

# A neutral approach to proof and refutation in MALL

Olivier Delandè and Dale Miller

LIX, École Polytechnique  
Palaiseau, France

LICS 2008  
June 27, 2008



## Outline

### Introduction

### Purely additive games

The additive fragment of MALL without atoms

Neutral expressions

A simple additive game

### A game for MALL

MALL without atoms

Neutral expressions

Focalization

Indeterminacy

Neutral graphs

The full game

### Conclusion

# Introduction and motivations

In proof search, one typically commits to the choice of a goal to prove and proceeds from it. A failed attempt to find a proof does not give, in general, enough information to build a proof of the negation.

In contrast, game semantics prove useful for such a neutral approach.

Additive behaviour is well captured by games, but what about multiplicative behaviour?

# Contribution

We define a game semantics for the multiplicative additive fragment of linear logic (MALL) without atoms.

- ▶ Our approach is distinct from well-known game semantics (Hyland & Ong, Blass, Abramsky); we don't model cut-elimination;
- ▶ our game is *positional*;
- ▶ winning strategies correspond to cut-free proofs;
- ▶ games can have three outcomes: winning strategy for player (proof), winning strategy for opponent (proof of the negation), no winning strategy for either player (unprovability of both).

# The additive fragment of MALL without atoms

## Syntax

$$F := F \oplus F \mid 0 \mid F \& F \mid \top$$

## Inference rules

$$\frac{\vdash F}{\vdash F \oplus G} \oplus_1 \quad \frac{\vdash G}{\vdash F \oplus G} \oplus_2 \quad \frac{\vdash F \quad \vdash G}{\vdash F \& G} \& \quad \frac{}{\vdash \top} \top$$

# Neutral expressions

Let us define a two-player game for this logic.

A neutral expression  $E$  represents a pair of dual formulae.

$$G := E + E \mid \mathbf{0}$$

$$E := G \mid \updownarrow G$$

A neutral expression  $E$  has a *positive* and a *negative* translation.

$E$	$E_1 + E_2$	$\mathbf{0}$	$\updownarrow G$
$[E]^+$	$[E_1]^+ \oplus [E_2]^+$	$0$	$[G]^-$
$[E]^-$	$[E_1]^- \& [E_2]^-$	$\top$	$[G]^+$

# A simple additive game

We define a two-player game in which positions are neutral expressions.

Consider the rewrite relation defined by

$$E_1 + E_2 \mapsto E_1 \quad E_1 + E_2 \mapsto E_2$$

When playing at position  $E$ , a player may move to  $F$  iff  $E \mapsto^* \uparrow F$ .  
A player loses at position  $E$  if there is no move from  $E$ .

## Theorem

The player (resp. the opponent) has a winning strategy in  $E$  iff  $[E]^+$  (resp.  $[E]^-$ ) is provable.

## Interpretation of moves

A rewrite step (aka “micro-move”)  $E_1 + E_2 \mapsto E_i$  is seen as

$$\frac{\vdash [E_i]^+}{\vdash [E_1]^+ \oplus [E_2]^+} \oplus_i$$

by the player  
(chooses which disjunct to  
prove)

$$\frac{\vdash [E_1]^- \quad \vdash [E_2]^-}{\vdash [E_1]^- \& [E_2]^-} \&$$

by the opponent  
(is prepared to prove either  
conjunct)

# Interpretation of moves

A rewrite step (aka “micro-move”)  $E_1 + E_2 \mapsto E_i$  is seen as

$$\frac{\vdash [E_i]^+}{\vdash [E_1]^+ \oplus [E_2]^+} \oplus_i$$

by the player  
(chooses which disjunct to  
prove)

$$\frac{\vdash [E_1]^- \quad \vdash [E_2]^-}{\vdash [E_1]^- \& [E_2]^-} \&$$

by the opponent  
(is prepared to prove either  
conjunct)

A move (aka “macro-move”)  $E \mapsto^* \updownarrow F$  is seen as a full layer of introductions of  $\oplus$  by the player and as a full layer of introductions of  $\&$  by the opponent (in *focused* proof systems, those layers are called *phases*).

# Features of the additive game

- ▶ The game is *determinate*,
- ▶ it is symmetric,
- ▶ the players view the game as *dual derivations*,
- ▶ a micro-move corresponds to the application of an inference rule,
- ▶ a macro-move corresponds to a full phase.

# Features of the additive game

- ▶ The game is *determinate*,
- ▶ it is symmetric,
- ▶ the players view the game as *dual derivations*,
- ▶ a micro-move corresponds to the application of an inference rule,
- ▶ a macro-move corresponds to a full phase.

## Objective

Define a similar game for the more expressive logic MALL without atoms.

Note: MALL without atoms is PSPACE-complete.

# MALL without atoms

$$F := F \oplus F \mid 0 \mid F \& F \mid \top \mid F \wp F \mid \perp \mid F \otimes F \mid 1$$

## Additives

$$\frac{\vdash F_i, \Delta}{\vdash F_1 \oplus F_2, \Delta} \oplus_i \quad \frac{\vdash F, \Delta \quad \vdash G, \Delta}{\vdash F \& G, \Delta} \& \quad \frac{}{\vdash \top, \Delta} \top$$

## Multiplicatives

$$\frac{\vdash F, G, \Delta}{\vdash F \wp G, \Delta} \wp \quad \frac{\vdash \Delta}{\vdash \perp, \Delta} \perp$$
$$\frac{\vdash F, \Delta_1 \quad \vdash G, \Delta_2}{\vdash F \otimes G, \Delta_1, \Delta_2} \otimes \quad \frac{}{\vdash 1} 1$$

Note: initial and cut are admissible.

## Neutral expressions

We add a multiplicative neutral connective and its unit.

$$G ::= E + E \mid \mathbf{0} \mid E \times E \mid \mathbf{1}$$

$$E ::= G \mid \updownarrow G$$

$E$	$E_1 + E_2$	$\mathbf{0}$	$E_1 \times E_2$	$\mathbf{1}$	$\updownarrow G$
$[E]^+$	$[E_1]^+ \oplus [E_2]^+$	$0$	$[E_1]^+ \otimes [E_2]^+$	$1$	$[G]^-$
$[E]^-$	$[E_1]^- \& [E_2]^-$	$\top$	$[E_1]^- \wp [E_2]^-$	$\perp$	$[G]^+$

## Parallelism vs permutability

Consider the two following dual derivations:

$$\frac{\frac{\vdash F}{\vdash F \oplus G} \oplus_1 \quad \bar{1} \quad 1}{\vdash (F \oplus G) \otimes 1} \otimes$$

$$\frac{\frac{\vdash F^\perp}{\vdash F^\perp, \perp} \perp \quad \frac{\vdash G^\perp}{\vdash G^\perp, \perp} \perp}{\vdash F^\perp \& G^\perp, \perp} \&}{\vdash (F^\perp \& G^\perp) \wp \perp} \wp$$

On the left, the player may apply  $\oplus_1$  and  $1$  in any order. On the right, applying  $\perp$  before  $\&$  would change the derivation.

## Parallelism vs permutability

Consider the two following dual derivations:

$$\frac{\frac{\vdash F}{\vdash F \oplus G} \oplus_1 \quad \bar{1} \quad 1}{\vdash (F \oplus G) \otimes 1} \otimes$$

$$\frac{\frac{\vdash F^\perp}{\vdash F^\perp, \perp} \perp \quad \frac{\vdash G^\perp}{\vdash G^\perp, \perp} \perp}{\vdash F^\perp \& G^\perp, \perp} \& \perp}{\vdash (F^\perp \& G^\perp) \wp \perp} \wp$$

On the left, the player may apply  $\oplus_1$  and  $1$  in any order. On the right, applying  $\perp$  before  $\&$  would change the derivation.

$$\frac{\vdash F}{\vdash (F \oplus G) \otimes 1}$$

$$\frac{\vdash F^\perp \quad \vdash G^\perp}{\vdash (F^\perp \& G^\perp) \wp \perp}$$

# Abstracting away from micro-moves

The internal structure of a macro-move vis-à-vis micro-moves is abstracted from the game.

Focusing allows a similar abstraction in proofs. In particular, permutation of inference rules within a phase are identified in focused proofs.

# Focalization (Andreoli)

Andreoli showed that we can classify the connectives and units into two groups

Synchronous	$\oplus$	0	$\otimes$	1
Asynchronous	$\&$	$\top$	$\wp$	$\perp$

Andreoli's original focused proof system constrained proofs as follows:

1. apply, in any order, asynchronous rules until none are applicable (they all permute over each other);
2. then choose to focus on a (synchronous) formula and apply synchronous rules to it and its descendants until they become asynchronous.

This strategy lacks symmetry: in a synchronous phase we select *one* formula, while in an asynchronous phase we select *all* of them.

# Multi-focalization for MALL without atoms

## Additives and multiplicatives

$$\frac{\vdash \Gamma \Downarrow F_i, \Delta}{\vdash \Gamma \Downarrow F_1 \oplus F_2, \Delta} [\oplus_i] \quad \frac{\vdash \Gamma \Uparrow F, \Delta \quad \vdash \Gamma \Uparrow G, \Delta}{\vdash \Gamma \Uparrow F \& G, \Delta} [\&] \quad \frac{}{\vdash \Gamma \Uparrow \top, \Delta} [\top]$$

$$\frac{\vdash \Gamma \Uparrow F, G, \Delta}{\vdash \Gamma \Uparrow F \wp G, \Delta} [\wp] \quad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Uparrow \perp, \Delta} [\perp]$$

$$\frac{\vdash \Gamma_1 \Downarrow F, \Delta_1 \quad \vdash \Gamma_2 \Downarrow G, \Delta_2}{\vdash \Gamma_1, \Gamma_2 \Downarrow F \otimes G, \Delta_1, \Delta_2} [\otimes] \quad \frac{}{\vdash \Downarrow 1} [1]$$

## Phase changes

$$\frac{\vdash \Gamma, F \Uparrow \Delta}{\vdash \Gamma \Uparrow F, \Delta} [R \Uparrow] \quad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Downarrow \Delta} [R \Downarrow] \quad \frac{\vdash \Gamma \Downarrow \Delta}{\vdash \Gamma, \Delta \Uparrow} [D]$$

$(F \text{ sync.}) \quad (\Delta \text{ async.}) \quad (\Delta \neq \emptyset)$

## What's difficult with multiplicatives?

The logic is not complete! The neutral expression  $\uparrow \mathbf{1} \times \uparrow \mathbf{1}$  translates to two unprovable formulae:

$$\perp \otimes \perp \quad \text{and} \quad 1 \wp 1$$

## What's difficult with multiplicatives?

The logic is not complete! The neutral expression  $\Downarrow \mathbf{1} \times \Downarrow \mathbf{1}$  translates to two unprovable formulae:

$$\perp \otimes \perp \quad \text{and} \quad \mathbf{1} \wp \mathbf{1}$$

$$\frac{\frac{\frac{\vdash A, \delta_1, \dots, \delta_k \quad \vdash B, \delta_{k+1}, \dots, \delta_n}{\vdash A \otimes B, \delta_1, \dots, \delta_n}}{\vdash (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n}}{\vdash (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp} \quad \frac{\frac{\vdash A^\perp, B^\perp}{\vdash A^\perp \wp B^\perp} \quad \vdash \delta_1^\perp \quad \dots \quad \vdash \delta_n^\perp}{\vdash (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp}}$$

On the left, the player chooses a partition of  $\delta_1, \dots, \delta_n$ . This information does not appear on the right.

On the right, you can tell  $A$  and  $B$  from the  $\delta_i$ . This information is lost on the left.

# Important consequences

- ▶ The game is no longer determinate. If neither  $[E]^+$  nor  $[E]^-$  are provable, then no one has a winning strategy in  $E$ .
- ▶ The state of the game cannot be a plain neutral expression any more. Previous work by Miller & Saurin investigated an intermediate situation in which multiplicatives were allowed between micro-moves, but not macro-moves.
- ▶ Derivations are now explicitly focused.

# Neutral graphs

In the additive case, a position in the game is a neutral expression.

- ▶ Its two translations are the frontiers of slices of two dual derivations;
- ▶ trivial invariant: applying cut to the two translations yields the empty sequent.

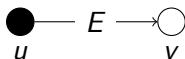
In order to preserve these properties in the full game, we need a position to record the multiplicative structure. A labeled graph structure, called a *neutral graph*, provides this additional information.

# Vertices as sequents, arcs as formulae

A neutral graph

- ▶ is a bipartite graph,
- ▶ whose arcs are labeled with neutral expressions,
- ▶ with no undirected cycles.

An arc



means that

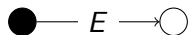
- ▶ the formula  $[E]^+$  occurs in the sequent associated with  $u$ ,
- ▶ the formula  $[E]^-$  occurs in the sequent associated with  $v$ .

## A basic example

The frontiers

$$\vdash [E]^+ \uparrow \mid \vdash \uparrow [E]^-$$

are represented by the neutral graph

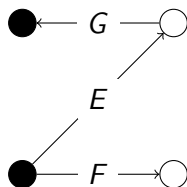


## A more complex example

The frontiers

$$\vdash [E]^+, [F]^+ \uparrow \quad \vdash \uparrow [G]^- \mid \vdash [G]^+ \uparrow [E]^- \quad \vdash \uparrow [F]^-$$

are represented by the neutral graph



# The full game

The players rewrite a neutral graph.

- ▶ A micro-move is seen as the application of a synchronous (resp. asynchronous) rule by the player (resp. the opponent);
- ▶ a macro-move is seen as a phase;
- ▶ some moves may make one (or both) of the proofs fail;
- ▶ a play goes on until both players fail (tie) or the graph becomes empty (win for the player who has not failed).

## Theorem

Player (resp. opponent) has a winning strategy from a neutral graph  $G$  iff the positive (resp. negative) translation of  $G$  is provable.

## A dynamic example

$$\begin{array}{c}
 \delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B \\
 \hline
 \delta_1, \dots, \delta_n \Downarrow A \otimes B \\
 \hline
 A \otimes B, \delta_1, \dots, \delta_n \Uparrow \\
 \hline
 \Uparrow A \otimes B, \delta_1, \dots, \delta_n \\
 \hline
 \Uparrow (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n
 \end{array}$$

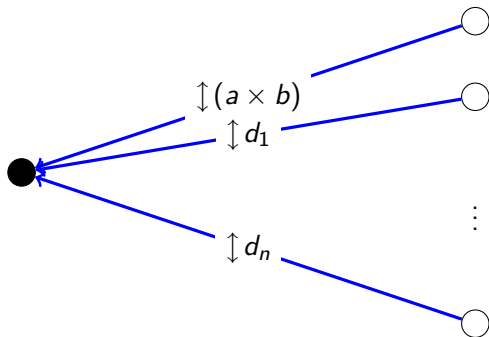
$$\begin{array}{c}
 \Uparrow A^\perp, B^\perp \\
 \hline
 \Uparrow A^\perp \wp B^\perp \quad \Uparrow \delta_1^\perp \quad \Uparrow \delta_n^\perp \\
 \hline
 \Downarrow A^\perp \wp B^\perp \quad \Downarrow \delta_1^\perp \quad \dots \quad \Downarrow \delta_n^\perp \\
 \hline
 \Downarrow (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp
 \end{array}$$

$$\bullet \leftarrow \Uparrow (a \times b) \times \Uparrow d_1 \times \dots \times \Uparrow d_n \text{ --- } \circ$$

## A dynamic example

$$\frac{\frac{\delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B}{\delta_1, \dots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \dots, \delta_n \Uparrow} \quad \frac{\uparrow A \otimes B, \delta_1, \dots, \delta_n}{\uparrow (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n}$$

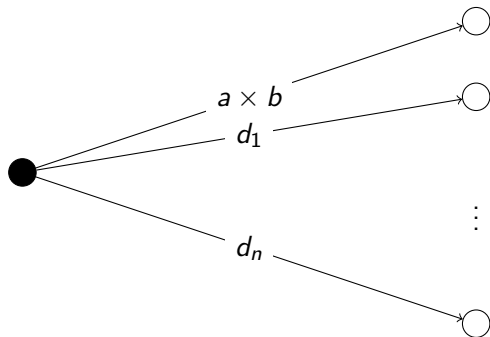
$$\frac{\frac{\uparrow A^\perp, B^\perp}{\uparrow A^\perp \wp B^\perp} \quad \frac{\uparrow \delta_1^\perp}{\downarrow \delta_1^\perp} \quad \dots \quad \frac{\uparrow \delta_n^\perp}{\downarrow \delta_n^\perp}}{\downarrow (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp}$$



## A dynamic example

$$\frac{\frac{\delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B}{\delta_1, \dots, \delta_n \Downarrow A \otimes B}}{\frac{A \otimes B, \delta_1, \dots, \delta_n \Uparrow}{\Uparrow A \otimes B, \delta_1, \dots, \delta_n}} \quad \Uparrow (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n$$

$$\frac{\frac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \wp B^\perp} \quad \Uparrow \delta_1^\perp \quad \dots \quad \Uparrow \delta_n^\perp}{\Downarrow A^\perp \wp B^\perp \quad \Downarrow \delta_1^\perp \quad \dots \quad \Downarrow \delta_n^\perp}}{\Downarrow (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp}$$



# A dynamic example

$$\frac{\delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B}{\delta_1, \dots, \delta_n \Downarrow A \otimes B}$$

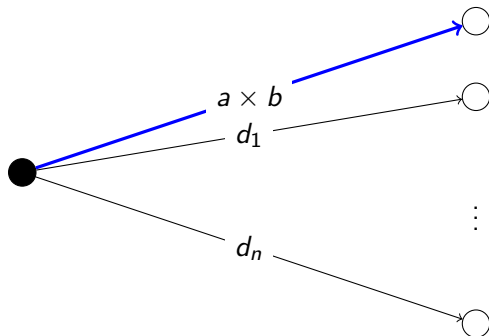
$$\frac{A \otimes B, \delta_1, \dots, \delta_n \Uparrow}{\Uparrow A \otimes B, \delta_1, \dots, \delta_n}$$

$$\frac{\Uparrow A \otimes B, \delta_1, \dots, \delta_n}{\Uparrow (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n}$$

$$\frac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \wp B^\perp}$$

$$\frac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp} \quad \dots \quad \frac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}$$

$$\frac{\Downarrow A^\perp \wp B^\perp \quad \Downarrow \delta_1^\perp \quad \dots \quad \Downarrow \delta_n^\perp}{\Downarrow (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp}$$



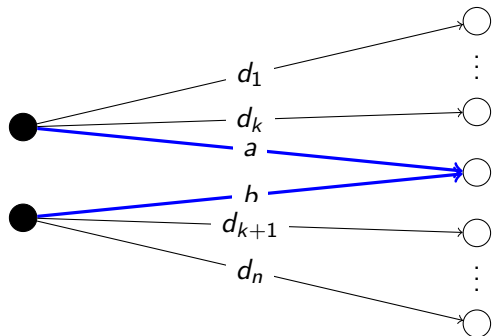
# A dynamic example

$$\frac{\frac{\delta_1, \dots, \delta_k \Downarrow A \quad \delta_{k+1}, \dots, \delta_n \Downarrow B}{\delta_1, \dots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \dots, \delta_n \Uparrow} \quad \frac{\quad}{\Uparrow A \otimes B, \delta_1, \dots, \delta_n} \quad \frac{\quad}{\Uparrow (A \otimes B) \wp \delta_1 \wp \dots \wp \delta_n}$$

$$\frac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \wp B^\perp} \quad \frac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp} \quad \dots \quad \frac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}$$


---


$$\Downarrow (A^\perp \wp B^\perp) \otimes \delta_1^\perp \otimes \dots \otimes \delta_n^\perp$$



## A typical macro-move

In a macro-move

$$G \xrightarrow{f_0, f_1} G'$$

the neutral graph  $G$  is rewritten to  $G'$ , and  $f_0$  (resp.  $f_1$ ) is a boolean value which is true iff player 0 (resp. 1) fails during the move.

A macro-move can be decomposed in micro-moves: an initial step selecting the neutral expressions to decompose, then small steps corresponding to single rule applications.

$$G \xrightarrow{D} G_0 \xrightarrow{f_0^{(1)}, f_1^{(1)}} G_1 \xrightarrow{f_0^{(2)}, f_1^{(2)}} \dots \xrightarrow{f_0^{(n)}, f_1^{(n)}} G_n = G'$$

and  $f_\sigma = \bigvee_{i=1}^n f_\sigma^{(i)}$  ( $\sigma \in \{0, 1\}$ ).

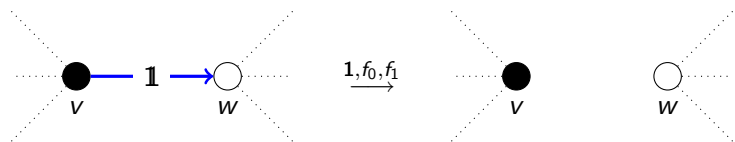
# Micro-moves

Micro-move	Sync reading	Async reading
$G \xrightarrow{D} G'$	$[D]$	none
$G \xrightarrow{R} G'$	$[R \Downarrow]$	$[R \Uparrow]$
$G \xrightarrow{+} G'$	$[\oplus]$	$[\&]$
$G \xrightarrow{\times} G'$	$[\otimes]$	$[\wp]$
$G \xrightarrow{0, f_0, f_1} G'$	none	$[T]$
$G \xrightarrow{1, f_0, f_1} G'$	$[1]$	$[\perp]$

## Remark

The micro-moves responsible for failure are those associated with units.

# Failure



is seen as the simultaneous application of

Player (black)

$$\frac{}{\vdash \Downarrow 1, \Delta_0} [1]$$

(requires  $\Delta_0 = \emptyset$ )

Opponent (white)

$$\frac{\vdash \Gamma_1 \Uparrow \Delta_1}{\vdash \Gamma_1 \Uparrow \perp, \Delta_1} [\perp]$$

(unprovable if  $\Gamma_1, \Delta_1 = \emptyset$ )

Hence player fails if  $v$  does not become isolated, opponent fails if  $w$  becomes isolated.

# Conclusion

- ▶ The neutral approach can have three outcomes: winning strategy for a player (proof), winning strategy for her opponent (refutation) or no winning strategy for either (no proof or refutation).
- ▶ This game with neutral graphs reveals the complexity of the multiplicatives.
- ▶ Every step in the game contributes simultaneously to building a proof and a refutation.
- ▶ This positional game yields relative completeness.

## Future work

- ▶ Extend the logic with fixed points, atoms, quantification.
- ▶ Develop connections with ludics.
- ▶ Change the framework to get full completeness (this may require switching to a move-based game).