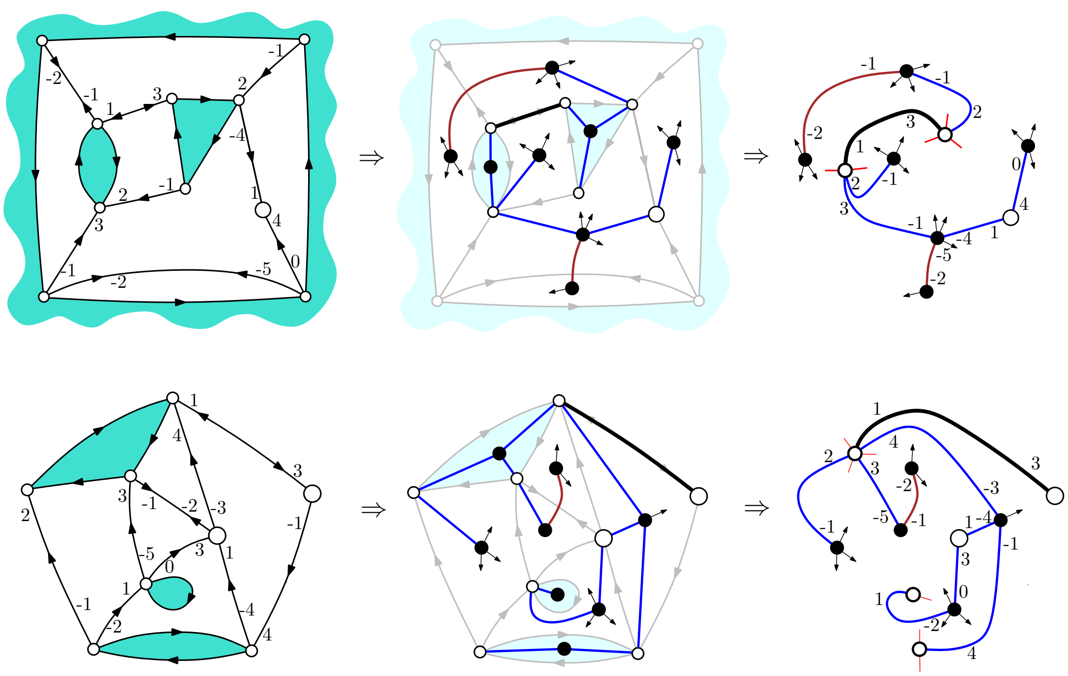


FIGURE 5. The master bijection $\widehat{\Phi}$ applied to two $\mathbb{Z}$-bioriented plane maps in $\widehat{\mathcal{O}}_d$, with $d = -4$ for the top-example, and $d = 5$ for the bottom-example. The weights of boundary-edges, which are always $(0, 1)$ by definition, are not indicated.

The bijection $\widehat{\Phi}$ is illustrated in Figure 5. As before several parameters can be tracked through the bijection. For a map M with boundaries endowed with a consistent $\mathbb{Z}$-biorientation, we define the *weight* (resp. the *indegree*) of a boundary C as the total weight (resp. total number) of ingoing half-edges incident to a vertex of C but not lying on an edge of C. For a boundary $\mathbb{Z}$-mobile, we define the *weight* of a white vertex v as the total weight of the half-edges (excluding legs) incident to v. It is easy to see that if $O \in \widehat{\mathcal{O}}_d$ and $T = \widehat{\Phi}(O)$, then

- every internal inner face of O corresponds to a black vertex in T of same degree and same weight,
- every internal vertex $v \in O$ corresponds to a non-boundary white vertex $v' \in T$ of the same weight and such that the indegree of v equals the degree of v',
- every inner boundary of length k, indegree r, and weight j in O corresponds to a boundary vertex in T with k legs, degree r, and weight j.

3. Bijection for quadrangulations with boundaries

In this section we obtain bijections for *quadrangulations with boundaries*, that is, maps with boundaries such that every internal face has degree 4.

3.1. **Quadrangulation with at least one boundary of length 2.**

We denote by $\mathcal{D}_\diamond$ the class of bipartite quadrangulations with boundaries, such that the outer face is a boundary face of degree 2. For $M \in \mathcal{D}_\diamond$, we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation with weights in $\{-1, 0, 1\}$ such that:

- every internal edge has weight 0 (hence is either 0-way with weights $(0, 0)$ or 1-way with weights $(-1, 1)$),
- every internal vertex has weight (and indegree) 1,
- every internal face (of degree 4) has weight -1 (hence has exactly one 1-way clockwise edge on its contour),
- every inner boundary of length $2i$ has weight (and indegree) $i + 1$, and the outer boundary, of length 2, has weight (and indegree) 0.

Proposition 1. *Every map $M \in \mathcal{D}_\diamond$ has a unique 1-orientation in $\widehat{\mathcal{O}}_{-2}$. We call it its* canonical biorientation.

The proof is delayed to Section 6.2. We denote by $\mathcal{T}_\diamond$ the set of boundary mobiles associated to maps in $\mathcal{D}_\diamond$ (endowed with their canonical biorientation) via the master bijection for maps with boundaries (Theorem 4). These are the boundary mobiles with weights in $\{-1, 0, 1\}$ satisfying the following properties:

- every edge has weight 0 (hence, is either black-black of weights $(0, 0)$, or black-white of weights $(-1, 1)$),
- every black vertex has degree 4 and weight -1, hence has a unique white neighbor,
- for all $i \geq 0$, every white vertex of degree $i + 1$ carries $2i$ legs.

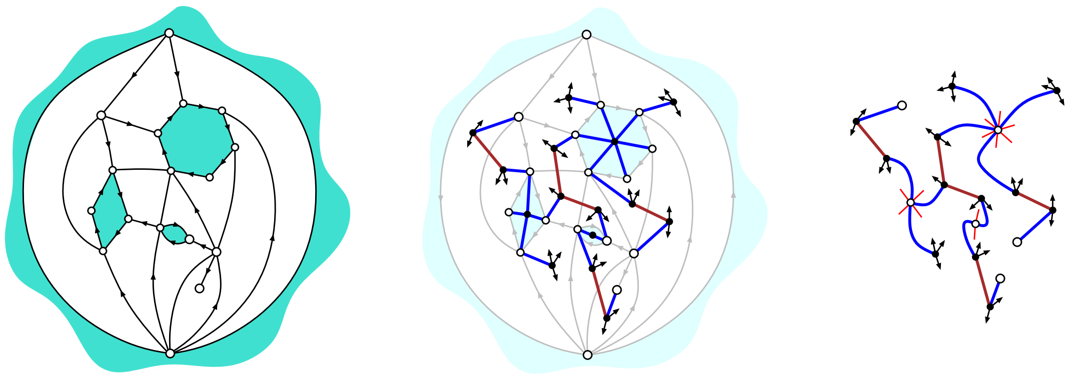
We omitted the condition that the excess is -2, because it can easily be checked to be a consequence of the above properties.

To summarize, Theorem 4 and Proposition 1 yield the following bijection (illustrated in Figure 6) for bipartite quadrangulations with a distinguished boundary of length 2.

Theorem 5. *The set $\mathcal{D}_\diamond$ is in bijection with the set $\mathcal{T}_\diamond$ via the master bijection $\widehat{\Phi}$. If $M \in \mathcal{D}_\diamond$ and $T \in \mathcal{T}_\diamond$ are associated by the bijection, then each inner boundary of length $2i$ in M corresponds to a white vertex in T of weight (and degree) $i + 1$, and each internal vertex of M corresponds to a white leaf in T.*

3.2. **Quadrangulations with arbitrary boundary lengths.**

For $a \geq 1$, we denote by $\mathcal{D}_\diamond^{(2a)}$ the set of bipartite quadrangulations with boundaries with a marked boundary face of degree $2a$. In the previous section we obtained a bijection for $\mathcal{D}_\diamond^{(2)}$. In order to get a bijection for $\mathcal{D}_\diamond^{(2a)}$ when $a > 1$, we will need to first mark an edge and decompose our marked maps



(a) (b) (c)

FIGURE 6. (a) A map in $\mathcal{D}_\diamond$ endowed with its canonical biorientation (where the 1-way edges are indicated as directed edges, the 0-way edges are indicated as undirected edges, and the weights, which are uniquely induced by the biorientation, are not indicated). (b) The quadrangulation superimposed with the corresponding mobile. (c) The reduced boundary mobile (with $2i$ legs at each white vertex of degree $i+1$), where again the weights (which are uniquely induced by the mobile) are not indicated.

into two pieces before applying the master bijection to each piece (such an approach was already used in [3, 4]).

Let $\overline{\mathcal{D}}_\diamond^{(2a)}$ be the set of maps obtained from maps in $\mathcal{D}_\diamond^{(2a)}$ by also marking an edge (either an internal edge or a boundary edge). Let $\mathcal{A}_\diamond^{(2a)}$ be the set of bipartite maps with 2 marked faces – a marked boundary face of degree $2a$ and a marked internal face of degree 2 – such that all the non-marked internal faces have degree 4. Given a map M in $\overline{\mathcal{D}}_\diamond^{(2a)}$, we obtain a map M' in $\mathcal{A}_\diamond^{(2a)}$ by "opening" the marked edge into an internal face of degree 2. This is clearly a bijection (preserving the number of internal vertices and the boundary lengths) for all $a > 1$, so that

$$\overline{\mathcal{D}}_\diamond^{(2a)} \simeq \mathcal{A}_\diamond^{(2a)}. \tag{3}$$

Note however that $\mathcal{A}_\diamond^{(2)}$ contains a map ϵ with 2 edges (a 2-cycle separating a boundary and an internal face) which is not obtained from a map in $\overline{\mathcal{D}}_\diamond^{(2)}$, so that $\overline{\mathcal{D}}_\diamond^{(2)} \cup \{\epsilon\} \simeq \mathcal{A}_\diamond^{(2)}$. We also denote $\vec{\mathcal{A}}_\diamond^{(2a)}$ the set of maps obtained from $\mathcal{A}_\diamond^{(2a)}$ by marking a corner in the marked boundary face.

We will now describe a canonical decomposition of maps in $\mathcal{A}_\diamond^{(2a)}$ illustrated in Figure 7(a)-(b). Let M be in $\mathcal{A}_\diamond^{(2a)}$, let C be a simple cycle of M, and let R_C and L_C be the regions bounded by C containing f_s and not containing f_s respectively. The cycle C is said to be *blocking* if the marked internal face is in L_C, and any boundary face incident to a vertex of C is in R_C. Note that the contour of the marked internal face is a blocking 2-cycle. It is easy to see that there exists a unique blocking 2-cycle C for which the region $C \cup R_C$ contains no other blocking 2-cycle; we call it the *canonical cycle* of M. The canonical cycle is indicated in Figure 7(a). The map M is called *reduced* if its canonical cycle is the contour of the marked internal face, and we denote by $\mathcal{B}_\diamond^{(2a)}$ and $\vec{\mathcal{B}}_\diamond^{(2a)}$ the subsets of $\mathcal{A}_\diamond^{(2a)}$ and $\vec{\mathcal{A}}_\diamond^{(2a)}$ corresponding to reduced maps.

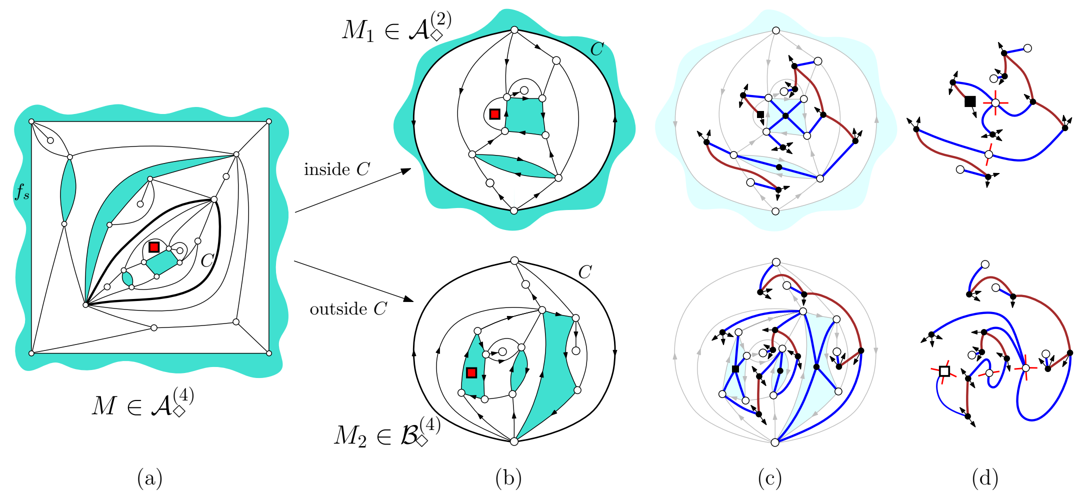


FIGURE 7. (a) A map in $\mathcal{A}_\diamond^{(4)}$: the marked boundary face is the outer face, the marked internal face is indicated by a square, and the canonical cycle C is drawn in bold. (b) The maps M_1 and M_2 resulting from cutting M along C, each represented as a plane map endowed with its canonical biorientation (the marked inner face in each case is indicated by a square). (c) The mobiles associated to M_1 and M_2. (d) The reduced boundary mobiles associated to M_1 and M_2, where the marked vertex (corresponding to the marked inner face) is indicated by a square.

We now consider the two maps obtained from a map M in $\mathcal{A}_\diamond^{(2a)}$ by cutting the sphere along the canonical 2-cycle C. We denote by M_1 the map in $\mathcal{A}_\diamond^{(2)}$ obtained from $C \cup L_C$ by considering the "hole" bounded by C as a marked boundary face of degree 2. We denote by M_2 be the map in $\mathcal{A}_\diamond^{(2a)}$ obtained from $C \cup R_C$ by considering the "hole" bounded by C as a marked internal face. Note that M_2 is necessarily reduced. Conversely, if we glue the marked boundary face of a map $N_1 \in \mathcal{A}_\diamond^{(2)}$ to the marked internal face of a reduced map $N_2 \in \mathcal{A}_\diamond^{(2a)}$, we obtain a map $M \in \mathcal{A}_\diamond^{(2a)}$ whose canonical cycle is the contour of the glued faces, so that $N_1 = M_1$ and $N_2 = M_2$. In order to make the preceding decomposition bijective, it is convenient to work with *rooted* maps. Given a map M in $\vec{\mathcal{A}}_\diamond^{(2a)}$, we define M_1 and M_2 as above, except that we mark a corner in the newly created boundary face of M_1. In order to fix a convention, we choose the corner of M_1 such that the vertices incident to the marked corners of M_1 and M_2 are in the same block of the bipartition of the vertices of M. The decomposition $M \mapsto (M_1, M_2)$ is now bijective and yields:

$$\vec{\mathcal{A}}_\diamond^{(2a)} \simeq \vec{\mathcal{A}}_\diamond^{(2)} \times \vec{\mathcal{B}}_\diamond^{(2a)}. \tag{4}$$

Next, we describe bijections for maps in $\vec{\mathcal{A}}_\diamond^{(2)}$ and $\vec{\mathcal{B}}_\diamond^{(2a)}$ by using a "master bijection" approach illustrated in Figure 7(b)-(d). For $M \in \mathcal{A}_\diamond^{(2)}$, we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation of M with weights in $\{-1, 0, 1\}$ such that:

- every internal edge has weight 0,

- every internal vertex has indegree 1,
- every non-marked internal face (of degree 4) has weight -1, while the marked internal face (of degree 2) has weight 0,
- every non-marked boundary of length $2i$ has weight (and indegree) $i+1$, while the marked boundary (of length 2) has weight (and indegree) 0.

Proposition 2. *Let M be a map in $\mathcal{A}_\diamond^{(2)}$ considered as a plane map by taking the outer face to be the marked boundary face. Then M admits a unique 1-orientation in $\widehat{\mathcal{O}}_{-2}$. We call it the* canonical biorientation *of M.*

Proof. This is a corollary of Proposition 1. Indeed, seeing M as a map D in $\mathcal{D}_\diamond$ where an edge e is opened into an internal face f_1 of degree 2, the canonical biorientation of M is directly derived from the canonical biorientation of D, using the rules shown in Figure 8. □

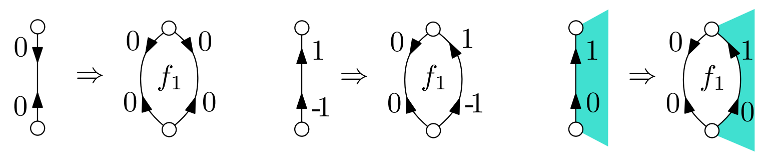


FIGURE 8. Transferring the biorientations and weights when blowing an edge into an internal face of degree 2.

For $M \in \mathcal{A}_\diamond^{(2a)}$, we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation with weights in $\{-1, 0, 1\}$ such that:

- every internal edge has weight 0,
- every internal vertex has weight (and indegree) 1,
- every non-marked internal face (of degree 4) has weight -1, while the marked internal face (of degree 2) has weight 0,
- every non-marked boundary of length $2i$ has weight (and indegree) $i+1$, while the marked boundary (of length $2a$) has weight (and indegree) $a-1$.

Proposition 3. *Let M be a map in $\mathcal{A}_\diamond^{(2a)}$ considered as a plane map by taking the outer face to be the marked internal face. Then M has a 1-orientation in $\widehat{\mathcal{O}}_2$ if and only if it is reduced (i.e., is in $\mathcal{B}_\diamond^{(2a)}$). In this case, M has a unique 1-orientation in $\widehat{\mathcal{O}}_2$, which we call the* canonical biorientation *of M.*

The proof is delayed to Section 6.3.

We denote by $\mathcal{U}_\diamond$ the family of mobiles corresponding to (canonically oriented) maps in $\mathcal{A}_\diamond^{(2)}$ via the master bijection. These are the boundary $\mathbb{Z}$-mobiles with weights in $\{-1, 0, 1\}$ satisfying the following properties (which imply that the excess is -2):

- every edge has weight 0 (hence, is either black-black of weights $(0, 0)$, or black-white of weights $(-1, 1)$),
- every black vertex has degree 4 and weight -1 (hence has a unique white neighbor), except for a unique black vertex of degree 2 and weight 0,
- for all $i \geq 0$, every white vertex of degree $i+1$ carries $2i$ legs.

We also denote $\vec{\mathcal{U}}_\diamond$ the set of mobiles obtained from mobiles in $\mathcal{U}_\diamond$ by marking one of the corners of the black vertex of degree 2.

For $a \geq 1$, we denote by $\mathcal{V}_{\diamond}^{(2a)}$ the family of mobiles corresponding to (canonically oriented) maps in $\mathcal{B}_{\diamond}^{(2a)}$. These are the boundary $\mathbb{Z}$-mobiles with weights in $\{-1,0,1\}$ satisfying the following properties (which imply that the excess is 2):

- every edge has weight 0 (hence is either black-black of weights $(0,0)$, or black-white of weights $(-1,1)$),
- every black vertex has degree 4 and weight -1 (hence has a unique white neighbor),
- there is a marked white vertex of degree $a-1$ which carries $2a$ legs,
- for all $i \geq 0$, every non-marked white vertex of degree $i+1$ carries $2i$ legs.

We also denote $\vec{\mathcal{V}}_{\diamond}^{(2a)}$ the family of rooted mobiles obtained from from mobiles in $\mathcal{V}_{\diamond}^{(2a)}$ by marking one of the $2a$ legs of the marked white vertex.

Propositions 2 and 3 together with the master bijection (Theorem 4) then yield:

Theorem 6. *The family $\mathcal{A}_{\diamond}^{(2)}$ (resp. $\vec{\mathcal{A}}_{\diamond}^{(2)}$) is in bijection with the family $\mathcal{U}_{\diamond}$ (resp. $\vec{\mathcal{U}}_{\diamond}$). The bijection is such that if the map M corresponds to the mobile T, then each inner boundary of length $2i$ in M corresponds to a white vertex in T of weight (and degree) $i+1$, and each internal vertex of M corresponds to a white leaf in T.*

Similarly, for all $a \geq 1$, the family $\mathcal{B}_{\diamond}^{(2a)}$ (resp. $\vec{\mathcal{B}}_{\diamond}^{(2a)}$) is in bijection with the family $\mathcal{V}_{\diamond}^{(2a)}$ (resp. $\vec{\mathcal{V}}_{\diamond}^{(2a)}$). The bijection is such that if the map M corresponds to the mobile T, then each non-marked boundary of length $2i$ in M corresponds to a non-marked white vertex in T of weight (and degree) $i+1$, and each internal vertex of M corresponds to a non-marked white leaf in T.

4. Bijection for triangulations with boundaries

In this section we adapt the strategy of Section 3 to triangulations with boundaries.

4.1. **Triangulations with at least one boundary of length 1.** Let $\mathcal{D}_{\Delta}$ be the set of plane triangulations with boundaries, such that the outer face is a boundary face of degree 1. For $M \in \mathcal{D}_{\Delta}$, we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation with weights in $\{-2,-1,0,1\}$ and with the following properties:

- every internal edge has weight -1 (i.e., is either 0-way of weights $(-1,0)$, or 1-way of weights $(-2,1)$),
- every internal vertex has weight 1,
- every internal face has weight -2,
- every inner boundary of length i has weight (and indegree) $i+1$, and the outer boundary has weight 0.

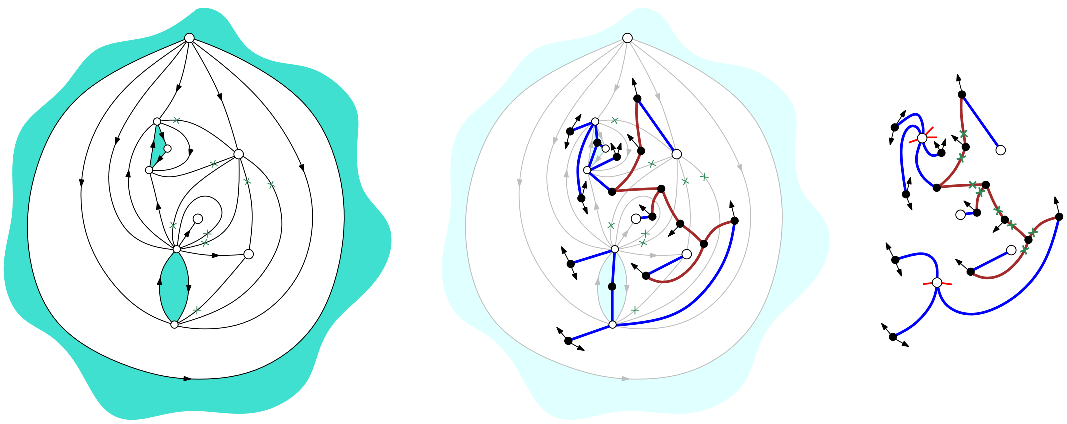
Similarly as in Section 3.1 we have the following proposition proved in Section 6.4.

Proposition 4. *Every $M \in \mathcal{D}_{\Delta}$ has a unique 1-orientation in $\widehat{\mathcal{O}}_{-1}$. We call it the* canonical biorientation *of M.*

We denote by $\mathcal{T}_{\Delta}$ the family of mobiles corresponding to (canonically oriented) maps in $\mathcal{D}_{\Delta}$ via the master bijection. These are the boundary $\mathbb{Z}$-mobiles with weights in in $\{-2,-1,0,1\}$ satisfying the following properties (which imply that the excess is -1):

- every edge has weight -1 (hence is either black-black of weights $(-1,0)$, or is black-white of weights $(-2,1)$),
- every black vertex has degree 3 and weight -2,
- for all $i \geq 0$, every white vertex of degree $i+1$ carries i legs.

To summarize, we obtain the following bijection for triangulations with a boundary of length 1 (see Figure 9 for an example):



(a) (b) (c)

FIGURE 9. (a) A triangulation in $\mathcal{D}_\triangle$ endowed with its canonical biorientation, where crosses indicate half-edges of weight -1 (1-way edges have weights $(-1,2)$ if internal and weights $(0,1)$ if boundary, 0-way edges have weights $(-1,0)$). (b) The triangulation superimposed with the corresponding mobile. (c) The reduced boundary mobile (with i legs at each white vertex of degree $i+1$, and with again the convention that half-edges of weight -1 are indicated by a cross).

Theorem 7. *The family $\mathcal{D}_\triangle$ is in bijection with the family $\mathcal{T}_\triangle$ via the master bijection. If $M \in \mathcal{D}_\triangle$ and $T \in \mathcal{T}_\triangle$ are associated by the bijection, then each inner boundary of length i in M corresponds to a white vertex in T of degree $i+1$, and each internal vertex of M corresponds to a white leaf in T.*

4.2. **Triangulations with arbitrary boundary lengths.** We now adapt the approach of Section 3.2 (decomposing maps in two pieces) to triangulations. For $a \geq 1$, we denote by $\mathcal{D}_\triangle^{(a)}$ the set of triangulations with boundaries with a marked boundary face of degree a. We also denote by $\overline{\mathcal{D}}_\triangle^{(a)}$ the set of maps obtained from maps in $\mathcal{D}_\triangle^{(a)}$ by also marking an arbitrary half-edge (either boundary or internal). Lastly, we denote by $\mathcal{A}_\triangle^{(a)}$ the set of maps with boundaries having 2 marked faces – a marked boundary face of degree a and a marked internal face of degree 1 – such that all the non-marked internal faces have degree 3.

Given a map M in $\overline{\mathcal{D}}_\triangle^{(a)}$, we obtain a map M' in $\mathcal{A}_\triangle^{(a)}$ by the operation illustrated in Figure 10. In words, we "open" the edge containing the marked half-edge h into a face f, and then at the corner of f corresponding to h we insert a loop bounding the marked internal face (of degree 1). This is clearly a bijection (preserving the number of internal vertices and the boundary lengths) for all $a > 1$ so that

$$\overline{\mathcal{D}}_\triangle^{(a)} \simeq \mathcal{A}_\triangle^{(a)}. \tag{5}$$

Note however that $\mathcal{A}_\triangle^{(1)}$ contains a map λ with 1 edges (a loop separating a boundary and an internal face) which is not obtained from a map in $\overline{\mathcal{D}}_\triangle^{(1)}$, so that $\overline{\mathcal{D}}_\triangle^{(1)} \cup \{\lambda\} \simeq \mathcal{A}_\triangle^{(1)}$. We also denote $\vec{\mathcal{A}}_\triangle^{(a)}$ the set of maps obtained from $\mathcal{A}_\triangle^{(2a)}$ by marking a corner in the marked boundary face.

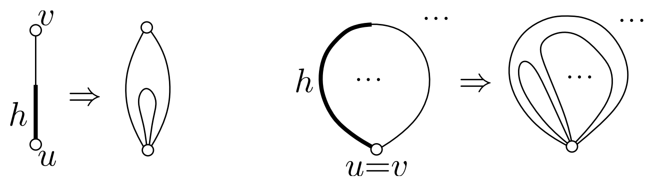


Figure 10. The operation of opening an half-edge h: it yields a new face of degree 1 surrounded by a new face of degree 3 (the case where h is on a loop is illustrated on the right).

Next we describe a canonical decomposition of maps in $\mathcal{A}_{\Delta}^{(a)}$ illustrated in Figure 11(a)-(b). For a 1-cycle C of a map $M \in \mathcal{A}_{\Delta}^{(a)}$ we denote by R_C and L_C the regions bounded by C containing and not containing the marked boundary face f_s respectively. The 1-cycle C is said to be *blocking* if the marked internal face is in L_C, and any boundary face incident to the vertex of C is in R_C. It is easy to see that there exists a unique blocking 1-cycle C for which the region $C \cup R_C$ contains no other blocking 1-cycle; we call it the *canonical cycle* of M. The canonical cycle is indicated in Figure 11(a). The map M is called *reduced* if its canonical cycle is the contour of the marked internal face, and we denote by $\mathcal{B}_{\Delta}^{(a)}$ and $\vec{\mathcal{B}}_{\Delta}^{(a)}$ the subsets of $\mathcal{A}_{\Delta}^{(a)}$ and $\vec{\mathcal{A}}_{\Delta}^{(a)}$ corresponding to reduced maps.

We now consider the two maps obtained from a map M in $\mathcal{A}_{\Delta}^{(a)}$ by cutting the sphere along the canonical 1-cycle C. We denote by M_1 the map in $\mathcal{A}_{\Delta}^{(1)}$ obtained from $C \cup L_C$ by considering the "hole" bounded by C as a marked boundary face of degree 2. We denote by M_2 the map in $\mathcal{B}_{\Delta}^{(a)}$ obtained from $C \cup R_C$ by considering the "hole" bounded by C as a marked internal face. The decomposition $M \mapsto (M_1, M_2)$ is bijective (both for rooted and unrooted maps because $\vec{\mathcal{A}}_{\Delta}^{(1)} \simeq \mathcal{A}_{\Delta}^{(1)}$): for all $a \geq 1$,

$$\vec{\mathcal{A}}_{\Delta}^{(a)} \simeq \mathcal{A}_{\Delta}^{(1)} \times \vec{\mathcal{B}}_{\Delta}^{(a)}. \tag{6}$$

Next, we describe bijections for maps in $\mathcal{A}_{\Delta}^{(1)}$ and $\mathcal{B}_{\Delta}^{(a)}$ by using the master bijection approach, as illustrated in Figure 11(b)-(d). For $M \in \mathcal{A}_{\Delta}^{(1)}$, we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation of M with weights in $\{-2, -1, 0, 1\}$ such that:

- every internal edge has weight -1,
- every internal vertex has weight (and indegree) 1,
- every non-marked internal face (of degree 3) has weight -2, and the marked internal face (of degree 1) has weight 0,
- every non-marked boundary of length i has weight (and indegree) $i+1$, and the marked boundary has weight (and indegree) 0.

As an easy consequence of Proposition 4 (similarly as Proposition 2 following easily from Proposition 1), we have:

Proposition 5. *Let M be a map in $\mathcal{A}_{\Delta}^{(1)}$, considered as a plane map by taking the marked boundary face as the outer face. Then M admits a unique 1-orientation in $\widehat{\mathcal{O}}_{-1}$. We call it the* canonical biorientation *of M.*

For $M \in \mathcal{A}_{\Delta}^{(a)}$ we call *1-orientation* of M a consistent $\mathbb{Z}$-biorientation with weights in $\{-2, -1, 0, 1\}$ such that:

- every internal edge has weight -1,
- every internal vertex has weight (and indegree) 1,

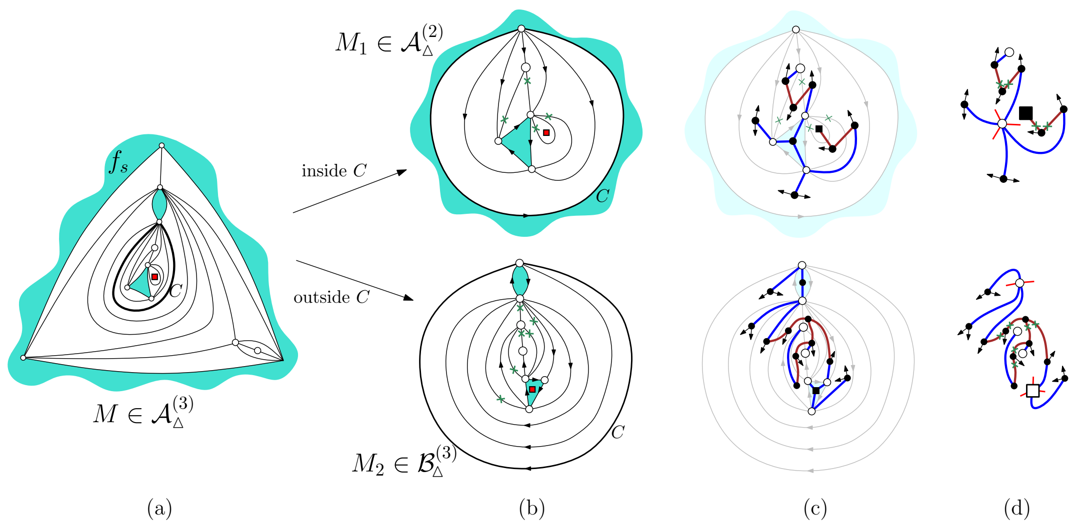


FIGURE 11. (a) A map in $\mathcal{A}_\Delta^{(3)}$: the marked boundary face is the outer face, the marked internal face is indicated by a square, and the canonical 1-cycle C is drawn in bold. (b) The maps M_1 and M_2 resulting from cutting M along C, each represented as a plane map endowed with its canonical biorientation (the marked inner face in each case is indicated by a square, each directed edge has weights $(-2, 1)$ and each undirected edge has weights $(-1, 0)$, with a cross on the half-edge of weight -1). (c) The mobiles associated to M_1 and M_2. (d) The reduced boundary mobiles associated to M_1 and M_2, where the marked vertex is represented by a square. Black-white edges have weights $(-2, 1)$, and black-black edges have weights $(-1, 0)$, with a cross on the half-edge of weight -1.

- every internal inner face (of degree 3) has weight -2, and the internal outer face (of degree 1) has weight 0.
- every non-marked boundary of length i has weight (and indegree) $i + 1$, while the marked boundary of length a, has weight (and indegree) $a - 1$.

Proposition 6. *Let M be a map in $\mathcal{A}_\Delta^{(a)}$ considered as a plane map by taking the outer face to be the marked internal face. Then M has a 1-orientation in $\widehat{\mathcal{O}}_1$ if and only if it is reduced (i.e., in $\mathcal{B}_\Delta^{(a)}$). In this case, M has a unique 1-orientation in $\widehat{\mathcal{O}}_1$, which we call the* canonical biorientation *of M.*

Again the proof is delayed to Section 6.5.

We denote by $\mathcal{U}_\Delta$ the family of mobiles corresponding to (canonically oriented) maps in $\mathcal{A}_\Delta^{(1)}$ via the master bijection. These are the boundary $\mathbb{Z}$-mobiles with weights in $\{-2, -1, 0, 1\}$ satisfying the following properties (which imply that the excess is -1):

- every internal edge has weight -1 (hence is either black-black of weights $(-1, 0)$, or black-white of weights $(-2, 1)$),
- every black vertex has degree 3 and weight -2, except for a unique black vertex of degree 1 and weight 0,
- for all $i \geq 0$, every white vertex of degree $i + 1$ carries i legs.

For $a \geq 1$, we denote by $\mathcal{V}_{\triangle}^{(a)}$ the family of mobiles corresponding to (canonically oriented) maps in $\mathcal{B}_{\triangle}^{(a)}$. These are the boundary $\mathbb{Z}$-mobiles with weights in $\{-2,-1,0,1\}$ satisfying the following properties (which imply that the excess is 1):

- every internal edge has weight -1
- every black vertex has degree 3 and weight -2,
- there is a marked white vertex of degree $a-1$ which carries a legs,
- for all $i \geq 0$, every non-marked white vertex of degree $i+1$ carries i legs.

We also denote $\vec{\mathcal{V}}_{\triangle}^{(a)}$ the family of mobiles obtained from from mobiles in $\mathcal{V}_{\triangle}^{(a)}$ by marking one of the a legs of the marked white vertex. Propositions 2 and 6 together with the master bijection (Theorem 4) then give:

Theorem 8. *The family $\mathcal{A}_{\triangle}^{(1)}$ is in bijection with the family $\mathcal{U}_{\triangle}$. The bijection is such that if the map M corresponds to the mobile T, then each inner boundary of length i in M corresponds to a white vertex in T of weight (and degree) $i+1$, and each internal vertex of M corresponds to a white leaf in T.*

Similarly, for all $a \geq 1$, the family $\mathcal{B}_{\triangle}^{(a)}$ (resp. $\vec{\mathcal{B}}_{\triangle}^{(a)}$) is in bijection with the family $\mathcal{V}_{\triangle}^{(a)}$ (resp. $\vec{\mathcal{V}}_{\triangle}^{(a)}$). The bijection is such that if the map M corresponds to the mobile T, then each non-marked boundary of length i in M corresponds to a non-marked white vertex in T of weight (and degree) $i+1$, and each internal vertex of M corresponds to a non-marked white leaf in T.

5. Counting results

5.1. **Proof of Theorem 2 for quadrangulations with boundaries.** Define a *planted mobile of quadrangulated type* as a tree P obtained as one of the two connected components after cutting a mobile $T \in \mathcal{T}_{\diamond}$ in the middle of an edge e; the half-edge h of e that belongs to P is called the *root half-edge* of P, and the vertex incident to h is called the *root-vertex* of P. The *root-weight* of P is the weight of h in T. For $j \in \{-1,0,1\}$, denote by $R_j \equiv R_j(t;x_0,x_1,x_2,\ldots)$ the generating function of planted mobiles of quadrangulated type and of root-weight j, where t is conjugate to the number of buds, and x_i is conjugate to the number of white vertices of degree $i+1$ (with $2i$ additional legs) for $i \geq 0$. We also denote $R := t+R_0$. The decomposition of planted trees at the root easily implies that $\{R_{-1},R_0,R_1\}$ are specified by the equation-system

$$
\left\{\begin{array}{rcl} R_{-1} &=& R^3, \\ R_0 &=& 3R_1R^2, \\ R_1 &=& \sum_{i\geq 0}\binom{3i}{i}{R_{-1}}^i, \end{array}\right. \tag{7}
$$

where (for instance) the factor $\binom{3i}{i}$ in the 3rd line accounts for the number of ways to place the $2i$ legs when the root-vertex has degree $i+1$ (the root half-edge plus i children), and the factor 3 in the second line accounts for choosing which of the 3 children of the root-vertex is white.

This gives $R = t + 3\sum_{i\geq 0}\binom{3i}{i}R^{3i+2}$, or equivalently,

$$
R = t\phi(R), \quad \text{with } \phi(y) = \Big(1 - 3\sum_{i\geq 0}\binom{3i}{i}y^{3i+1}\Big)^{-1}. \tag{8}
$$

Let $U_{\diamond} \equiv U_{\diamond}(t;x_0,x_1,\ldots)$ be the generating function of mobiles in $\vec{\mathcal{U}}_{\diamond}$ with t conjugate to the number of buds and x_i conjugate to the number of white vertices of degree $i+1$ for $i \geq 0$. And for $a \geq 1$, let $V_{\diamond}^{(2a)} \equiv V_{\diamond}^{(2a)}(t;x_0,x_1,\ldots)$ be the generating function of mobiles in

$\vec{\mathcal{V}}_\diamond^{(2a)}$ with t conjugate to the number of buds and x_i conjugate to the number of non-marked white vertices of degree $i+1$ for $i \geq 0$. The decomposition at the marked vertex gives

$$U_\diamond = R^2, \qquad V_\diamond^{(2a)} = \binom{3a-2}{a-1} R_{-1}{}^{a-1} = \binom{3a-2}{a-1} R^{3a-3}.$$

Let $\vec{A}_\diamond^{(2a)} \equiv \vec{A}_\diamond^{(2a)}(x_0, x_1, \ldots)$ and $\vec{B}_\diamond^{(2a)} \equiv \vec{B}_\diamond^{(2a)}(x_0, x_1, \ldots)$ be the respective generating functions of $\vec{\mathcal{A}}_\diamond^{(2a)}$ and $\vec{\mathcal{B}}_\diamond^{(2a)}$, where x_0 is conjugate to the number of internal vertices and for $i \geq 1$, x_i is conjugate to the number of unmarked boundaries of length $2i$. Theorem 6 implies that

$$\vec{A}_\diamond^{(2)} = U_\diamond|_{t=1}, \qquad \vec{B}_\diamond^{(2a)} = V_\diamond^{(2a)}|_{t=1},$$

and Equation (4) implies that $\vec{A}_\diamond^{(2a)} = \vec{A}_\diamond^{(2)} \cdot \vec{B}_\diamond^{(2a)}$, so that we obtain

$$\vec{A}_\diamond^{(2a)} = \binom{3a-2}{a-1} R^{3a-1}|_{t=1}.$$

Now denote by $\beta_a(m; n_1, \ldots, n_h)$ the number of maps in $\vec{\mathcal{D}}_\diamond^{(2a)}$ with a distinguished corner in the marked boundary face, with m internal vertices, n_i non-marked boundaries of length $2i$ for $1 \leq i \leq h$, and no inner boundary of length larger than $2h$. The half total boundary length is $b = a + \sum_i i n_i$, the total number of boundaries is $r = 1 + \sum_i n_i$, and the number of edges is (by the Euler relation) $e = 3b + 2r + 2m - 4$, which is $3b + 2k$ with $k := r + m - 2$. Then (3) yields

$$e\, \beta_a(m; n_1, \ldots, n_h) = [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \vec{A}_\diamond^{(2a)} = \binom{3a-2}{a-1} [x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^{3a-1}|_{t=1}.$$

Since R is specified by $R = t\phi(R)$, with $\phi(y) = (1 - \sum_{i \geq 0} x_i \binom{3i}{i} y^{3i+1})^{-1}$, the Lagrange inversion formula [17, Thm 5.4.2] gives for any positive integers n, q,

$$[t^n] R^q = \frac{q}{n} [y^{n-q}] \phi(y)^n.$$

Note that $\widetilde{R} := R/t$ satisfies $\widetilde{R} = 1/(1 - \sum_{i \geq 0} x_i t^{3i+1} \binom{3i}{i} \widetilde{R}^{3i+1})$ hence for any $k \geq 1$,

$$\begin{aligned}
[x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^k|_{t=1} &= [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \widetilde{R}^k|_{t=1} \\
&= [t^{m + \sum_{i=1}^h (3i+1) n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}] \widetilde{R}^k \\
&= [t^{k + m + \sum_{i=1}^h (3i+1) n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}] R^k.
\end{aligned}$$

Hence, denoting $p := 3a - 1 + m + \sum_{i=1}^h (3i+1) n_i = m + r + 3b - 2 = k + 3b$, we have (using the Lagrange inversion formula from the 1st to the 2nd line):

$$\begin{aligned}
[x_0 x_1^{n_1} \cdots x_h^{n_h}] R^{3a-1}|_{t=1} &= [t^p x_0^m x_1^{n_1} \ldots x_h^{n_h}] R^{3a-1} \\
&= \frac{3a-1}{p} [x_0^m \cdots x_h^{n_h}] [y^{p-3a+1}] \Big(1 - 3 \sum_{i=0}^h x_i \binom{3i}{i} y^{3i+1}\Big)^{-p} \\
&= \frac{3a-1}{p} [x_0^m \cdots x_h^{n_h}] \Big(1 - 3 \sum_{i=0}^h x_i \binom{3i}{i}\Big)^{-p} \\
&= \frac{3a-1}{p} 3^{m+r-1} \binom{p-1+m+r-1}{p-1, m, n_1, \ldots, n_h} \prod_{i=1}^h \binom{3i}{i}^{n_i}
\end{aligned}$$

so that we obtain (using $k=m+r-2$, $p=k+3b$, $e=p+k$, and $(3a-1)\binom{3a-2}{a-1}=\frac{2}{3}a\binom{3a}{a}$)

$$\beta_a(m;n_1,n_2,\ldots,n_h)=3^k\frac{(e-1)!}{m!(k+3b)!}2a\binom{3a}{a}\prod_{i=1}^{h}\frac{1}{n_i!}\binom{3i}{i}^{n_i}, \tag{9}$$

which, multiplied by $\prod_{i=1}^h n_i!(2i)^{n_i}$ (to account for numbering the inner boundary faces and marking a corner in each of these faces), gives (2).

5.2. **Proof of Theorem 1 for triangulations with boundaries.** We proceed similarly as in Section 5.1. We call *planted mobile of triangulated type* any tree P equal to one of the two connected components obtained from some $T\in\mathcal{T}_\vartriangle$ by cutting an edge e in its middle; the half-edge h of e belonging to P is called the *root half-edge* of P, and the weight of h in T is called the *root-weight* of P. For $j\in\{-2,-1,0,1\}$, denote by $S_j\equiv S_j(t;x_0,x_1,\ldots)$ the generating function of planted mobiles of triangulated type and root-weight j, with t conjugate to the number of buds and x_i conjugate to the number of white vertices of degree $i+1$ for $i\geq 0$. We also denote $S:=t+S_{-1}$. The decomposition of planted trees at the root easily implies that $\{S_{-2},S_{-1},S_0,S_1\}$ are specified by the following equation-system:

$$\left\{\begin{array}{rcl} S_{-2} &=& S^2,\\ S_{-1} &=& 2SS_0,\\ S_0 &=& 2SS_1+S_0^2,\\ S_1 &=& \sum_{i\geq 0}x_i\binom{2i}{i}{S_{-2}}^i. \end{array}\right. \tag{10}$$

The second line (and $S=t+S_{-1}$) gives $S=t/(1-2S_0)$, so that the 3rd line gives $S_0(1-S_0)(1-2S_0)=2tS_1$. If we now define $A=1-2S_0$, we have $S_0=(1-A)/2$ and $1-S_0=(1+A)/2$, so that $A(1-A^2)=8tS_1$. Hence $A=8tS_1+A^3$, hence $X:=1/A$ satisfies $X^2=1+8tS_1X^3$. Since $S=tX$, S satisfies $S^2=t^2+8S_1S^3$, hence $S=t(1-8S_1S)^{-1/2}$, i.e., S satisfies the equation:

$$S=t\phi(S),\quad\text{where } \phi(y)=\Big(1-8\sum_{i\geq 0}x_i\binom{2i}{i}y^{2i+1}\Big)^{-1/2}. \tag{11}$$

Let $U_\vartriangle\equiv U_\vartriangle(t;x_0,x_1,\ldots)$ be the generating function of mobiles from $\mathcal{U}_\vartriangle$, with t conjugate to the number of buds and x_i conjugate to the number of white vertices of degree $i+1$ for $i\geq 0$. And for $a\geq 1$ let $V_\vartriangle^{(a)}\equiv V_\vartriangle^{(a)}(t;x_0,x_1,\ldots)$ be the generating function of mobiles in $\vec{\mathcal{V}}_\vartriangle^{(a)}$, with t conjugate to the number of buds and x_i conjugate to the number of non-marked white vertices of degree $i+1$ for $i\geq 0$. A decomposition at the marked vertex gives

$$U_\vartriangle=S,\qquad V_\vartriangle^{(a)}=\binom{2a-2}{a-1}{S_{-2}}^{a-1}=\binom{2a-2}{a-1}S^{2a-2}.$$

Let $\vec{A}_\vartriangle^{(a)}\equiv\vec{A}_\vartriangle^{(a)}(x_0,x_1,\ldots)$ and $\vec{B}_\vartriangle^{(a)}\equiv\vec{B}_\vartriangle^{(a)}(x_0,x_1,\ldots)$ be respectively the generating functions of $\vec{\mathcal{A}}_\vartriangle^{(a)}$, and $\vec{\mathcal{B}}_\vartriangle^{(a)}$, where x_0 is conjugate to the number of internal vertices and for $i\geq 1$, x_i is conjugate to the number unmarked boundaries of length i. Theorem 8 implies that

$$\vec{A}_\vartriangle^{(1)}=U_\vartriangle|_{t=1},\qquad \vec{B}_\vartriangle^{(a)}=V_\vartriangle^{(a)}|_{t=1},$$

and Equation 6 implies that $\vec{A}_\vartriangle^{(a)}=\vec{A}_\vartriangle^{(1)}\cdot\vec{B}_\vartriangle^{(a)}$, so that we obtain

$$\vec{A}_\vartriangle^{(a)}=\binom{2a-2}{a-1}S^{2a-1}|_{t=1}. \tag{12}$$

Define now $\eta_a(m;n_1,n_2,\ldots,n_h)$ as the number of triangulations with a marked boundary of length a having a distinguished corner, with m internal vertices, n_i non-marked boundaries of length n_i for $1 \leq i \leq h$, and no non-marked boundary of length larger than h. The total boundary-length is $b := a + \sum_i i n_i$, the number of boundaries is $r = 1 + \sum_i n_i$, and the associated number of edges is (by the Euler relation) $e = 2b + 3r + 3m - 6$, which is $2b + 3k$ with $k := r + m - 2$. Then (5) yields

$$2e\, \eta_a(m; n_1, n_2, \ldots, n_h) = [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \vec{A}_{\Delta}^{(a)} = [x_0^m x_1^{n_1} \cdots x_h^{n_h}] \binom{2a-2}{a-1} S^{2a-1}|_{t=1}.$$

And a similar argument as in Section 3.2 (using $\widetilde{S} = S/t$) ensures that for $k \geq 1$,

$$[x_0^m x_1^{n_1} \cdots x_h^{n_h}] S^k|_{t=1} = [t^{k+m+\sum_{i=1}^h (2i+1)n_i} x_0^m x_1^{n_1} \cdots x_h^{n_h}] S^k$$

Hence, by the Lagrange inversion formula, and using the notations $p := 2a - 1 + m + \sum_{i=1}^h (2i+1) n_i = 2b + k$, $s = m + \sum_{i \geq 1} n_i = k + 1$ and $B = 8 \sum_{i=0}^h x_i \binom{2i}{i}$:

$$\begin{aligned}
[x_0^m x_1^{n_1} \ldots x_h^{n_h}] S^{2a-1}|_{t=1} &= [t^p x_0^m x_1^{n_1} \ldots x_h^{n_h}] S^{2a-1} \\
&= \frac{2a-1}{p} [y^{p-2a+1} x_0^m x_1^{n_1} \ldots x_h^{n_h}] \Big(1 - 8 \sum_{i=0}^h x_i \binom{2i}{i} y^{2i+1}\Big)^{-p/2} \\
&= \frac{2a-1}{p} [x_0^m x_1^{n_1} \ldots x_h^{n_h}] \Big(1 - 8 \sum_{i=0}^h x_i \binom{2i}{i}\Big)^{-p/2} \\
&= \frac{2a-1}{p} [x_0^m x_1^{n_1} \ldots x_h^{n_h}] [u^s] (1 - Bu)^{-p/2} \\
&= \frac{2a-1}{p} [x_0^m x_1^{n_1} \ldots x_h^{n_h}] B^s \cdot [u^s] (1-u)^{-p/2} \\
&= \frac{2a-1}{p} 8^s \binom{s}{m, n_1, \ldots, n_h} \Big[\prod_{i=1}^h \binom{2i}{i}^{n_i}\Big] \cdot \frac{(p+2s-2)!!}{(p-2)!! s! 2^s} \\
&= \frac{2a-1}{m!} 4^s \Big[\prod_{i=1}^h \frac{1}{n_i!} \binom{2i}{i}^{n_i}\Big] \cdot \frac{(p+2s-2)!!}{p!!},
\end{aligned}$$

so that we obtain, using $e = p + 2s - 2$,

$$\eta_a(m; n_1, n_2, \ldots, n_h) = \frac{(2a-1)!}{(a-1)!^2} 4^{k+1} \left[\prod_{i=1}^h \frac{1}{n_i!} \binom{2i}{i}^{n_i}\right] \cdot \frac{(e-2)!!}{2m!\, p!!}.$$

Using $\frac{(2a-1)!}{(a-1)!^2} = \frac{1}{2} a \binom{2a}{a}$, and $p = 2b + k$, this rewrites as

$$\eta_a(m; n_1, n_2, \ldots, n_h) = 4^k \frac{(e-2)!!}{m!\,(2b+k)!!} a \binom{2a}{a} \left[\prod_{i=1}^h \frac{1}{n_i!} \binom{2i}{i}^{n_i}\right].$$

Multiplying this last expression by $\prod_{i=1}^h n_i! i^{n_i}$ (to account for numbering the inner boundary faces and marking a corner in each of these faces) gives (1).

5.3. **Solution of the dimer model on quadrangulations and triangulations.** A *dimer-configuration* on a map M is a subset X of the non-loop edges of M such that every vertex of M is incident to at most one edge in X. The edges of X are called *dimers*, and the vertices not incident to a dimer are called *free*. The *partition function* of the *dimer model* on a class $\mathcal{B}$ of maps is the generating function of maps in $\mathcal{B}$ endowed with a dimer configuration, counted according to the number of dimers and free vertices. The partition function of the dimer model is well-known for rooted 4-valent maps [18, 5] (and more generally p-valent maps).

We observe that counting (rooted) maps with dimer configurations is a special case of counting (rooted) maps with boundaries. More precisely, upon blowing each dimer into a boundary face of degree 2, a rooted map with a dimer-configuration can be seen as a rooted map with boundaries, such that all boundaries have length 2, and the rooted corner is in an internal face. Based on this observation we easily obtain from Theorem 2 that, for all $m, r \geq 0$ with $m + 2r \geq 3$, the number $q_{m,r}$ of dimer-configurations on rooted quadrangulations with r dimers and $m + 2r$ vertices is

$$q_{m,r} = 4(m + 2r - 2)\frac{3^{2r+m-2}(5r + 2m - 5)!}{r!m!(4r + m - 2)!}. \tag{13}$$

Similarly, Theorem 1 implies that, for all $m, r \geq 0$ with $m + 2r \geq 3$, the number $t_{m,r}$ of dimer-configurations on rooted triangulations with r dimers and $m + 2r$ vertices is

$$t_{m,r} = (m + 2r - 2)\frac{2^{2m+3r-3}3^{r+1}(7r + 3m - 8)!!}{r!m!(5r + m - 2)!!}. \tag{14}$$

In the context of statistical physics it would be useful to have an expression for the partition function, that is, the generating function of the coefficients $q_{m,r}$ or $t_{m,r}$. It should be possible to lift the expressions in (13) and (14) to generating function expressions, however we find it easier to obtain directly an exact expression from the bijections of Section 3.1 (for quadrangulations) and Section 4.1 (for triangulations), without a possibly technical lift from the coefficient expressions. Here this works by considering generating functions for the model with a slight restriction at the root edge.

For quadrangulations, we consider the generating function $Q(x, w)$ of rooted quadrangulations endowed with a dimer-configuration, with the constraint that both extremities of the root edge are free, where x is conjugate to the number of free vertices minus 2, and w is conjugate to the number of dimers. These objects are clearly in bijection (by opening the root-edge and every dimer into a boundary face of degree 2) with the set $\mathcal{Q}$ of rooted quadrangulation with boundaries all of length 2, such that the root-corner is in a boundary face. So $Q(x, w)$ is the generating function of maps in $\mathcal{Q}$, where x is conjugate to the number of internal vertices and w is conjugate to the number of inner boundaries. Note that $\mathcal{Q}$ can be seen as a subset of $\mathcal{D}_\diamond$, except that we are marking a corner in the outer face. Thus, applying the bijection of Section 3.1, we can interpret $Q(x, w)$ in terms of the set $\mathcal{T}'_\diamond$ of mobiles from $\mathcal{T}_\diamond$ such that every boundary vertex has 2 legs. More precisely, upon remembering that mobiles in $\mathcal{T}_\diamond$ have excess -2, it is not hard to see that $Q(x, w) = Q_1 - Q_2$, where Q_1 (resp. Q_2) is the generating function of mobiles from $\mathcal{T}'_\diamond$ with a marked bud (resp. with a marked leg or half-edge at a white vertex) with x counting white leaves, and w counting boundary vertices. From the series expressions obtained in Section 5.1 we get $Q_1 = R_0 = R - 1$ and $Q_2 = x\, R_{-1} + 6w\, {R_{-1}}^2$, under the specialization $\{t = 1, x_0 = x, x_1 = w, x_i = 0\ \forall i \geq 2\}$. Hence

$$Q(x, w) = R - 1 - x\, R^3 - 6w\, R^6, \quad \text{where } R = 1 + 3xR^2 + 9w\, R^5. \tag{15}$$

Note that $\widetilde{Q}(z, w) := Q(z, z^2 w)$ is the generating function for the same objects, with z conjugate to the number of vertices minus 2 (which by the Euler relation is also the number of faces) and w conjugate to the number of dimers. Now, if we are interested in the *phase transition* of this model, we need to determine how the asymptotic behavior of the coefficients $c_n = [z^n]\widetilde{Q}(z, w)$ (for $n \to \infty$) depends on the parameter w. Hence [10], we need to study the dominant singularities of $\widetilde{Q}(z, w)$ *considered as a function of* z. A companion maple worksheet can be found on the webpages of the authors. Denote by $\sigma(w)$ the dominant singularity of $\widetilde{Q}(z,w)$, and let $Z=\sigma(w)-z$. For all $w\geq 0$, the singularity type of $\widetilde{Q}(z,w)$ is $Z^{3/2}$ (as for maps without dimers), and no phase-transition occurs. However we find a singular value of w at $w_0=-3/125$, where $\sigma(w_0)=4/45$ and the singularity of $\widetilde{Q}(z,w_0)$ is of type $Z^{4/3}$ (as a comparison, it is shown in [5, Sec.6.2] that for the dimer model on rooted 4-valent maps endowed the critical value of the dimer-weight is $w_0=-1/10$ and the singularity type is the same: $Z^{4/3}$).

For triangulations we consider the generating function $T(x,w)$ of rooted triangulations endowed with a dimer-configuration, with the constraint that the root-vertex is free, where x is conjugate to the number of free vertices minus 1, and w is conjugate to the number of dimers. These objects are in bijection (up to opening the dimers into boundaries and opening the root half-edge as in Figure 10) with the set $\mathcal{T}$ of triangulations with boundaries, with one boundary of degree 1 taken as the outer face and all the other boundaries (inner boundaries) of length 2, and such that there are at least two inner faces. Let τ be the unique triangulation with one boundary face of length 1 (the outer face) and one inner face. By the preceding, $T(x,w)+x$ is the generating function of maps in $\mathcal{T}'=\mathcal{T}\cup\{\tau\}$. The bijection of Section 4.1 applies to the set $\mathcal{T}'$ and allows us to express $T(w,x)$ in terms of the set $\mathcal{T}'_\vartriangle$ of mobiles from $\mathcal{T}_\vartriangle$ such that every boundary vertex has 2 legs. More precisely, upon remembering that mobiles in $\mathcal{T}'_\vartriangle$ have excess -1, this bijection gives $T(x,w)+x=T_1-T_2$, where T_1 (resp. T_2) is the generating function of mobiles from $\mathcal{T}'_\vartriangle$ with a marked bud (resp. a marked leg or half-edge incident to a white vertex) with x counting white leaves and w counting boundary vertices. From the series expressions obtained in Section 5.2, we get $T_1=S_0=(S-1)/(2S)$ and $T_2=x\,S_{-2}+10w\,{S_{-2}}^3$, under the specialization $\{t=1,x_0=x,x_2=w,x_i=0\ \forall i\notin\{0,2\}\}$. Hence

$$T(x,w)=\frac{S-1}{2S}-x-x\,S^2-10\,w\,S^6,\quad \text{where } S^2=1+8xS^3+48\,w\,S^7. \tag{16}$$

Again we note that $\widetilde{T}(z,w):=T(z,z^2w)$ is the generating function for the same objects, with z conjugate to the number of vertices minus 1 (which by the Euler relation is also one plus half the number of faces) and w conjugate to the number of dimers. We now discuss the phase transition. We use the notations $\sigma(w)$ for the dominant singularity of $\widetilde{T}(z,w)$, and $Z=\sigma(w)-z$. We find that for all $w\geq 0$, the singularity of $\widetilde{T}(z,w)$ is of type $Z^{3/2}$, so that no phase-transition occurs. However, we find a singular value $w_0=-8\sqrt{105}/5145\approx -0.0159$, for which $\sigma(w_0)=5\sqrt{105}/1008\approx 0.0508$ and $\widetilde{T}(z,w_0)$ has singularity type $Z^{4/3}$.

6. Proofs of the existence and uniqueness of the canonical orientations

6.1. Preliminary results.

In this section we gives some definitions and preliminary results about orientations.

Let M be a map with boundaries, let V be its set of internal vertices, and let B be its set of boundaries. Given a mapping α from $V\cup B$ to $\mathbb{N}$, we call *α-orientation* of M an orientation which is *consistent* (that is, every boundary edge is oriented with the boundary face on its right), and such that each internal vertex v has indegree $\alpha(v)$, and each boundary f has indegree $\alpha(f)$. The following lemma is an immediate consequence of the results in [9] (upon seeing boundaries as "big" vertices).

Lemma 1. *Let M be a plane map with boundaries, let V be its set of internal vertices, and let B be its set of boundaries. Let $\alpha: V\cup B\to\mathbb{N}$ be such that there exists an α-orientation X of M. If the outer face of M is an internal face, then M admits a unique minimal α-orientation X_0. If the outer face of M is a boundary face and the outer boundary C_0 satisfies*

$\alpha(C_0) = 0$, then M admits a unique almost-minimal α-orientation X_0. In addition, in both cases, X is accessible from a vertex v if and only if X_0 is accessible from v.

Next, we state a parity lemma for orientations in $\widehat{\mathcal{O}}_d$.

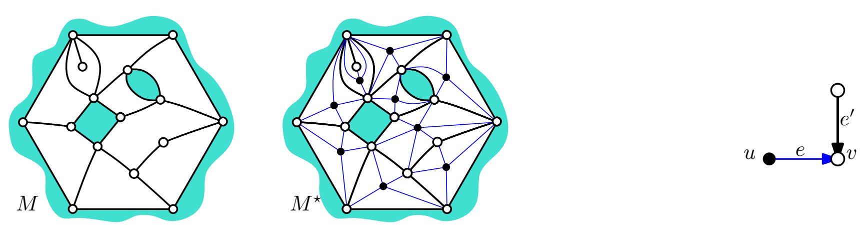
Lemma 2. *Let O be a consistent $\mathbb{Z}$-biorientation in $\widehat{\mathcal{O}}_d$ (for some $d \in \mathbb{Z}\backslash\{0\}$), such that every internal edge, internal vertex, internal face, boundary, has even weight. Then every internal half-edge also has even weight.*

Proof. Let T be the boundary mobile associated with O by the master bijection (Theorem 4). The parity conditions of O imply that all edges and vertices of T have even weight. In particular an edge e of T either has its two half-edges of odd weight, in which case e is called *odd*, or has its two half-edges of even weight, in which case e is called *even*. Let F be the subforest of T formed by the odd edges. Since every vertex of T has even weight, it is incident to an even number of edges in F. Hence F has no leaf, so that F has no edge. Thus all edges of T are even, and by the local rules of the master bijection it implies that all internal half-edges of O have even weight. □

Next, we build on results we proved in [2], to prove the existence of some canonical orientations for simple bipartite maps with boundaries. We start with some definitions. Let M be a map with boundaries. We call *star map* of M, and denote $M^\star$, the map obtained from M by inserting a vertex v_f, called *star vertex*, in each internal face f, and joining v_f by an edge to each corner of f. The star map $M^\star$ is considered as a map with boundaries (same boundaries as M). A star map is shown in Figure 12. The vertices and edges of $M^\star$ which are in M are called M-vertices and M-edges, while the others are called star-vertices and star-edges. If M is bipartite and f_s is a boundary, we call *2-regular orientation of* $(M^\star, f_s)$ a consistent orientation of $M^\star$ such that

(i) every internal M-vertex has indegree 2,
(ii) every star-vertex v has indegree $\deg(v)/2 + 2$,
(iii) every boundary $f \neq f_s$ has indegree $\deg(f)/2 + 2$,
(iv) the special boundary f_s has indegree $\deg(f_s)/2 - 2$.

Lastly we say that a vertex (resp. face) x of M is *d-blocked from a face f* if there is a simple d-cycle C not containing x (resp. not equal to the contour of f) such that x and f are on different sides of C, and moreover any boundary face incident to C lies in the same side as f.



(a) (b)

FIGURE 12. (a) A map with boundaries M, and the corresponding star-map $M^\star$ (with star-vertices colored black and M-vertices colored white). (b) Condition for an orientation of $M^\star$ to be *transferable*: if e is oriented toward v, then e' is oriented toward v.

Lemma 3. *Let M be a simple bipartite map with boundaries, and let f_s be a boundary face. Then there exists a 2-regular orientation of $(M^\star, f_s)$. Moreover, any such orientation is accessible from the vertices incident to f_s, and also from any M-vertex v which is not 4-blocked from f_s.*

The proof of Lemma 3 uses the two following claims about simple quadrangulations with boundaries. The first claim has a simple proof based on the Euler formula. The second one is a generalization of a classical result about simple quadrangulations without boundaries (see for instance [14]), and is a special case of a result proved in [2].

Claim 1. *Let Q be a quadrangulation with a single boundary of length ℓ. Then the numbers v and e of internal vertices and internal edges of Q are related by: $e = 2v + \ell/2 - 2$.*

Proof. The number f of internal faces is related to e and v by: $4f = 2e+\ell$ (incidence between faces and edges) and $f - e + v = 1$ (Euler relation). Eliminating f proves the claim. □

Claim 2. *Let M be a simple quadrangulation with a single boundary. Then M admits a consistent orientation such that every internal vertex has indegree 2. Moreover any such orientation is accessible from every boundary vertex.*

Proof. Consider the $\mathbb{Z}$-biorientation given by the case $d = 4$ of [2, Proposition 19]. Since M is bipartite we know that every weight is even by [2, Proposition 21]. Upon dividing every weight by 2, we get an unweighted orientation of M with the claimed properties. □

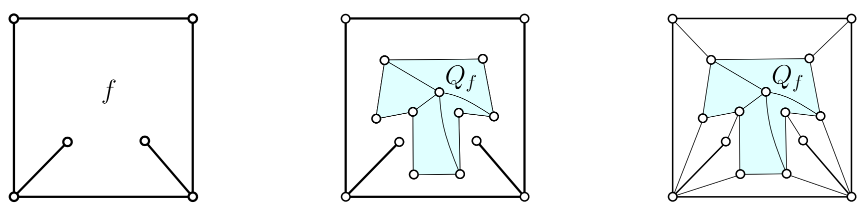


FIGURE 13. Quadrangulating a face f of M without creating double edges: insert a quadrangulation Q_f with one boundary f' of length $\deg(f)$ inside f (middle picture), and then connect each corner of f to a distinct corner of f' (right picture).

Proof of Lemma 3. We start by defining an orientation of $M^\star$, and we will show later that it has the desired properties. We first construct from M a simple quadrangulation N with a single boundary f_s, by quadrangulating each face $f \neq f_s$ in the manner illustrated in Figure 13. Precisely, for each face $f \neq f_s$ of M we

- insert a simple quadrangulation Q_f with one boundary f' of length $\deg(f)$ (the insertion is such that f and f' are facing each other; see Figure 13),
- join by an edge each corner of f to a distinct corner of f' (without creating edge crossings).

It is clear that the quadrangulation N is simple; and moreover we can also impose that Q_f has no internal edges joining boundary vertices, which ensures that any 4-cycle of N having an edge not in M is contained in one of the faces of M (this property will be used later on). Since N is simple, Claim 2 ensures the existence of a consistent orientation X of N such that every internal vertex has indegree 2. We then obtain an orientation Y of $M^\star$ by the following process:

(a) For each boundary $f \neq f_s$ of M, we delete Q_f and the incident edges, and we reorient the boundary edges so that they have f on their right.

(b) For each internal face f of M, we contract the quadrangulation Q_f to a single (star) vertex v_f.

We now prove that Y is a 2-regular orientation of $(M^\star, f_s)$. First, we observe that the internal M-vertices have indegree 2 as wanted. Next, we prove property (iv) of 2-regular orientations. By Claim 1, the number v and e of internal vertices and internal edges of N are related by $e = 2v + \deg(f_s)/2 - 2$. Since $2v$ internal edges of N are oriented toward internal vertices, there are $\deg(f_s)/2 - 2$ internal edges of N oriented toward f_s. Thus the indegree of f_s is $\deg(f_s)/2 - 2$ in X, hence also in Y. It remains to prove properties (ii) and (iii). Let f be a face of M distinct from f_s, and let v be the total number of vertices of Q_f. By definition of X, there are $2v$ edges of M oriented toward a vertex of Q_f, and by Claim 1, $2v - \deg(f)/2 - 2$ of these edges are in Q_f. Hence, in X there are $\deg(f)/2 + 2$ edges oriented from a corner of f toward Q_f. This implies that for any internal face f, the star-vertex v_f has indegree $\deg(v_f)/2 + 2$ in Y. This also implies that any boundary $f \neq f_s$ of M has indegree $\deg(f)/2 + 2$ in Y. Indeed, for any boundary $f \neq f_s$, there are $2\deg(f)$ edges of N oriented toward a vertex of f in X, from which $\deg(f)$ are boundary edges and $\deg(f)/2 - 2$ are edges oriented from Q_f toward f. Hence, there are

$$2\deg(f) - \deg(f) - (\deg(f)/2 - 2) = \deg(f)/2 + 2$$

edges of N strictly outside of f and oriented toward f. Thus Y is a 2-regular orientation. This proves the first statement of Lemma 3.

Next, we know from Claim 2 that the orientation X of N is accessible from the vertices incident to f_s. Since the accessibility properties can only be improved by applying the operations (a) and (b), the orientation Y of $M^\star$ is also accessible from the vertices incident to f_s.

Lastly, we consider a vertex v such that Y is not accessible from v, and we want to prove that there exists a 4-cycle of M blocking v from f_s. Let N' be the quadrangulation with boundaries obtained from N by performing (a), and let X' be the orientation of N' obtained from X. Note that X' is not accessible from v (since operation (b) can only improve accessibility). Hence, there is no directed path from v to f_s in X' (since X' is accessible from the vertices incident to f_s). Let U be the set of vertices of N' from which there is a directed path toward f_s, and let N'' be the submap of N made of U and the edges with both endpoints in U. The vertex v lies strictly inside a face f_1 of N''. Let us consider the sets V and E of vertices and edges lying strictly inside f_1. By definition of N'', any edge e in E having an endpoint on f_1 has its other endpoint in V and is oriented toward this endpoint. Hence, the total indegree of the vertices in V is $|E|$. Since each vertex has indegree 2, this gives $|E| = 2|V|$. Moreover, Claim 1 gives $|E| = 2|V| + \deg(f_1)/2 - 2$. Hence $\deg(f_1) = 4$. Thus the contour C of f_1 is a 4-cycle of N'. Moreover C must have all its edges in M (because, by construction of N, any 4-cycle with an edge not in M lies inside one of the internal faces of M, and hence can not separate v from f_s). Thus C is a 4-cycle of M. Lastly, any boundary face incident to C lies on the same side as v because any boundary face with at least one vertex in N'' has all its vertices and edges in N''. This proves that v is 4-blocked from f_s, hence the second statement of Lemma 3. □

We say that an orientation of $M^\star$ is *transferable* if for any star-edge e oriented from a star-vertex u to a vertex v of M, the M-edge e' following e in clockwise order around v is also oriented toward v; see Figure 12(b).

Lemma 4. *Let M be a plane map with boundaries, and let d be its outer degree. Let α be a function assigning an integer to each vertex and boundary face of $M^\star$, such that there exists an α-orientation of $M^\star$ which is accessible from the outer vertices of M. If the outer face f_0 of M is a boundary, we can consider $M^\star$ as a plane map with outer face f_0. In this case, if $\alpha(f_0) = 0$ and the (unique) almost-minimal α-orientation of $M^\star$ is transferable, then there exists a unique $\mathbb{Z}$-orientation in $\widehat{\mathcal{O}}_{-d}$ such that*

(i) the weights of half-edges are in $\{-1, 0, 1, 2\}$, and any edge has weight 1,
(ii) any vertex v has weight $\alpha(v)$,
(iii) any boundary face f has weight $\alpha(f)$, and any internal face f has weight $-\deg(f) + \alpha(v_f)$.

If the outer face f_0 of M is an internal face, we can consider $M^\star$ as a plane map by choosing a face f' incident to the star vertex v_{f_0} to be the outer face of $M^\star$. In this case, if $\alpha(f_0) = \deg(f_0)$ and the (unique) minimal α-orientation of $M^\star$ is transferable, then there exists a unique $\mathbb{Z}$-orientation of M in $\widehat{\mathcal{O}}_d$ satisfying the conditions (i-iii) above.

The main arguments of the following proof were already given in [3] (although the result was not stated in this generality there).

Proof. We treat the case where f_0 is an internal face; the case where f_0 is a boundary face is almost identical. Let Y be the unique minimal α-orientation of $M^\star$ (uniqueness is granted by Lemma 1). Let Z be the $\mathbb{Z}$-orientation of M obtained by keeping the orientation of the M-edges in Y, and putting weights according to the following *transfer rule*:

(*) for any M-edge e put weights -1 and 2 on the outgoing and ingoing half-edges of e if e is an internal edge such that the star-edge preceding e clockwise around the end v of e is oriented toward v, and otherwise put weights 0 and 1.

Since Y is a transferable consistent α-orientation of $M^\star$, it is easy to see that Z is a consistent orientation of M satisfying conditions (i-iii).

We want to prove that Z is in $\widehat{\mathcal{O}}_d$. First, the minimality of Y clearly implies the minimality of Z (since any ccw-cycle of Z would be a ccw-cycle of Y). Next we prove that Z is accessible from every outer vertex of M. Let v_0 be an outer vertex of M, let b_0 be the star-vertex inside f_0 and let $b_1, \ldots, b_k$ be the other star-vertices of $M^\star$. Let Y_0 be the orientation obtained from Y by deleting b_0 (and the incident star-edges). By hypothesis, Y is accessible from v_0, and no edge of Y is going out of b_0. Thus Y_0 is accessible from v_0. We now show, by induction on $i \in [k]$ that the orientation Y_i obtained by from Y by deleting $b_0, \ldots, b_i$ is accessible from v_0. We assume that Y_{i-1} is accessible from v_0 and suppose for contradiction that a vertex w is not accessible from v_0 in Y_i. In this case, any directed path P from v_0 to w in Y_{i-1} goes through b_i. Let P be such a path, and let e_0 and e_1 be the edges arriving at and leaving b_i along P. The situation is represented in Figure 14(a). We define the *left-degree* of P to be the number of edges of Q between e_0 and e_1 in clockwise order around b_i. We choose P so as to minimize the left-degree. Let P_0 be the portion of P before b_i, and let P_1 be the portion of P after b_i. Let u be the origin of e_0, and let v be the end of e_1. Since Y is transferable, the M-edge e' preceding e_1 around v is oriented toward v. Let v' be the origin of e'. Note that $v' \neq u$ (otherwise there would be a path of Y_{i-1} from v_0 to w avoiding b_i) and that the edge $e'_0 = \{b_i, v'\}$ preceding e_1 in clockwise order around b_i must be directed from v' to b_i (otherwise one could replace in P the portion $u \to b_i \to v$ by $u \to b_i \to v' \to v$, yielding a path with smaller left-degree). Since Y_{i-1} is accessible from v_0, there exists a directed path P' in Y_{i-1} from v_0 to v'. However, there is no directed path in Y_i from v_0 to v' (otherwise there would be a path to w). Hence there is a simple directed path P'_1 in Y_{i-1} from b_i to v'. Moreover P'_1 does not meet with P_0 except at b_i (otherwise there would be a path from v_0

to w avoiding b_i), and the first edge of b_i is not strictly between e_0 and e_1 in clockwise order around b_i (otherwise it would contradict the minimality of the left-degree of P). This implies that $P'_1 \cup \{e'_0\}$ is a ccw cycle: a contradiction. This concludes the proof that Y_i is accessible from v_0. Thus $Z = Y_k$ is accessible from every outer vertex. Lastly, we show that the outer edges of Z all have the outer face f_0 on their left. Suppose by contradiction that there is an outer edge $e = (u, v)$ having f_0 on its right. Since u is accessible from the outer vertex u, there is a directed path from u to v, which forms a ccw-cycle with e. This contradicts the minimality of Z, hence every outer edge has the outer face f_0 on its left. This proves that Z is in $\widehat{\mathcal{O}}_d$.

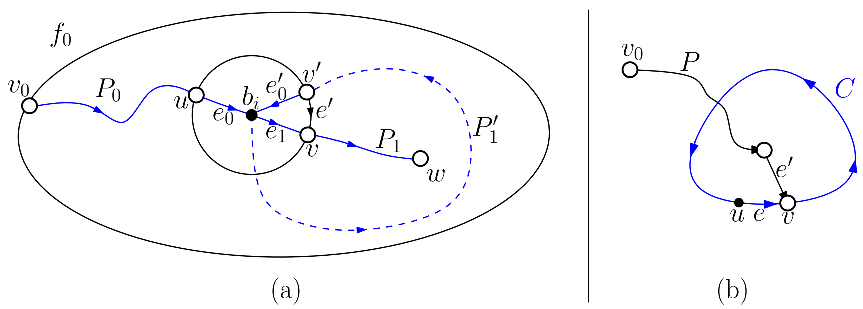


FIGURE 14. (a) Inductive proof that Y_i is accessible from v_0. (b) Proof of the uniqueness of the orientation in $\widehat{\mathcal{O}}_d$.

It only remains to prove that there is no orientation $Z' \neq Z$ in $\widehat{\mathcal{O}}_d$ satisfying (i-iii). Suppose, by contradiction, that Z' is such an orientation. It is easy to see that there exists a transferable consistent α-orientation $Y' \neq Y$ of $M^\star$ such that Z' is obtained from Y' by the transfer rule (*). Since Y and Y' are both α-orientations, Lemma 1 ensures that Y' is accessible from the outer vertices of M but is not minimal. Since Y' is not minimal, there exists a ccw-cycle C of Y' which encloses no other ccw-cycles. Since Z' is minimal, and the orientation of M-edges in Y' and Z' coincide, there must be a star-vertex u on C. The star-edge e of C going out of u ends at an M-vertex v. And since Y' is transferable, the M-edge e' preceding e around v is oriented toward v in Y'. Note that e' is enclosed by C; see Figure 14(b). Now consider a directed path P in Y going from an outer vertex of M to the origin of e'. It is clear that $C \cup P \cup \{e'\}$ contains a ccw-cycle enclosed in C. This contradicts our choice of C, and completes the proof. □

Let M be a simple bipartite map with boundaries, and let f_s be a boundary face. We call *2-orientation* of (M, f_s) a consistent $\mathbb{Z}$-biorientation of M with weights in $\{-1, 0, 1, 2\}$ such that:

- each internal edge has weight 1,
- each internal vertex has weight 2,
- each internal face f has weight $2 - \deg(f)/2$,
- each boundary $f \neq f_s$ has weight $\deg(f)/2 + 2$, while f_s has weight $\deg(f_s)/2 - 2$,

Corollary 1. *Let M be a simple bipartite plane map with boundaries, having outer degree 4. Let f_0 be the outer face and let f_s be a boundary face. If $f_0 = f_s$ (resp. f_0 is an internal face which is not 4-blocked from f_s), then Lemma 3 and Lemma 1 ensure the existence and uniqueness of an almost-minimal (resp. a minimal) 2-regular orientation X of $(M^\star, f_s)$.*

Moreover if X is transferable, then there exists a unique 2-orientation of (M, f_s) in $\widehat{\mathcal{O}}_{-4}$ (resp. $\widehat{\mathcal{O}}_4$)

Proof. This is a direct consequence of Lemma 3 and Lemma 4. The only point left to justify, is that if the outer face f_0 is an internal face which is not 4-blocked from f_s, then X is accessible from the outer vertices. But this is clear from the accessibility statement of Lemma 3, because if one of the outer vertices was 4-blocked from f_s, then f_0 would also be 4-blocked from f_s. □

6.2. **Proof of Proposition 1.** Let M be in $\mathcal{D}_\diamond$. We want to prove that M admits a unique 1-orientation in $\widehat{\mathcal{O}}_{-2}$. Let N be the map with boundaries obtained from M by inserting a vertex, called *edge-vertex*, in the middle of each edge (the boundary faces of N identify to those of M). Note that M is loopless (since it is bipartite), so that N is simple. We want to apply Corollary 1 to N. Let X be the unique almost-minimal 2-regular orientation of $(N^\star, f_s)$, where f_s is the outer face of N. The proof of the following claim is actually the only place where we use the fact that the internal faces of M have degree 4.

Claim 3. *The orientation X of $N^\star$ is transferable.*

Proof. We consider an edge e of $M^\star$ oriented from a star-vertex u to an N-vertex v, and consider the N-edge $e' = \{w, v\}$ following e in clockwise direction around v. We want to show that e' is oriented toward v. In Figure 15, we suppose by contradiction that e' is oriented toward w.

Let us suppose first that v is a vertex of M, as in Figure 15(a). We first observe that the star-edge $\{u, w\}$ must be oriented toward w to avoid creating a ccw-cycle. Since w is either a boundary vertex or a vertex of indegree 2, the N-edge $e'' \neq e'$ incident to w is oriented away from w. As shown in Figure 15(a), this implies the existence of a ccw-cycle; a contradiction.

Let us now suppose that v is an edge-vertex of N, as in Figure 15(b). As before, the star-edge $\{u, w\}$ must be oriented toward w. Moreover, since w is a vertex of M, we have shown above that the edge e'' following $\{u, w\}$ around w must be directed toward w. This implies that the origin v_1 of e'' is internal. Moreover the star-edge $\{v_1, u\}$ must be away from v_1 (because u is out-saturated), hence the other star-edge $e_1 = \{v_1, u_1\}$ incident to v_1 is oriented toward v_1 (because v_1 is out-saturated). At this point, we can apply to u_1, v_1, w the reasoning we just applied to u, v, w. Iterating the process until $v_k = v$ shows the existence of a ccw-cycle: a contradiction. □

We can now conclude the proof of Proposition 1. By Corollary 1, there exists a unique 2-orientation X of (N, f_s) in $\widehat{\mathcal{O}}_{-4}$. We then use the rules indicated in Figure 16 in order to obtain, from X, an orientation Y of M. It is easily seen that Y is a consistent orientation of M in $\widehat{\mathcal{O}}_{-2}$ such that

- each internal edge of M has weight 0,
- each internal vertex has weight 2,
- each internal face f has weight -2,
- each boundary face $f \neq f_s$ has weight $\deg(f) + 2$, while f_s has weight 0.

By Lemma 2, all internal half-edges have even weight. Dividing all these weights by 2, we obtain a 1-orientation Z of M in $\widehat{\mathcal{O}}_{-2}$. Lastly, there can be no other 1-orientation Z' of M in $\widehat{\mathcal{O}}_{-2}$, otherwise doubling its weights and applying the rule of Figure 16 in reverse direction would yield another 2-orientation $X \neq X'$ of (N, f_s) in $\widehat{\mathcal{O}}_{-4}$, thereby contradicting the uniqueness of X.

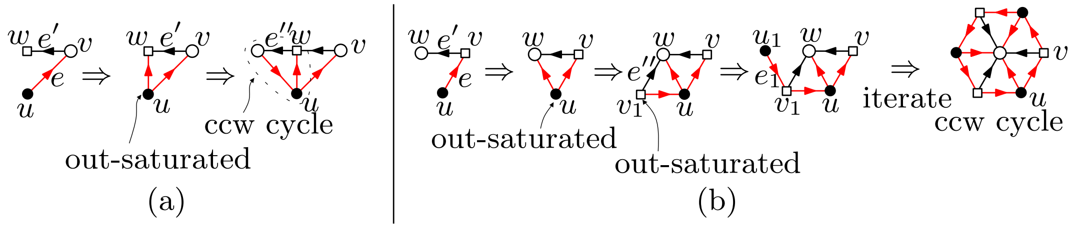


FIGURE 15. Proof by contradiction that e' is oriented toward v. The case where v is a vertex of M is treated in (a), while the case where v is an edge-vertex is treated in (b). Star-vertices, vertices of M and edge-vertices are represented as black circles, white circles and white squares respectively. At each step the possible configurations are forced by the avoidance of a ccw-cycle, or by the fact that a vertex indegree or outdegree is saturated.

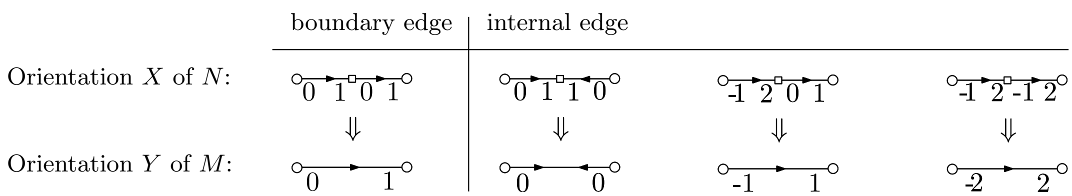


FIGURE 16. Top: possible orientations for X around an edge-vertex of N. Bottom: the associated orientation Y of M.

6.3. **Proof of Proposition 3.** We first prove the necessity of being reduced.

Lemma 5. *Let $a \geq 1$. If $M \in \mathcal{A}_{\diamond}^{(2a)}$ has a 1-orientation in $\widehat{\mathcal{O}}_2$, then M is reduced (i.e., in $\mathcal{B}_{\diamond}^{(2a)}$).*

Proof. Assume by contradiction that M is not in $\mathcal{B}_{\diamond}^{(2a)}$. Then M has a 2-cycle C blocking the outer face f_0 from the special boundary face f_s. We first prove that no 1-way edge outside of C has its end on C. By definition of blocking cycles, any edge (strictly) outside of C with an endpoint on C is internal. Let v, q, n_i (for $i \geq 1$) be respectively the numbers of internal vertices, quadrangular internal faces, and boundaries of length $2i$ that are (strictly) outside of C. The Euler relation easily implies that $q = v + \sum_i (i+1)n_i$. By definition of 1-orientations, q is also the number of 1-way internal edges e such that the face on the right of e is outside of C. Also by definition of 1-orientations, $v + \sum_i (i+1)n_i$ is the number of 1-way internal edges whose extremity is (strictly) outside C. Thus, no 1-way edge outside of C has its end on C. Since by definition of $\widehat{\mathcal{O}}_2$ the outer edges of M are 1-way, the outer vertices are not on C. Thus, there can be no directed path from the outer vertices to the vertices on C. This contradicts the accessibility of the orientation and completes the proof. $\square$

It remains to prove that any map $M \in \mathcal{B}_{\diamond}^{(2a)}$ has a unique 1-orientation in $\widehat{\mathcal{O}}_2$. The proof is similar to the proof of Proposition 1. Namely, we consider the map N obtained from M by inserting an edge-vertex in the middle of each edge. Note that N is a simple bipartite map. Moreover since M is reduced, the outer face f_0 of N (corresponding to the outer face of M) is not 4-blocked from the face f_s of N corresponding to the marked face of M of degree $2a$. As stated in Corollary 1, there is a unique minimal 2-regular orientation X of $(N^\star, f_s)$, where f_s is the face of N corresponding to the marked face of M of degree $2a$. Moreover X is shown to be transferable by the same proof as for Claim 3. Thus, by Corollary 1, there exists a unique 2-orientation X of (N, f_s) in $\widehat{\mathcal{O}}_4$. We then uses the rules indicated in Figure 16 in order to obtain, from X, an orientation Y of M. It is easily seen that Y is a consistent orientation of M in $\widehat{\mathcal{O}}_2$ such that

- each internal edge of M has weight 0,
- each internal vertex has weight 2,
- each inner internal face f has weight -2,
- each boundary face $f \neq f_s$ has weight $\deg(f) + 2$, while f_s has weight $2a - 2$.

Then, using Lemma 2 similarly as in Section 6.2, we conclude (dividing by 2 the weights Y) that M has a unique 1-orientation in $\widehat{\mathcal{O}}_2$. This completes the proof of Proposition 3.

6.4. **Proof of Proposition 4.** The proof follows similar lines as the proof of Proposition 1 with one main difference: edges of M are subdivided into 4 edges instead of 2.

Let $M \in \mathcal{D}_\Delta$. Let N be the map with boundaries obtained from M by inserting 3 vertices, called *edge-vertices*, on each edge of M. Note that N is a simple bipartite map. We want to apply Corollary 1 to N. Let X be the unique almost-minimal 2-regular orientation of $(N^\star, f_s)$, where f_s is the outer face of N.

Claim 4. *The orientation X of $N^\star$ is transferable.*

Proof. We consider an edge e of $M^\star$ oriented from a star-vertex u to an N-vertex v, and consider the N-edge $e' = \{w, v\}$ following e in clockwise direction around v. We want to show that e' is oriented toward v. In Figures 17 and 18, we suppose by contradiction that e' is oriented toward w.

Let us suppose first that v is a vertex of M, as in Figure 17. We observe that the star-edge $\{u, w\}$ must be oriented toward w to avoid creating a ccw-cycle. Since w is either a boundary vertex or a vertex of indegree 2, the N-edge $e'' \neq e'$ incident to w is oriented away from w. Iterating this reasoning twice, and remembering that the star-vertex u has outdegree 4 (since it has degree 12) implies the existence of a ccw-cycle; a contradiction.

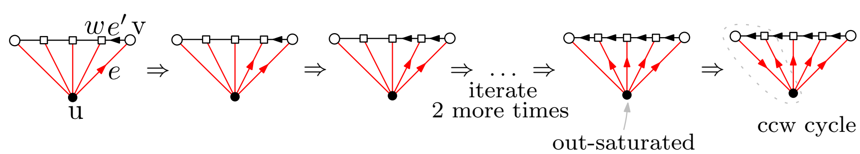


Figure 17. Proof that M is transferable: case where v is a vertex of M.

We now suppose that v is an edge-vertex of N, as in Figure 18. Let f be the face of N containing u, let a be the edge of M on which v lies, let b be edge of M preceding a clockwise around f, and let t be the vertex of M preceding v clockwise around f. Let P be the path of N made of the edges between v and t on a. Reasoning as above we conclude that P is oriented toward t and the star edges between P and u are oriented toward P. We now consider the first star-edge ϵ_1 oriented toward u following e in counterclockwise order around u, and we denote v_1 its origin. Since v_1 is not on P and the star-vertex u has outdegree 4, v_1 must be an edge-vertex on b; see Figure 18. Moreover, the edge of N preceding ϵ_1 in clockwise order around v_1 must be oriented toward v_1 (to avoid a ccw-cycle). This implies that v_1 is internal, and that the other star-edge e_1 incident to v_1 must be oriented toward v_1 (because v_1 is out-saturated). At this point we can apply to e_1 the reasoning we just applied to e.

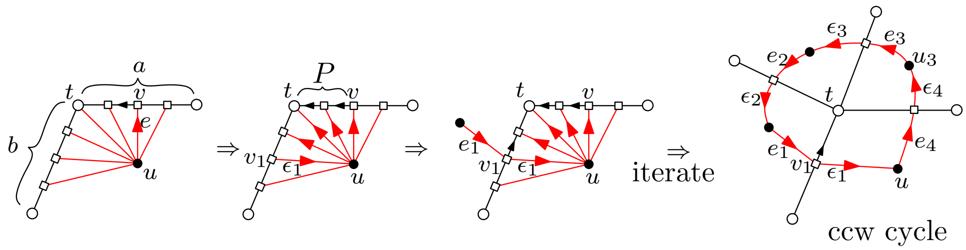


FIGURE 18. Proof that M is transferable: case where v is an edge-vertex of N.

Iterating this process proves the existence of some edges $\epsilon_1, e_1, \ldots, \epsilon_{\deg(t)}, e_{\deg(t)}$, forming a ccw-cycle around t; a contradiction. $\square$

We can now conclude the proof of Proposition 4. By Corollary 1, there exists a unique 2-orientation X of (N, f_s) in $\widehat{\mathcal{O}}_{-4}$. We then uses the rules indicated in Figure 19 in order to obtain, from X, an orientation Y of M.

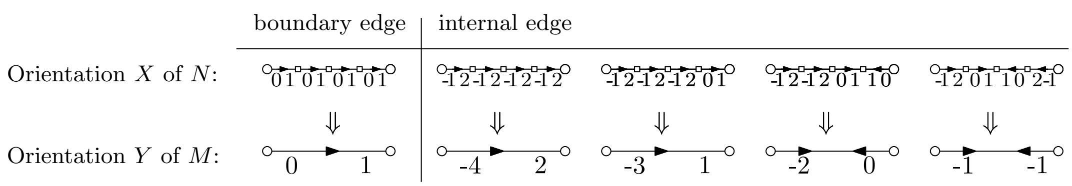


FIGURE 19. Top: possible orientations for X at a subdivided edge of N. Bottom: the associated orientation Y of M.

It is easily seen that Y is a consistent orientation of M in $\widehat{\mathcal{O}}_{-1}$ such that

- each internal edge of M has weight -2,
- each internal vertex has weight 2,
- each internal face f has weight -4,
- each boundary face $f \neq f_s$ has weight $2\deg(f) + 2$, while f_s has weight 0.

By Lemma 2, all internal half-edges have even weight. Dividing all these weights by 2, we obtain a 1-orientation Z of M in $\widehat{\mathcal{O}}_{-1}$. Lastly, there can be no other 1-orientation Z' of M in $\widehat{\mathcal{O}}_{-1}$, otherwise doubling its weights and applying the rule of Figure 19 in reverse direction would yield another 2-orientation $X \neq X'$ of (N, f_s) in $\widehat{\mathcal{O}}_{-4}$, thereby contradicting the uniqueness of X. This completes the proof of Proposition 4.

6.5. **Proof of Proposition 6.** The proof of Proposition 6 follows the same lines as the proof of Proposition 3, except that, as in Section 6.4, we subdivide edges into 4 edges instead of 2.

7. Additional results and perspectives

Recall that the *girth* of a graph is the smallest length of its cycles. In [3] we obtained bijections for maps *without boundaries*, with control on the girth and the face-degrees. This was done using a master bijection approach based on canonical orientations (some of these results were later recovered in [7] using another approach based on so-called slices). Ideally we would like to extend this bijective approach in order to count maps *with boundaries* with control on the girth and on the degrees of both boundary and internal faces. However there seem to be some obstacles to this goal.

First, for maps with boundaries without girth constraint, we do not know how to extend our bijective approach to maps having internal faces of larger degree. For bipartite maps we could only allow internal faces of degree at most 4, while for general maps we could only allow internal faces of degree at most 3. This limitation was crucial to ensure the *transferability* property of orientations in Claim 3 and Claim 4. So, our current proof cannot be easily extended, and furthermore we do not know if there are suitable canonical biorientations characterizing maps having internal faces of larger degree.

Second, we do not know how to extend our bijective approach to maps with girth constraints, say of girth d for $d \geq 2$. Based on [2, 3] we expect that the easiest case would be that of d-angulations of girth d, for $d \geq 3$. The canonical orientations obtained in [2] for d-angulation of girth d without boundary seem a useful starting point. Indeed, using them, one can prove that d-angulations with boundaries of girth d admits a (unique minimal) weighted biorientation such that every inner boundary of length k has weight $k + d$, every inner vertex has weight d, every internal face has weight 0, and every internal edge has weight $d - 2$. However, it seems that some d-angulations with boundaries of girth less than d also admit this type of orientation. In other words, the most natural "canonical orientations" seem to characterize a larger class of maps. In the near future, we plan to show that they actually characterize d-angulations with boundaries of *blocking girth* d. The *blocking girth* is the smallest length of a cycle C bounding a simply connected region R not containing the outer face such that every face in R incident to a vertex of C is internal.

Third, we could try to characterize maps with boundaries of *contractible girth* d, where the *contractible girth* means the smallest length of a cycle enclosing a region with no boundary face (note that *girth* $\leq$ *blocking girth* $\leq$ *contractible girth*). In this direction, it is possible to use a generating function approach (by substitution) to compute the generating function of triangulations with boundaries of contractible girth $d = 2$ or 3 (starting from our results for $d = 1$), and the generating function of bipartite quadrangulations with boundaries of contractible girth $d = 4$ (starting from our results for $d = 2$). But we do not know if this can be extended further, or if a bijective approach would work. However, there seems to be some hopes for all d when the maps have at most 2 boundaries (to be investigated...).

Acknowledgments. The authors thank Jérémie Bouttier and Juanjo Rué for very interesting discussions. Olivier Bernardi was partially supported by NSF grant DMS-1400859. Éric Fusy was partly supported by the ANR grant "Cartaplus" 12-JS02-001-01 and the ANR grant "EGOS" 12-JS02-002-01.

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