BIJECTIONS FOR WEYL CHAMBER WALKS ENDING ON AN AXIS, USING ARC DIAGRAMS AND SCHNYDER WOODS

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Abstract. In the study of lattice walks there are several examples of enumerative equivalences which amount to a trade-off between domain and endpoint constraints. We present a family of such bijections for simple walks in Weyl chambers which use arc diagrams in a natural way. One consequence is a set of new bijections for standard Young tableaux of bounded height. A modification of the argument in two dimensions yields a bijection between Baxter permutations and walks ending on an axis, answering a recent question of Burrill et al. (2016). Some of our arguments (and related results) are proved using Schnyder woods. Our strategy for simple walks extends to any dimension and yields a new bijective connection between standard Young tableaux of height at most 2k and certain walks with prescribed endpoints in the k-dimensional Weyl chamber of type D.

Keywords: Lattice paths, excursions, Schnyder woods, Dyck paths, Weyl Chambers, Young Tableaux.

1. Introduction

In the context of directed 2D lattice paths with unit steps, there is a classic bijection between meanders and bridges of equal length. This maps lattice walks with steps (1, 1) and (1, −1) starting at the origin, staying above the x-axis (meanders) to those ending at height zero (bridges) – see Figure 1. This example illustrates a common trade-off in lattice walks between domain constraints and endpoint constraints [15, 6]. Note that the natural bijection shown in Figure 1 proceeds via an intermediate class of walks (Dyck walks) where both the stronger domain and endpoint restrictions are imposed, and the elements of this class carry additional “decorations” (here, marked down-steps reaching the x-axis).

In this work, a similar but new strategy can successfully be applied to several models of Weyl chamber walks in arbitrary dimension. In particular, for two classical step sets (simple walks and hesitating walks), we have found explicit bijections that exchange a domain constraint with an endpoint constraint. In the two-dimensional case, these bijections match

Figure 1. An example of the classical bijection between meanders and bridges. From a Dyck walk $D$ with some marked steps $d_1, \ldots, d_k$ reaching the $x$-axis, one gets (bijectively) a meander by turning every marked step into an up-step. To get (bijectively) a bridge from $D$, for $1 \leq i \leq k$ we let $u_i$ be the up-step matched with $d_i$, and we switch every step between $u_i$ and $d_i$ (included).

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walks in the quadrant \( \{ x \geq 0, y \geq 0 \} \) ending at the origin (excursions\(^1\)), and walks in the octant \( \{ x \geq y \geq 0 \} \) and ending on the \( x \)-axis (axis-walks). For both step sets, these bijections pass through decorated excursions restricted to the octant. Deciding exactly how to mark the steps in the decorated intermediary is less obvious than the Dyck walk example. We do this by using open arc diagrams that are associated to the walks via the robust bijection of Chen et al. [11]. Rather, we require the extension to open arc diagrams due to Burrill et al. [8]. In their full generality, these bijections map open diagrams with no \((k+1)\)-crossing\(^2\) to walks in the \( k \)-dimensional Weyl chamber of type C \( \{ (x_1, \ldots, x_k) : x_1 \geq \cdots \geq x_k \geq 0 \} \) that end on the \( x_1 \)-axis, where the number of open arcs gives the abscissa of the endpoint.

It is at the level of arc diagrams that the marking of the object is easiest to describe: we map walks that end on the \( x \)-axis to open arc diagrams, mark the location of the open arcs, remove them, and then apply the inverse bijection to get marked excursions. The schematic outline of our core idea is illustrated in Figure 2. The advantage of our approach is that it very easily generalizes to walks in arbitrary dimension.

Once these decorated excursions are obtained, it remains to process the marks. This processing is handled differently for simple walks and for hesitating walks (where a further step of transfer of decorations is needed), but in both cases the marks are used to produce an unmarked walk in a larger domain.

Part of the bijection for simple walks has the nice feature that it extends to higher dimension, unveiling a new bijective connection with standard Young tableaux of even-bounded height (which are known to be related to Weyl chamber axis-walks [22, 8, 26]).

1.1. **Bijection for 2D simple walks.** A lattice model is said to be simple if the step set consists of all of the elementary vectors and their negatives. In two dimensions, the steps correspond to the compass directions, that we denote \( N, E, S, W \). Our first main result is the following Theorem, which is proved in Section 3.

**Theorem 1.** There exists an explicit bijection (preserving the length) between simple axis-walks of even length staying in the first octant, and simple excursions staying in the first quadrant.

As announced in the introduction, our strategy uses open arc diagrams to turn the simple axis-walk of length \( 2n \) into a decorated excursion. This is then transformed to a simple walk of length \( 2n \) in the tilted quadrant \( \{ (x, y) : x \geq 0, |y| \leq x \} \) starting and ending at \((1/2, 1/2)\),

\(^1\)More generally, we use the term excursion to indicate the set of walks with a prescribed start and end point. When they are not specified, the prescribed start and end is assumed to be the origin.

\(^2\)A \( k \)-crossing is a set of \( k \) mutually crossing arcs.
and finally mapped to a pair of Dyck paths of respective lengths $2n$ and $2n + 2$. These are known [12, 2] to be in bijection with simple excursions of length $2n$ in the quadrant. Compare this to the following result recently proved by Elizalde:

**Theorem 2** (Elizalde [15]). There exists an explicit bijection (preserving the length) between simple walks staying in the first octant and ending on the diagonal, and simple excursions staying in the first quadrant.

In Section 5 we provide an alternative proof of Theorem 2 using Schnyder woods. Note that Theorems 1 and 2 together yield a bijection for simple walks of length $2n$ staying in the octant, mapping those ending on the $x$-axis to those ending on the diagonal. This answers an open question of Bousquet-Mélou and Mishna [6].

Moreover, in Section 6 we give an extension for dimension $k \geq 1$ of the aforementioned bijection between simple axis-walks in the octant and simple walks from $(1, \frac{1}{2})$ to itself in the tilted quadrant. This yields a new bijective connection between standard Young tableaux of height at most $2k$ and simple excursions in the $k$-dimensional Weyl chamber of type $D$.

Grabner and Magyar gave explicit enumeration formulas for excursions in Weyl Chambers, and hence this bijection permits a straightforward application of their results. In Section 6.2 we use their results to illustrate a new derivation of Gessel’s formulas for standard Young tableaux of even-bounded height.

### 1.2. Bijection for 2D hesitating walks

A (2-dimensional) hesitating walk is a sequence of steps $s_1, \ldots, s_{2n}$ such that every step of odd index is either in $\{N, E\}$ (a positive step) or is $0 = (0, 0)$, every step of even index is either in $\{W, S\}$ (a negative step) or is $0$, and for every $i \in \{1, \ldots, n\}$, $s_{2i-1}$ and $s_{2i}$ cannot both be zero. It is convenient to not represent the null step in the drawings, but rather to group the steps by pairs of the form $(s_{2i-1}, s_{2i})$. In Section 4 we show the analogous, although more difficult, result for hesitating walks, which answers a recent question of Burrill et al. [8]:

**Theorem 3.** There exists an explicit bijection (preserving the length) between hesitating axis-walks in the first octant, and hesitating excursions in the first quadrant.

It is then easy to derive a bijection between Baxter permutations of size $n + 1$ (known to be in bijection with hesitating excursions of half-length $n$ in the quadrant) and open matching-diagrams with $n$ points and no enhanced 3-nesting (known to be in bijection with hesitating axis-walks of half-length $n$ in the octant). This answers a conjecture given in [9].

In order to show Theorem 3, again the first step is to use the strategy of Figure 2 to turn the axis-walks into decorated hesitating excursions, where the decoration consists in marking some W-steps on the $x$-axis. A further ingredient here is to turn the decoration into marked steps leaving the diagonal, after which the decorated excursions in the octant are known [8] to be equivalent to hesitating excursions in the quadrant.

Hesitating excursions of half-length $n - 1$ in the quadrant are known to be counted by the Baxter numbers $B_n = \frac{2}{n(n+1)^2} \sum_{k=1}^{n} \binom{n+1}{k+1} \binom{n+1}{k} \binom{n+1}{k-1}$. Indeed, as shown in [8], they are in easy bijection with the classical Baxter family of non-intersecting triples of directed lattice walks. On the other hand it has been first shown in [31] (and more recently in [8]) that hesitating axis-walks of half-length $n$ in the octant are also counted by $B_{n+1}$. Both of these proofs involve an equality of generating functions, and neither proof retains significant combinatorial intuition. Our result is the first bijective proof that these walks are counted.
by $B_{n+1}$. Such a result is not obvious to find since the family of hesitating axis-walks in the octant does not seem to be naturally endowed with the classical (bivariate) symmetric generating tree common to the Baxter families such as Baxter permutations, twin pairs of binary trees, 2-oriented plane quadrangulations, and plane bipolar orientations [14, 1, 17]. These families share the same generating tree, and hence there exists a “canonical” bijection relating them [5]. We cannot rely on such a systematic bijective strategy here.

Theorem 3 can be extended to a similar kind of plane walks, namely vacillating walks, leading to some new enumerative results on such walks.

In the case of simple walks, we saw that both the excursions in the quadrant and the axis-walks in the octant are in bijection with the simple walks in the octant ending on the diagonal. Does this hold for hesitating walks? In fact, a computational enumeration (up to half-length 50) suggests the following conjecture (where the conjectural part is the bijective link to the third family):

**Conjecture 4.** The following families are in bijection:

- hesitating excursions of length $2n$ in the quadrant,
- hesitating axis-walks of length $2n$ in the octant,
- hesitating walks of length $2n$ in the octant, ending on the thick diagonal: $\{(n,n), n \in \mathbb{N}\} \cup \{(n+1,n), n \in \mathbb{N}\}$.

If we denote by $u_n$ (resp. $v_n$) the number of hesitating walks of length $2n$ in the octant ending on $\{x = y\}$ (resp. ending on $\{x = y + 1\}$), then Conjecture 4 would imply that $u_n + v_n$ equals the Baxter number $B_{n+1}$. We have not been able yet to find a computational proof that $u_n + v_n = B_{n+1}$, but have numerically checked that both $u_n$ and $v_n$ seem to be $\mathbb{P}$-recursive (of order 2). We have not found any natural subfamily of hesitating axis-walks of length $2n$ (for instance with a parity constraint on the ending point) that are counted by $u_n$.

2. OPEN ARC DIAGRAMS

Arc diagrams are a graphic representation of combinatorial structures such as partitions or matchings, which enables a convenient visualization of certain patterns, such as crossings. A *partition diagram* is defined for a set partition $\pi$ of $\{1, \ldots, n\}$: draw $n$ points on a line, labeled from 1 to $n$; for each (ordered) block $\{a_1, \ldots, a_k\}$ of $\pi$, we draw an arc from $a_i$ to $a_{i+1}$ for $1 \leq i \leq k-1$. A *matching diagram* is a partition diagram where the underlying set partition is a matching (i.e. every block has size 2).

These diagrams are called *matching diagrams* and *partition diagrams*. A point of a partition diagram can be an *opening point*, if it is the first point of a block of size $\geq 2$; a *closing point*, if it is the last point of a block of size $\geq 2$; a *transition point*, if it is a non-extremal point in a block of size $\geq 3$; or a *fixed point*, if it is a block of size 1. A matching diagram only has opening and closing points.

A 3-crossing pattern in an arc diagram is a set of three mutually crossing arcs, i.e. three arcs $(i_1, j_1), (i_2, j_2)$ and $(i_3, j_3)$ with $i_1 < i_2 < i_3 < j_1 < j_2 < j_3$. An *enhanced 3-crossing* is a 3-crossing where arcs sharing an endpoint are also considered to be crossing. More formally, three arcs $(i_1, j_1), (i_2, j_2), (i_3, j_3)$ form an enhanced 3-crossing if $i_1 < i_2 < i_3 \leq j_1 < j_2 < j_3$. These definitions are naturally generalized to $k$-crossings for $k \geq 2$. 
Arc diagrams can be extended to \textit{open arc diagrams}, by allowing arcs with only a left endpoint, and no right endpoint (see Figures 3 for examples). In terms of crossings, an open arc is considered to end at an imaginary point to the right, and two open arcs are assumed to be non-crossing. The two examples of Figure 3 have multiple 2-crossings but no 3-crossing. For all types of arc diagrams the \textit{size} is defined as the number of points in the diagram.

In [11], Chen \textit{et al.} describe a bijection between arc diagrams with no \((k+1)\)-crossings, and excursions staying in the \(k\)-dimensional Weyl chamber of type \(C\). It was subsequently extended in [7, 8] by Burrill \textit{et al.} to map open arc diagrams to axis-walks.

**Theorem 5** (Burrill \textit{et al.} [8], restricted to 3-crossings). \textit{There exists an explicit combinatorial bijection between open matching (resp. open partition) diagrams of size \(n\), with \(m\) open arcs and no 3-crossing (resp. no enhanced 3-crossing), and simple (resp. hesitating) walks of length \(n\) (resp. of half-length \(n\)) staying in the first octant \(\{(x, y), 0 \leq y \leq x\}\), starting at the origin and ending at \((m, 0)\).}

We refer the reader to [8] for a description of the bijection in its complete form, which is based on the Robinson-Schensted insertion algorithm on tableaux, but can also conveniently be reformulated in terms of growth diagrams [25].

Here are important properties of this correspondence that we use in our bijections. The first part is extracted from Proposition 3 of [8], and the second part follows straightforwardly from the description of the bijection.

**Property 6.** Let \(\pi\) be a closed matching (resp. partition) diagram of size \(n\) with no 3-crossing (resp. enhanced 3-crossing), and \(\omega\) the simple (resp. hesitating) walk of length \(n\) corresponding to \(\pi\) via the bijection from [8].

Open arcs can be inserted into intervals at positions \(i_1, i_2, \ldots, i_k\) in \(\pi\) without forming a 3-crossing if and only if for every \(j \in \{1, \ldots, k\}\) the \(y\)-coordinate after \(i_j\) steps in \(\omega\) is zero.

Given a partition diagram \(\pi\), the fixed points of \(\pi\) correspond to the factors \(\text{EW}\) in \(\omega\), and the closing points of \(\pi\) correspond to the factors \(\{0W, 0S\}\) in \(\omega\) (with \(0\) denoting the zero step). In addition, an open arc can be added on a fixed point or a closing point without creating an enhanced 3-crossing if and only if an open arc could be added into the interval just to the left of that point without creating a 3-crossing.

3. \textbf{Proof of Theorem 1: Simple Walks}

The first main ingredient lies in the results from [12, 2], where the respective authors describe a correspondence between simple excursions of length \(2n\) in the quadrant, and pairs of Dyck paths of lengths \(2n\) and \(2n + 2\). To have a bijective proof of Theorem 1, we then need to connect such pairs of Dyck paths to simple axis-walks of even length in the octant. This is given by the following theorem, with an extension to odd length.
Theorem 7. Let \( C_n \) be the set of Dyck paths of length \( 2n \), and let \( \mathcal{U}_n \) be the set of simple axis-walks of length \( n \) in the first octant. There is an explicit bijection for each \( n \geq 0 \) between \( \mathcal{U}_{2n} \) and \( C_n \times C_{n+1} \), and between \( \mathcal{U}_{2n+1} \) and \( C_{n+1} \times C_{n+1} \).

This section provides a bijective proof of this result, as illustrated by Figure 4. Gouyou-Beuchamps [22] showed that the cardinality of simple axis-walks in the octant is indeed \( \text{Cat}_n \cdot \text{Cat}_{n+1} \) or \( \text{Cat}_n^2 \), depending on the parity, where \( \text{Cat}_n \) is the \( n \)th Catalan number. However, his proof uses a reflection principle argument of Gessel Viennot which involves subtractions and cancellations of terms.

As described in the introduction, our strategy relies on the bijection of Theorem 5, which allows us to turn simple axis-walk into simple excursion with decorations consisting of weights assigned to each visit to the \( x \)-axis.

Lemma 8. Simple axis-walks of length \( n \) in the octant ending at \((m,0)\) are in bijection with simple excursions of length \( n - m \) in the octant, where each visit to the \( x \)-axis carries a non-negative integer weight, such that the sum of the weights is \( m \).

Proof. Using Theorem 5, such an axis-walk is mapped to an open matching diagram of size \( n \) with \( m \) open arcs and without 3-crossing. We then remove the open arcs to obtain a (closed) matching diagram \( \pi \) of size \( n - m \), and we record their former positions as follows: for each interval of \( \pi \) that contained at least one open arc, we assign to the interval a positive weight equal to the number of open arcs it formerly contained (see Figure 4(b) to (c)). The sum of these weights is thus \( m \). Only specific intervals can carry weights since adding arcs in some intervals might create a 3-crossing.

By Theorem 5 (again), the diagram \( \pi \) is mapped to an excursion in the octant. By Property 6, we know that the intervals of \( \pi \) where insertion of open arcs is possible exactly correspond to the visits of the excursion to the \( x \)-axis. We then transfer the weights to the corresponding positions (see Figure 4(c) to (d)). \( \square \)
As a final step, we transform the weighted excursions in the octant into pairs of Dyck paths. To do so, we define an intermediary class of walks in the tilted quadrant

\[ \tilde{Q} = \{(x, y) : x \geq 0, |y| \leq x\}, \]

domain which corresponds to the duplication of the octant \( \{(x, y) : x \geq y \geq 0\} \) by a symmetry with respect to \( y = 0 \).

**Lemma 9.** For \( n, m \) both even (resp. both odd), the set of simple decorated excursions of length \( n - m \) in the octant with a total weight \( m \) on the visits to the \( x \)-axis is in bijection with the set of simple walks of length \( n \) in the tilted quadrant \( \tilde{Q} \) from \( (\frac{1}{2}, \frac{1}{2}) \) to \( (\frac{1}{2}, \frac{1}{2}) \) (resp. to \( (\frac{1}{2}, -\frac{1}{2}) \)) where exactly \( m \) steps change the sign of \( y \) in the walk. This set is in bijection with \( \mathcal{C}_{\lfloor (n+1)/2 \rfloor} \times \mathcal{C}_{\lfloor (n+1)/2 \rfloor} \), in such a way that if the two Dyck paths are drawn with respective starting points \((0, 0), (-1, 0)\), they cross exactly \( m \) times.

*Proof.* The idea is illustrated by Figure 4(d)-(f). The weights indicate a switch from one copy of the octant to the other within \( \tilde{Q} \) (one copy is for \( y > 0 \), the other one for \( y < 0 \)). We use the convention that the walk in the lower copy is upside-down.

Concerning the second bijection, we map any walk \((x_i, y_i)_{i \in \{0,\ldots,n\}} \) of \( \tilde{Q} \) to the pair \( P_1, P_2 \) of paths

\[ ((x_i + y_i)_{i \in \{0,\ldots,n\}}, (x_i - y_i)_{i \in \{0,\ldots,n\}}) , \]

i.e., the successive heights of \( P_1 \) (resp. of \( P_2 \)) are the successive values of \( x_i + y_i \) (resp. of \( x_i - y_i \)). We easily see that the constraint of staying in \( \tilde{Q} \) is mapped to the constraint that both \( P_1 \) and \( P_2 \) remain nonnegative. In addition, for even length \( 2n \) the endpoint conditions ensure that \( P_1 \) starts and ends at \( 1 \), while \( P_2 \) starts and ends at \( 0 \); so \( Q \) is a Dyck path of length \( 2n \) and \( P_1 \) identifies to a Dyck path of length \( 2n + 2 \), upon prepending an up-step and appending a down-step. For odd length \( 2n + 1 \), the endpoint conditions ensure that \( P \) starts at \( 1 \) and ends at \( 0 \), while \( P_2 \) starts at \( 0 \) and ends at \( 1 \); hence both \( P_1 \) and \( P_2 \) identify to a Dyck path of length \( 2n + 2 \) (upon prepending a down-step to \( P_1 \) and appending a down-step to \( P_2 \)).

We obtain the bijection for Theorem 7 by composing Lemma 8 with Lemma 9.

4. **Proof of Theorem 3:** Hesitating Walks

We next give the details of the bijection between hesitating axis-walks in the octant, and hesitating excursions to prove Theorem 3. The initial part is similar to before, and we provide two variants for the second.

4.1. **Transformation into decorated hesitating excursions in the octant.** The general strategy is the same as for simple walks: turn an axis-walk in the octant into an open partition diagram, remove the open arcs while marking their locations, and transform the decorated diagram back to a decorated excursion. In contrast, the second part is different from the simple walk case, since the marking does not easily induce an excursion in a larger domain. That is why we need an additional step of decoration transfer.

**Lemma 10.** Hesitating walks of length \( 2n \) staying in the octant and ending at \((m, 0)\) are in bijection with hesitating excursions of length \( 2n \) staying in the octant in which \( m \) \( W \)-steps on the \( x \)-axis have been marked.
Proof. Using Property 6, it is easy to check that for $\pi$ a partition-diagram of size $n$ and $\omega$ the corresponding hesitating excursion of length $2n$ in the octant, the (closing or fixed) points of $\pi$ where an open arc can be added exactly correspond to the $W$-steps of $\omega$ on the $x$-axis. If we mark $m$ such steps we obtain an open partition diagram of size $n$ with $m$ open arcs and no enhanced 3-crossing, which itself corresponds (by Theorem 5) to an hesitating walk of length $2n$ in the octant that ends at $(m, 0)$. \hfill \square

A similar property can be deduced for excursions in the first quadrant: the following result is proved in [8] and can be seen as a consequence of the reflection principle with respect to the diagonal.

**Lemma 11.** Hesitating (resp. simple) excursions of length $2n$ in the first quadrant are in bijection with hesitating (resp. simple) excursions of length $2n$ in the first octant with marked steps leaving the diagonal $y = x$.

The number of marked steps of the second object corresponds either to a parameter called the switch-multiplicity of the walk, which is roughly speaking the number of times that the walk crosses the diagonal, or similarly to the number of times the walk goes over the diagonal.

4.2. Moving the marks around. Theorem 3 holds when there is an equidistribution of the parameter counting the steps leaving the diagonal, and the parameter counting $W$-steps on the $x$-axis. This is true, and furthermore, they are symmetrically distributed.

**Proposition 12.** There is an explicit involution over the set of hesitating excursions of length $2n$ in the octant that exchanges the number of $W$-steps on the $x$-axis and the number of steps leaving the diagonal.

**Proof.** The proof passes through four main intermediaries: We first map a hesitating walk to a simple walk, tracking enough information to recover the hesitating walk. We use a classic mapping of steps to convert a simple excursion to a pair of Dyck paths. Then, we apply an involution on pairs of Dyck paths which swaps a key parameter. The final deduction comes from tracing parameters through these bijections, and back. This gives the stated result.

We now give the details of the individual steps.

**From hesitating walks to simple walks.** We call a sailing point a positive step, $E$ or $N$, followed by a negative step, $W$ or $S$. We transform every hesitating walk into a simple walk in which some sailing points are marked. To do so, we gather the steps of the hesitating walk in pairs, discarding the zero-steps, and marking every sailing point induced by the gathering of two non-zero steps. Thus, every hesitating excursion in the octant is identified to a simple excursion in the octant where some sailing points are marked.

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From simple walks to pairs of Dyck paths. Simple excursions in the octant are mapped to non-crossing pairs of Dyck paths by the transformation \((x_i, y_i) \rightarrow ((x_i + y_i), (x_i)\).
If we restrict the axis-walk to be an excursion, then under the previous bijection, \(hw\) is also an excursion, and \(\text{laststep}\) is necessarily 0 (hence the factor 2).

We now use the bijection of Theorem 3, which matches the hesitating axis-walks in the first octant and the hesitating excursions in the first quadrant. Translated in terms of vacillating walks via the above bijection, it means that the vacillating axis-walks of the first octant are in bijection with the pairs formed by a vacillating excursion in the first quadrant, and a step in \(\{E,0\}\).

As for the counting formula for vacillating excursions of half-length \(n\) in the first quadrant, it directly follows from the fact that hesitating excursions of half-length \(k\) in the first quadrant are counted by \(B_{k+1}\).

\[\square\]

5. Alternative Proofs Using Schnyder Woods

5.1. Schnyder woods and pairs of non-crossing Dyck paths. A rooted triangulation is the embedding of a simple planar graph in the plane such that all faces are triangles, with a distinguished external face and a distinguished root vertex on the external face. We are interested in a particular kind of edge-coloring and orientation of triangulations, known as Schnyder woods. Schnyder woods were introduced by Schnyder in [28] and [29] for triangulations and were later extended to different families of maps. There are several applications of these colorings, ranging from graph drawing to map encoding.

We describe a Schnyder wood using some conventions of the triangulation. The size of a triangulation is the number of vertices minus 3 (we do not count the vertices of the external face). The root vertex is labeled \(v_0\), and the two other vertices in counterclockwise order around the external face are labeled \(v_1\) and \(v_2\). The vertices that are not incident to the external face and edges that do not bound the external face are called internal.

A Schnyder wood of a simple triangulation is an edge-coloring into 3 colors, along with an orientation of all internal edges, such that the following properties are satisfied:

- for each \(i \in \{0,1,2\}\), the set of \(i\)-colored edges forms a spanning tree \(T_i\), rooted and oriented towards \(v_i\);
- if, for \(i \in \{0,1,2\}\), an \(i\)-tail (resp. \(i\)-head) with respect to a vertex denotes an edge colored by \(i\) oriented away from (resp. toward) this vertex, then in clockwise order around any internal vertex, there are: one 0-tail, some 1-heads, one 2-tail, some 0-heads, one 1-tail, some 2-heads.

Schnyder woods are in bijective correspondence with several combinatorial families. A pair \(P,Q\) of Dyck paths of length \(2n\) (both starting at \((0,0)\) and ending at \((2n,0)\)) is called non-crossing if for each \(0 \leq i \leq 2n\), the height of \(P\) at abscissa \(i\) is at most the height of \(Q\) at abscissa \(i\).

**Theorem 15** (Bonichon [4]). Non-crossing pairs of Dyck paths of length \(2n\) are in bijection with Schnyder woods of size \(n\).

Theorem 15 was first proved by Bonichon in [4]. Bernardi and Bonichon improved the description of this bijection a few years later [3]. It is the latter form that we use here. We give a quick description of the map \(\Psi\) from Schnyder woods to pairs of non-crossing Dyck paths. See the bottom-right part of Figure 6 for an example of the bijection. First, remember the classical bijection \(\Omega\) between plane trees and Dyck paths: take a plane tree, turn around it clockwise, starting and ending at the root; the first time an edge is visited,
write an up-step, the second time, write a down step. From a Schnyder wood, we now generate a pair \((P,Q)\) of non-crossing Dyck paths. The bottom path \(Q = UD^{\alpha_1} \ldots UD^{\alpha_n}\) is the path representing the tree \(T_0\) of color 0: \(Q = \Omega(T_0)\). The tour around \(T_0\) induces an order on the internal vertices, that we subsequently call \(u_1, \ldots, u_n\) (the first vertex visited by the tour is \(v_0\)). Let \(\beta_i\) be the number of 1-heads incident to \(u_i\) and let \(\beta_{n+1}\) be the number of 1-heads incident to \(v_1\). Note that \(\beta_1\) has to be 0. The upper path is now defined as \(P = UD^{\beta_2} \ldots UD^{\beta_{n+1}}\), where \(U\) stands for up-step and \(D\) stands for down-step. We refer the reader to [3] for a proof that \(P\) is positive and does not cross \(Q\), a description of the reverse map \(\Phi\), and a proof that \(\Psi\) and \(\Phi\) are reciprocal bijections.

For our own purpose, we need to review how certain parameters are transformed by \(\Psi\), as shown in Table 1 where the first column corresponds to Schnyder woods, and the second column corresponds to pairs of Dyck paths. The bijections \(\Psi\) and \(\Phi\) map each parameter to its counterpart on the same row. In the proof of Proposition 12, we defined the upper bounces as common down-step. Similarly, a reversed upper bounce is a common up-step.

**Lemma 16.** The bijection \(\Phi\) satisfies the correspondence of parameters given by Table 1.

**Proof.** This proof uses references and notation from [3]. Row 1 is trivial, and Rows 2 and 4 are well-known properties on the bijection between Dyck paths and trees. Row 5 is a direct consequence of the construction of the bijection, and Row 3 is a direct consequence of the fact that an upper peak is a descent \(i\) of positive length \(\beta_i > 0\), which corresponds to an internal node of \(T_1\).

Row 6 is a bit less obvious. It corresponds to the tight case in [3] when proving that the pair of walks is non-crossing; we sketch the main arguments. Let \(u_i\) be an internal vertex of \(T\) (for \(1 \leq i \leq n\)), and let \(h_i\) be the tail of color 2 at \(u_i\). Then one can check that \(u_i\) is a neighbor of \(v_2\) in \(T_2\) if and only if \(h_i\) is not below a 1-arc (an arc of color 1), i.e., there is no 1-arc \(e\) such that \(h_i\) is inside the (unique) cycle formed by \(e\) and \(T_0\). In such a situation, \(h_i\) comes after the tail of \(e\) and before the head of \(e\) during a clockwise tour around \(T_0\). The number of 1-tails before \(h_i\) in such a tour is \(\sum_{j < i} \alpha_j\), while the number of 1-heads before \(h_i\) is \(\sum_{j < i} \beta_j\). Consequently, given that a 1-tail always comes in the tour before its corresponding 1-head, \(u_i\) is a neighbor of \(v_2\) in \(T_2\) if and only if the previous numbers coincide, i.e., \(\sum_{j < i} \alpha_j = \sum_{j < i} \beta_j\). Finally, we remark that \(-i + \sum_{j < i} \alpha_j\) (resp. \(-i + \sum_{j \leq i} \beta_j\)) gives the height of the \(i\)th up-step of the lower (resp. upper) Dyck path; hence the two numbers
match if and only if the $i$th up-steps of the lower and upper Dyck paths form a reversed upper bounce.

5.2. Consequences. From Property 16, we can prove several non-trivial properties on non-crossing pairs of Dyck paths, and subsequently on excursions in the octant. Schnyder woods have more evident symmetries than pairs of Dyck paths, and the very expression of these symmetries gives involutions that are not easily phrased in terms of pairs of Dyck paths.

We give an alternative proof of Theorem 2 (see Figure 6 for an outline of the bijection).

Proof of Theorem 2. First, we use Lemma 11 to map an excursion in the quadrant to an excursion in the octant with $k$ marked steps leaving the diagonal. This excursion is then mapped to a pair of non-crossing Dyck paths with $k$ marked lower bounces.

We apply $\Phi$, move the root from $v_0$ to $v_1$ and change the orientation (meaning that “clockwise” becomes “counterclockwise”). Moving the root and changing the orientation amounts to exchanging the roles of $T_0$ and $T_1$. Hence, according to Lemma 16, when we then apply $\Psi$ to get back to a pair of non-crossing Dyck paths, we get $k$ marked steps on the descent of the upper path.

We reverse these $k$ steps, and the upper path now ends at height $2k$. We map the pair of paths back to a walk in octant that ends at coordinates $(k, k)$.

The following theorem is a stronger version of Claim 13, and thus completes the proof of Theorem 3. It is proved in a similar way.

Theorem 17. The pairs of non-crossing Dyck paths of length $2n$ with $p$ peaks, $\ell$ lower bounces and $u$ upper bounces are in bijection with the pairs of non-crossing Dyck paths of length $2n$ with $p$ peaks, $u$ lower bounces and $\ell$ upper bounces.
Figure 7. A minimal counterexample to the conjectural identity $N(n, p_1, \ldots p_k) = N(n, n-p_k+1, \ldots n-p_1+1)$ (which holds for $k \in \{1, 2\}$). We have $3 = N(4, 2, 3, 2) \neq N(4, 3, 2, 3) = 2$.

Proof. We take a pair $(P, Q)$ of non-crossing Dyck paths, reverse them to transform upper bounces into reversed upper bounces, while keeping the same number of peaks and lower bounces, and apply $\Phi$ to get a Schnyder wood. We then move the root of the map from $v_0$ to $v_2$, and flip the orientation of the plane. Finally we apply $\Psi$ to get back to a pair of non-crossing Dyck paths, and reverse again the two paths. Lemma 16 is enough to conclude the proof. \hfill \Box

5.3. An extension of the Narayana symmetry. The Narayana number $N(n, p)$ is defined as the number of Dyck paths of length $2n$ with $p$ peaks. These numbers refine the Catalan numbers $\text{Cat}_n$, in the sense that $\sum_{p=1}^{n} N(n, p) = \text{Cat}_n$. The following statement is well-known.

Property 18. The Narayana numbers satisfy the following symmetry property

$$N(n, p) = N(n, n-p+1).$$

The symmetry can be obtained from a classical bijection between Dyck paths of length $2n$ and rooted binary trees with $n+1$ leaves: the number of peaks of the Dyck path is mapped to the number of left leaves, and the symmetry follows by applying a reflection to the tree.

The Narayana numbers can be extended to any $k$-tuple of non-crossing Dyck paths in the following way: $N(n, p_1, \ldots, p_k)$ is the number of non-crossing $k$-tuples $D_1, \ldots, D_k$ of Dyck paths (ordered from bottom to top) such that $D_i$ has $p_i$ peaks.

Theorem 19. For $k = 2$, the extended Narayana numbers satisfy the symmetry property

$$N(n, p, q) = N(n, n-q+1, n-p+1).$$

Proof. The method we use is similar both to the case of classical Narayana numbers and to the previous subsection. Starting from a non-intersecting pair of Dyck paths, we apply $\Phi$, move the root from $v_0$ to $v_1$, change the orientation, and apply $\Psi$ back to a pair of paths. This yields an involution on non-crossing pairs of Dyck paths that has the desired peak-parameter correspondence, according to Lemma 16. \hfill \Box

However, a similar symmetry does not seem to hold for higher values of $k$. For example, one could expect that $N(n, p_1, \ldots, p_k) = N(n, n-p_k+1, \ldots, n-p_1+1)$, but we present a minimal counterexample to that in Figure 7.

6. A new bijection for Young tableaux of even-bounded height

As we have seen in Section 3, the main step in the proof of Theorem 1 is an explicit bijection between simple axis-walks of length $n$ staying in the octant $\{x \geq y \geq 0\}$, and simple walks of length $n$ from $(\frac{1}{2}, \frac{1}{2})$ to $(\frac{1}{2}, \frac{(-1)^n}{2})$ staying in the tilted quadrant. As it turns
out, this bijection can be easily generalized to any dimension, and infers new connections with standard Young tableaux with even-bounded height.

6.1. Walks in higher dimensional Weyl Chambers. For \( k \geq 1 \), we define the \( k \)-dimensional Weyl chamber\(^3\) of type C as
\[
W_C(k) := \{(x_1, x_2, \ldots, x_k) \mid x_1 \geq x_2 \geq \cdots \geq x_k \geq 0\},
\]
and the \( k \)-dimensional Weyl chamber of type D as
\[
W_D(k) := \{(x_1, x_2, \ldots, x_k) \mid x_1 \geq x_2 \geq \cdots \geq x_k \geq 0 \mid x_k\}.
\]
In this context, an axis-walk is any walk starting at the origin and ending on the \( x_1 \)-axis. With these definitions, the generalization of Theorem 1 reads as follows.

**Theorem 20.** For \( k \geq 1 \) and \( n \geq 0 \), there is an explicit bijection between simple axis-walks of length \( n \) staying in \( W_C(k) \) and simple excursions of length \( n \) staying in \( W_D(k) \), starting from \((\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})\), and ending at \((\frac{1}{2}, \ldots, \frac{1}{2}, (-1)^n)\). The ending \( x_1 \)-coordinate of a walk from \( W_C(k) \) corresponds to the number of steps that change the sign of \( x_k \) in its bijective image.

Note that the case \( k = 1 \) is precisely our introductory example, and the case \( k = 2 \) is the first part of Lemma 9. The arguments to show Theorem 20 are very similar to those in the proofs of Lemma 8 and the first part of Lemma 9. They use the general formulation of Theorem 5 (specifically, the bijection between open matching diagrams without \((k + 1)\)-crossing and simple axis-walks in \( W_C(k) \)), and the property that the intervals where an open arc can be added (without creating a \((k + 1)\)-crossing) correspond to the visits of the walk to the hyperplane defined by \( x_k = 0 \).

Moreover, it has been recently shown [8, 26] that the set of standard Young tableaux of size \( n \) with height at most \( d \) is in bijection with simple axis-walks of length \( n \) in \( W_C(k) \), with the ending \( x_1 \)-coordinate mapped to the number of columns of odd length. Composing this bijection with Theorem 20 infers the following result.

**Corollary 21.** For \( n, k \geq 1 \), there is an explicit bijection between the standard Young tableaux of size \( n \) with height at most \( d \), and the simple walks of length \( n \) staying in \( W_D(k) \), starting from \((\frac{1}{2}, \ldots, \frac{1}{2}, \frac{1}{2})\), and ending at \((\frac{1}{2}, \ldots, \frac{1}{2}, (-1)^n)\). The number of odd columns corresponds to the number of steps that change the sign of \( x_k \).

6.2. Recovering Gessel’s formula. Thanks to the lattice path enumeration techniques of Grabiner and Magyar [23], the previous corollary has an interesting consequence: a combinatorial interpretation of the determinant expression of Gessel [20] for the generating function of standard Young tableaux of even-bounded height.

**Proposition 22** [Gessel [20]]. Let \( Y_d[n] \) be the number of Young tableaux of size \( n \) with at most \( d \) rows, and \( Y_d(x) = \sum_{n \geq 0} \frac{1}{n!} Y_d[n] x^n \) the associated generating function. Then for each \( k \geq 1 \),
\[
Y_{2k}(x) = \det \left( I_{i-j}(2x) + I_{i+j-1}(2x) \right)_{1 \leq i, j \leq k},
\]
where (for \( m \in \mathbb{Z} \)) \( I_m(2x) = \sum_{i \geq 0} \frac{1}{(m+1)!} x^{m+2i} \).

\(^{3}\)For convenience we define the chambers using non-strict inequalities, our bijective statements can equivalently be given under strict inequalities, upon applying the coordinate shift \( \tilde{x}_i = x_i + k + 1 - i \).
Let us now explain how we can recover this result from Corollary 21. First, it proves to be convenient to take here the Weyl chamber of type D under the form
\[ \hat{W}_D(k) := \{|x_1| < x_2 < \cdots < x_k\}. \]

For each point \( \lambda = (\lambda_1, \ldots, \lambda_k) \) (in \( \mathbb{R}^k \)) we denote by \( \lambda' \) the point \((-\lambda_1, \lambda_2, \ldots, \lambda_k)\). Let \( \rho = (1/2, 3/2, \ldots, k - 1/2) \). Then Corollary 21 states that for \( n \) even (resp. odd), \( Y_{2k}[n] \) is the number of walks of length \( n \) in \( \hat{W}_D(k) \) from \( \rho \) to \( \rho' \) (resp. to \( \rho \)).

For every points \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \ldots, \mu_k) \) both in \( \hat{W}_D(k) \), let \( N_{\lambda,\mu}[n] \) be the number of simple walks of length \( n \) from \( \lambda \) to \( \mu \) staying in \( \hat{W}_D(k) \), and let \( N_{\lambda,\mu}(x) := \sum_{n \geq 0} \frac{1}{n!} N_{\lambda,\mu}[n] x^n \) be the associated generating function.

**Lemma 23** (from [23]). For every points \( \lambda = (\lambda_1, \ldots, \lambda_k) \) and \( \mu = (\mu_1, \ldots, \mu_k) \) in \( \hat{W}_D(k) \) such that \( \lambda_i \) and \( \mu_i \) belong to \( 1/2 + \mathbb{Z} \) for \( i \in \{1, \ldots, k\} \), we have
\[
N_{\lambda,\mu}(x) + N_{\lambda,\mu'}(x) = \det(I_{\lambda_i-\mu_j}(2x) + I_{\lambda_i+\mu_j}(2x))_{1 \leq i,j \leq k},
\]
where, for \( m \in \mathbb{Z} \), \( I_m(2x) \) is defined as \( I_m(2x) = \sum_{i \geq 0} \frac{1}{(m+i)!} x^{m+2i} \).

The proof techniques in [23] rely on a general reflection principle (see [21, Theorem 1]) for Weyl chamber walks, which results in determinant expressions for the relevant generating functions (the determinant is naturally expressed in terms of \( I_m(2x) \), which is the exponential generating function of simple 1d walks that start at 0 and end at \( m \)).

By Corollary 21 we have
\[ Y_{2k}(x) = N_{\rho,\rho}(x) + N_{\rho,\rho'}(x), \]
where \( N_{\rho,\rho}(x) \) gathers the coefficients of even power and \( N_{\rho,\rho'}(x) \) gathers the coefficients of odd power. Hence, applying Lemma 23 to \( \lambda = \mu = \rho \), we recover Proposition 22.

### 6.3. Related formulas.
For odd \( d \), the expression for the generating function \( Y_d(x) \) found in [20] is
\[ Y_{2k+1}(x) = e^x \det(I_{i-j}(2x) - I_{i+j}(2x))_{1 \leq i,j \leq k}. \]
In that case the combinatorial derivation is easier. Indeed, a Young tableau with at most \( 2k+1 \) rows identifies (via the Robinson-Schensted correspondence) to an involutive permutation without \((2k+2)\)-decreasing subsequence, which itself identifies to a partial matching diagram without \((k+1)\)-nesting (partial here means that there can be isolated points along the line). This implies that
\[
\sum_{n \geq 0} \frac{1}{n!} Y_{2k+1}(n) x^n = e^x M_k(x),
\]
where \( M_k(x) \) is the exponential generating function for matching diagrams without \((k+1)\)-nesting. By the Chen et al. bijection [11], the series \( M_k(x) \) is the exponential generating function of simple walks in the Weyl chamber of type \( C \) starting and ending at the origin; and it is shown in [23, Section 6.2] that this generating function is \( \det(I_{i-j}(2x) - I_{i+j}(2x))_{1 \leq i,j \leq k} \).

Similarly, a determinant formula is known for the enumeration of pairs of Young tableaux of bounded height. More precisely, let \( u_d[n] \) be the number of pairs of Young tableaux of the same shape with at most \( d \) rows (also by the Robinson-Schensted correspondence, the number of permutations in \( S_n \) with no \((d+1)\)-increasing subsequence). Then for every
d ≥ 1, we have as shown in [20] (note that this time the expression is uniform in d, with no dependence on the parity)

\[ \sum_{n \geq 0} \frac{u_d[n]}{n!^2} x^{2n} = \det (I_{i,j}(2x))_{1 \leq i,j \leq k}. \]

A combinatorial proof of this expression has been given in [19] via simple walks ending at so-called Toeplitz points, see also [30] for a combinatorial derivation based on arc diagrams.

Regarding asymptotic enumeration, it should in principle be possible to use recent results by Denisov and Wachtel for the asymptotic enumeration of walks in cones [13, Theo. 6] in order to recover from Theorem 21 the expression found by Regev [27] for the asymptotic number of Young tableaux of size \( n \) with at most \( 2k \) rows, which is, for each fixed \( k \geq 1 \),

\[ Y_{2k}[n] \sim_{n \to \infty} (2/\pi)^{k/2}(2k)^n(k/n)^{k(k-1)/2} \prod_{i=0}^{k-1} (2i)!. \]

(The relevant constants for the Weyl chamber of type D can be computed from [24] and from Selberg integrals [18, Eq. 1.20].)

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