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Non deterministic computation over the real numbers

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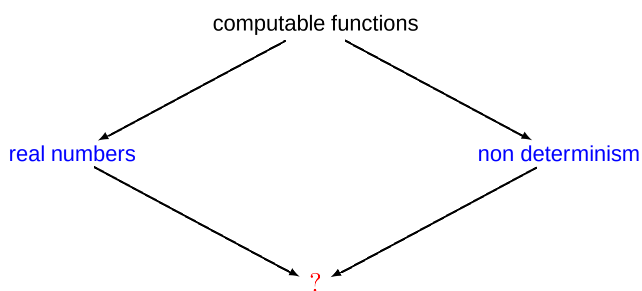

Some motivations

The usual formulation of the laws of nature: propositions

This meeting: algorithmic formulation of the laws of nature

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Algorithms on discrete structures ($\{0,1\}^*$, $\mathbb{N}$, $\mathbb{Q}$, ...) or continuous ones ($\mathbb{R}$, $\mathbb{C}$, ...) ?

No answer to this question here

$\mathbb{R}$-algorithms needed anyway to compare algorithmic and propositional formulations

Deterministic or non-deterministic algorithms?

algorithmic formulations of the laws of nature: no more predictions than propositional ones

... that may be non deterministic (e.g. Qϕ)

Non deterministic computation (over discrete structures, e.g. $\mathbb{N}$)

Extend the notion of computable function to that of computable relation

Finitely computable $\subsetneq$ Decidable $\subsetneq$ Semi-decidable = Effectively enumerable

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Decidable: computable function f s.t. $f(x,y) = 1$ if $x\ R\ y$ and $f(x,y) = 0$ otherwise

Semi-decidable: computable fn f s.t. $f(x,y) = 1$ if $x\ R\ y$ and $f(x,y)$ undefined otherwise

Effectively enumerable: a computable fn f s.t. $x\ R\ y$ if $\exists i \in \Omega\ (y = f(x,i))$

Semi-decidable (a.k.a. effectively enumerable) relations: the good notion

Corresponds to operational definitions of non deterministic algorithms

(Deterministic) computation over the real numbers

$g : \mathbb{R} \to \mathbb{R}$ computable if there exists

$G : (\mathbb{Q} \times (\mathbb{Q}^+ \setminus \{0\})) \to (\mathbb{Q} \times (\mathbb{Q}^+ \setminus \{0\} \cup \{\infty\}))$ computable s.t.

$$(G(q,\eta) = (r,\varepsilon) \text{ and } |x - q| \leq \eta) \Rightarrow |g(x) - r| \leq \varepsilon$$

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For all x, $(q_n)_n$, $(\eta_n)_n$ s.t.

for all n, $|x - q_n| \leq \eta_n$ and $(\eta_n)_n \to 0$

then $(\varepsilon_n)_n \to 0$ where $(\varepsilon_n)_n$ defined by $(\_, \varepsilon_n) = G(q_n, \eta_n)$

Continuity

Merging: decidable relations

R decidable relation if there exists

$f : \mathbb{R} \times \mathbb{R} \to \{0,1\}$ computable s.t. $f(x,y) = 1$ if $x \, R \, y$ and $f(x,y) = 0$ otherwise

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f continuous hence constant

Only two decidable relations (empty and full)

Not enough to express the laws of nature...

Try something else: semi-decidable

Needs a notion of partial computable function over the real numbers

Partial functions

Allow G to be partial

Not a very good idea: terminates or not depending on the chosen approximation of the same x

Relax the convergence condition

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$(q_n)_n$,$(\eta_n)_n$ shrinking approximations of x, $(\varepsilon_n)_n$ defined by $(\_, \varepsilon_n) = G(q_n, \eta_n)$

then either $(\varepsilon_n)_n \longrightarrow 0$ or $\varepsilon_n = \infty$ for all n

Example: $\chi(x) = 1$ if $x > 0$ and $\chi(x)$ undefined otherwise

computed by $G(q, \eta) = (1, \eta)$ if $q - \eta > 0$ and $G(q, \eta) = (1, \infty)$ otherwise

Merging: semi-decidable relations

R semi-decidable relation if there exists

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f computable s.t. $f(x, y) = 1$ if $x \; R \; y$ and $f(x, y)$ undefined otherwise

Better $y > x$ semi-decidable $f(x, y) = \chi(y - x)$

But $y = x$ is not: the domain of a partial computable function is always an open set

Merging: effectively enumerable relations

R effectively enumerable relation if there exists

$f : \mathbb{R} \times \Omega \to \mathbb{R}$ computable s.t. $x \; R \; y$ if $\exists i \in \Omega \; (y = f(x, i))$

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$y = x$: $\Omega = \mathbb{N}$, $f(x, i) = x$

$x \; R \; y$ if $(y = x$ or $y = x + 1)$: $\Omega = \mathbb{N}$, $f(x, 0) = x$, $f(x, n + 1) = x + 1$

More effectively enumerable relations than semi-decidable ones

On discrete structures (e.g. $\mathbb{N}$)

Every effectively enumerable relation is semi-decidable

Enumerate all the $i \in \Omega$ compute $f(x, i)$ if y return 1 else continue enumeration

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Assumes a decidable equality: correct for discrete structures, but not continuous ones

Semi-decidable $\subsetneq$ Effectively enumerable

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Only effectively enumerable relations can pretend to be used to describe the (non deterministic) laws of nature

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A test: functional relations

If a relation happens to be functional it is effectively enumerable if and only if it is computable as a function

$$f(x, i) = g(x)$$

Not true for semi-decidable relations: $x \mapsto x$ computable but $y = x$ not semi-decidable

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Ω

Indices introduced for reason internal to computability theory

Reminiscent of several notion

Used in various domains (but common point: potentiality)

- Elementary event (Kolmogorov's notion of probability space)
- Possible world (Kripke's semantics of modal logics)
- Universe (Everett's interpretation of Qϕ)
- Hidden variable (Einstein's interpretation of Qϕ)

Some of them are real (Everett's universes, local hidden variables, ...)

Some are not (Kolmogorov's events, Everett's universes according to Feynman, ...)

A question for philosophers: what does "real" mean in the sentence above?

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Probabilities

Moving from a description of possible events to a quantification of their the propension

(Almost) same results

But still a lot to be done

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What can we expect from a computational description of a (non deterministic) experiment?

Preparation of a system choosing x, evolution for a time T, measurement: y

An algorithm that indicates if y is a possible result or not? No way

An algorithm that indicates the probability that for y to be the result of the experiment? Neither

For each x, a “random variable” $f_x : \Omega \to \mathbb{R}$ s.t. $x, i \mapsto f_x(i)$ computable

Questions for physicists:

- Are all the $f_x(i)$ computed in parallel?
- What relation between the computation time (of $f_x(i)$) and the physical time T?

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A physical Church thesis for a non deterministic universe

For each experiment ~~the function mapping x to y~~ is ~~computable~~

For each experiment the relation between x and y is effectively enumerable

Effectively enumerable + functional = computable, thus

For deterministic experiments collapses exactly to the usual physical Church thesis