A neutral approach to proof and refutation in MALL

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Outline

Introduction and motivations

- In logic programming, a single computation yields a proof or a refutation. We would like to achieve the same effect in proof theory.
- Game semantics seem well adapted to this neutral approach.
- Additive behaviour is well captured by games, but what about multiplicative behaviour?

Neutral approach to proof and refutation

In (say) Prolog, if we input a query such as

```
?- G.
```

Prolog will make a single computation and answer `yes` or `no`, yielding a proof of G or a refutation of G in a suitable proof system.

In proof search, we fix a goal and proceed from it. We would like to recover Prolog's neutral approach, i.e. to be able to start with $\vdash G$ and $\vdash \neg G$ at the same time.

Purely additive games (1/2)

Hintikka showed that truth in first-order logic can be characterized by games. Let $\mathcal{M}$ be a model. Two players, P and O, play with a closed formula.

- If it is atomic, then P (resp. O) wins if the formula is false (resp. true) in $\mathcal{M}$,
- if it is a conjunction, then P picks one of the conjuncts,
- if it is a disjunction, then O picks one of the disjuncts,
- if it is universally quantified, then P picks an instance,
- if it is existentially quantified, then O picks an instance.

Purely additive games (2/2)

- P (resp. O) has a winning strategy for a formula iff it is false (resp. true) in $\mathcal{M}$,
- the game is *determined*,
- the object manipulated by the players is a single formula,
- there is a symmetry: de Morgan dual connectives only differ by the player who decomposes them.

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Objective

Define a similar game for a more expressive logic, MALL without atoms.

MALL without atoms

$$F, G := F \oplus G \mid 0 \mid F \mathbin{\&} G \mid \top \mid F \parr G \mid \bot \mid F \otimes G \mid 1$$

Additives

$$\frac{\vdash F_i, \Delta}{\vdash F_1 \oplus F_2, \Delta}\,[\oplus_i] \quad \frac{\vdash F, \Delta \quad \vdash G, \Delta}{\vdash F \mathbin{\&} G, \Delta}\,[\&] \quad \frac{}{\vdash \top, \Delta}\,[\top]$$

Multiplicatives

$$\frac{\vdash F, G, \Delta}{\vdash F \parr G, \Delta}\,[\parr] \quad \frac{\vdash \Delta}{\vdash \bot, \Delta}\,[\bot]$$

$$\frac{\vdash F, \Delta_1 \quad \vdash G, \Delta_2}{\vdash F \otimes G, \Delta_1, \Delta_2}\,[\otimes] \quad \frac{}{\vdash 1}\,[1]$$

Problem

Represent two dual derivations being developed simultaneously.

Parallelism vs. permutability

Consider the two following dual derivations:

$$\dfrac{\dfrac{\vdash A}{\vdash A \oplus B}[\oplus_1] \quad \dfrac{}{1}[1]}{\vdash (A \oplus B) \otimes 1}[\otimes]$$

$$\dfrac{\dfrac{\dfrac{\vdash A^\perp}{\vdash A^\perp, \perp}[\perp] \quad \dfrac{\vdash B^\perp}{\vdash B^\perp, \perp}[\perp]}{\vdash A^\perp \,\&\, B^\perp, \perp}[\&]}{\vdash (A^\perp \,\&\, B^\perp) \parr \perp}[\parr]$$

On the left, the player may apply $[\oplus_1]$ and [1] in any order. On the right, applying $[\perp]$ before [&] would change the derivation.

Parallelism vs. permutability

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$$\dfrac{\dfrac{\dfrac{\vdash A^\perp}{\vdash A^\perp, \bot}\,[\bot] \qquad \dfrac{\vdash B^\perp}{\vdash B^\perp, \bot}\,[\bot]}{\vdash A^\perp \mathbin{\&} B^\perp, \bot}\,[\&]}{\vdash (A^\perp \mathbin{\&} B^\perp) \parr \bot}\,[\parr]$$

On the left, the player may apply $[\oplus_1]$ and $[1]$ in any order. On the right, applying $[\bot]$ before $[\&]$ would change the derivation.

Remark

It does not really matter since $[\bot]$ and $[\&]$ can be easily permuted. Could we see the right derivation modulo this permutability?

Focalization (Andreoli)

Andreoli showed that we can classify the connectives and units in two groups

Synchronous	$\oplus$	0	$\otimes$	1
Asynchronous	$\&$	$\top$	$\parr$	$\bot$

and constrain proofs to follow a strategy:

1. apply asynchronous rules until it is not possible any more, in any order (they all permute);
2. then choose a (synchronous) formula and *focus* on it, i.e. apply synchronous rules on it and its descendants until they become asynchronous.

A proof becomes an alternation of big steps: asynchronous and synchronous phases.

Multi-focalization for MALL without atoms

Additives and multiplicatives

$$\frac{\vdash \Gamma \Downarrow F_i, \Delta}{\vdash \Gamma \Downarrow F_1 \oplus F_2, \Delta}\ [\oplus_i] \quad \frac{\vdash \Gamma \Uparrow F, \Delta \quad \vdash \Gamma \Uparrow G, \Delta}{\vdash \Gamma \Uparrow F \,\&\, G, \Delta}\ [\&] \quad \frac{}{\vdash \Gamma \Uparrow \top, \Delta}\ [\top]$$

$$\frac{\vdash \Gamma \Uparrow F, G, \Delta}{\vdash \Gamma \Uparrow F \parr G, \Delta}\ [\parr] \quad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Uparrow \bot, \Delta}\ [\bot]$$

$$\frac{\vdash \Gamma_1 \Downarrow F, \Delta_1 \quad \vdash \Gamma_2 \Downarrow G, \Delta_2}{\vdash \Gamma_1, \Gamma_2 \Downarrow F \otimes G, \Delta_1, \Delta_2}\ [\otimes] \quad \frac{}{\vdash \Downarrow 1}\ [1]$$

Change phases

$$\frac{\vdash \Gamma, F \Uparrow \Delta}{\vdash \Gamma \Uparrow F, \Delta}\ [R\Uparrow] \quad \frac{\vdash \Gamma \Uparrow \Delta}{\vdash \Gamma \Downarrow \Delta}\ [R\Downarrow] \quad \frac{\vdash \Gamma \Downarrow \Delta}{\vdash \Gamma, \Delta \Uparrow}\ [D]$$

$(F \text{ sync.})$ $\quad$ $(\Delta \text{ async.})$ $\quad$ $(\Delta \neq \emptyset)$

Neutral expressions

A neutral expression represents a pair of dual formulas of linear logic. Miller & Saurin defined a language of neutral expressions. We present a modified version for the propositional fragment here:

$$G ::= \mathbb{0} | \mathbb{1} | E + E | E \times E$$
$$E ::= G | \updownarrow G$$

Each neutral expression has a *positive* and a *negative* translation:

E	$\mathbb{0}$	$E_1 + E_2$	$\mathbb{1}$	$E_1 \times E_2$	$\updownarrow G$
$[E]^+$	0	$[E_1]^+ \oplus [E_2]^+$	1	$[E_1]^+ \otimes [E_2]^+$	$[G]^-$
$[E]^-$	$\top$	$[E_1]^- \& [E_2]^-$	$\bot$	$[E_1]^- \parr [E_2]^-$	$[G]^+$

Dual derivations

$$\dfrac{\dfrac{}{\vdash\Downarrow 1}\,[1] \qquad \vdash\Downarrow F}{\vdash\Downarrow 1 \otimes F}\,[\otimes] \qquad \Bigg| \qquad \dfrac{\dfrac{\vdash\Uparrow F^{\perp}}{\vdash\Uparrow \bot, F^{\perp}}\,[\bot]}{\vdash\Uparrow \bot \parr F^{\perp}}\,[\parr]$$

Here is a common representation:

$$\mathbb{1} \times f \mapsto \mathbb{1}, f \mapsto f$$

A game on simple expressions

Define a game in which each player rewrites a multiset of neutral expressions at her turn:

$$\mathbb{1}, \Gamma \mapsto \Gamma \qquad N_1 \times N_2, \Gamma \mapsto N_1, N_2, \Gamma$$
$$N_1 + N_2, \Gamma \mapsto N_i, \Gamma$$

- Main question: does $N \mapsto^* \{\updownarrow M_1, \ldots, \updownarrow M_m\}$ hold?
- Restrict the language to the *simple expressions*, for which $m \leq 1$.
- Game rules: if $N \mapsto^* \{\}$, win. If $N \mapsto^* \updownarrow M$, pass M to the other player.

Main result: There is a winning strategy for N iff $\vdash [N]^+$. There is a counter winning strategy for N iff $\vdash [N]^-$.

Restriction on multiplicatives

A purely multiplicative proof corresponding to a winning strategy is of the form

$$
\dfrac{\dfrac{}{\vdash \Downarrow A_1} \qquad \cdots \qquad \dfrac{\dfrac{\dfrac{\dfrac{\vdots}{\vdash \Downarrow C}}{\vdash C \Uparrow}}{\vdash \Uparrow B_1, \ldots, \Uparrow B_n}}{\dfrac{\vdash \Uparrow B}{\vdash \Downarrow A_i}} \qquad \cdots \qquad \dfrac{}{\vdash \Downarrow A_m}}{\vdash \Downarrow A}
$$

- Proofs are focused. A move corresponds to a phase.
- Restriction: simple expressions prevent interactions between multiplicatives ($\otimes$ and $⅋$).

What's wrong with multiplicatives?

The logic is not complete! $\updownarrow \mathbb{1} \times \updownarrow \mathbb{1}$ translates to two unprovable formulas:

$$\bot \otimes \bot \quad \text{and} \quad 1 \parr 1$$

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In terms of games, players have "generic" attacks:

$$\dfrac{\dfrac{\vdash A, \delta_1, \ldots, \delta_k \quad \vdash B, \delta_{k+1}, \ldots, \delta_n}{\vdash A \otimes B, \delta_1, \ldots, \delta_n}}{\vdash (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n} \qquad \Bigg| \qquad \dfrac{\dfrac{\vdash A^\bot, B^\bot}{\vdash A^\bot \parr B^\bot} \quad \vdash \delta_1^\bot \quad \ldots \quad \vdash \delta_n^\bot}{\vdash (A^\bot \parr B^\bot) \otimes \delta_1^\bot \otimes \ldots \otimes \delta_n^\bot}$$

On the left, the player chooses a partition of $\delta_1, \ldots, \delta_n$. This information does not appear on the right.
On the right, you can tell A and B from the δ_i. This information is lost on the left.

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On the right, you can tell A and B from the δ_i. This information is lost on the left.

Problem

Find a common representation containing all the information.

Extending to non simple neutral expressions

We extend Miller & Saurin's games (at least for the propositional fragment).

- The two players still manipulate an object with two dual translations.
- Nothing really new with additives ($\oplus$ and $\&$).
- The objects are much more complex than neutral expressions, because they record the interactions between multiplicatives.
- The game is no longer determined (recall $\updownarrow \mathbb{1} \times \updownarrow \mathbb{1}$).
  When a player fails, the game continues until the other player wins or fails as well (tie).

Example

$$\dfrac{\dfrac{\dfrac{\vdash A^\perp \otimes B, C \otimes D \Uparrow}{\vdash\Uparrow A^\perp \otimes B, C \otimes D}}{\dfrac{\vdash\Uparrow (A^\perp \otimes B) \parr (C \otimes D)}{\vdash\Downarrow (A^\perp \otimes B) \parr (C \otimes D)} \quad \dfrac{\vdash\Uparrow E}{\vdash\Downarrow E}}}{\dfrac{\vdash\Downarrow ((A^\perp \otimes B) \parr (C \otimes D)) \otimes E}{\vdash ((A^\perp \otimes B) \parr (C \otimes D)) \otimes E \Uparrow}}$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\vdash E^\perp \Uparrow A \parr B^\perp}{\vdash E^\perp \Downarrow A \parr B^\perp} \quad \dfrac{\vdash\Uparrow C^\perp \parr D^\perp}{\vdash\Downarrow C^\perp \parr D^\perp}}{\vdash E^\perp \Downarrow (A \parr B^\perp) \otimes (C^\perp \parr D^\perp)}}{\dfrac{\vdash (A \parr B^\perp) \otimes (C^\perp \parr D^\perp), E^\perp \Uparrow}{\vdash\Uparrow (A \parr B^\perp) \otimes (C^\perp \parr D^\perp), E^\perp}}}{\vdash\Uparrow ((A \parr B^\perp) \otimes (C^\perp \parr D^\perp)) \parr E^\perp}$$

Neutral graphs

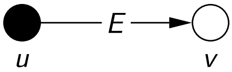
- The object manipulated by the players represents the frontiers of the two derivations.
- Invariant: the empty sequent can be infered from those frontiers via multiple cuts.
- The object is a directed bipartite graph. Vertices represent sequents, arcs represent pairs of dual formulas.

Vertices as sequents, arcs as formulas

In a neutral graph

- vertices are labeled with players,
- arcs are labeled with neutral expressions,
- there are no undirected cycles.

An arc



means that

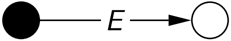
- the formula $[E]^+$ occurs in the sequent associated with u,
- the formula $[E]^-$ occurs in the sequent associated with v.

A simple neutral graph

The frontiers

$$\vdash [E]^{+} \Uparrow \Big| \vdash \Uparrow [E]^{-}$$

are represented by the neutral graph

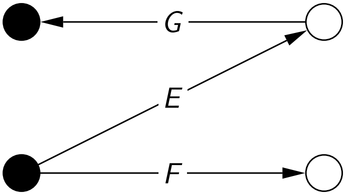


Another example

The frontiers

$$\vdash [E]^+, [F]^+ \Uparrow \qquad \vdash\Uparrow [G]^- \mid \vdash [G]^+ \Uparrow [E]^- \qquad \vdash\Uparrow [F]^-$$

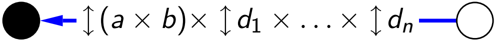
are represented by the neutral graph



A dynamic example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \qquad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\color{red}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}}
$$

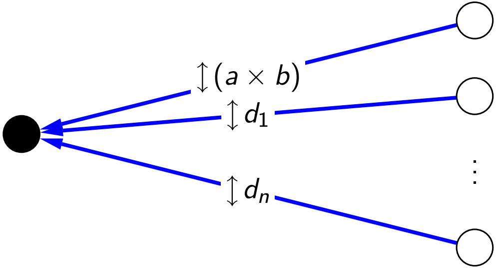
$$
\dfrac{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \parr B^\perp}}{\Downarrow A^\perp \parr B^\perp} \qquad \dfrac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp} \qquad \ldots \qquad \dfrac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}}{\color{red}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}}
$$



A dynamic example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \quad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\color{red}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}
$$

$$
\dfrac{\color{red}{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \parr B^\perp}}{\Downarrow A^\perp \parr B^\perp}} \quad \color{red}{\dfrac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp}} \quad \ldots \quad \color{red}{\dfrac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}}}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}
$$



A dynamic example

$$
\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \quad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{\color{red}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}
$$

$$
\dfrac{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\color{red}{\Uparrow A^\perp \parr B^\perp}}}{\Downarrow A^\perp \parr B^\perp} \quad \dfrac{\color{red}{\Uparrow \delta_1^\perp}}{\Downarrow \delta_1^\perp} \quad \ldots \quad \dfrac{\color{red}{\Uparrow \delta_n^\perp}}{\Downarrow \delta_n^\perp}}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}
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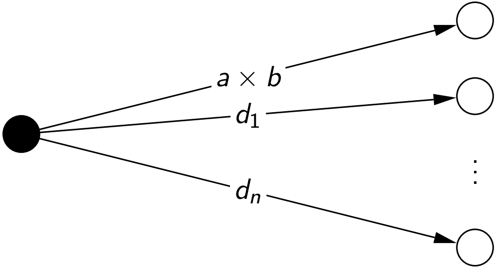
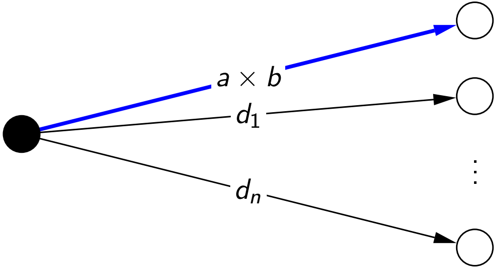


A dynamic example

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \quad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\color{red}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}$$

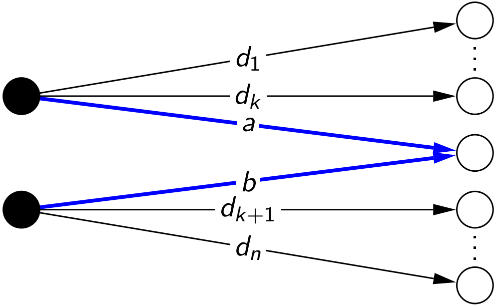
$$\dfrac{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\color{red}{\Uparrow A^\perp \parr B^\perp}}}{\Downarrow A^\perp \parr B^\perp} \quad \dfrac{\color{red}{\Uparrow \delta_1^\perp}}{\Downarrow \delta_1^\perp} \quad \ldots \quad \dfrac{\color{red}{\Uparrow \delta_n^\perp}}{\Downarrow \delta_n^\perp}}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}$$



A dynamic example

$$\dfrac{\dfrac{\dfrac{\dfrac{\delta_1, \ldots, \delta_k \Downarrow A \qquad \delta_{k+1}, \ldots, \delta_n \Downarrow B}{\delta_1, \ldots, \delta_n \Downarrow A \otimes B}}{A \otimes B, \delta_1, \ldots, \delta_n \Uparrow}}{\Uparrow A \otimes B, \delta_1, \ldots, \delta_n}}{\Uparrow (A \otimes B) \parr \delta_1 \parr \ldots \parr \delta_n}$$

$$\dfrac{\dfrac{\dfrac{\Uparrow A^\perp, B^\perp}{\Uparrow A^\perp \parr B^\perp}}{\Downarrow A^\perp \parr B^\perp} \qquad \dfrac{\Uparrow \delta_1^\perp}{\Downarrow \delta_1^\perp} \qquad \ldots \qquad \dfrac{\Uparrow \delta_n^\perp}{\Downarrow \delta_n^\perp}}{\Downarrow (A^\perp \parr B^\perp) \otimes \delta_1^\perp \otimes \ldots \otimes \delta_n^\perp}$$



Rules of the game

- A move of player σ corresponds to a synchronous phase from σ's sequents and to an asynchronous phase from her opponent's sequents.
- Some choices may make one (or both) of the proofs fail.
- A play goes on until both players fail (tie) or the graph becomes empty (win for the player who has not failed).
- Main result: there is a proof iff there is a winning strategy.

A typical move

A move usually corresponds to a synchronous phase for the current player. We write

$$G \overset{f_0, f_1}{\rightsquigarrow} G'$$

f_0 (resp. f_1) is a boolean value which is true iff player 0 (resp. 1) fails during the phase.
A move can be decomposed in a sequence of "micro-moves". First a decision step, then small steps corresponding to single rule applications.

$$G \xrightarrow{D} G_0 \xrightarrow{f_0^{(1)}, f_1^{(1)}} G_1 \xrightarrow{f_0^{(2)}, f_1^{(2)}} \cdots \xrightarrow{f_0^{(n)}, f_1^{(n)}} G_n = G'$$

and $f_\sigma = \bigvee_{i=1}^{n} f_\sigma^{(i)}$ $(\sigma \in \{0, 1\})$.

Conclusion

- The neutral approach gives three possible outcomes: win for a player (proof), win for her opponent (proof of the dual formula) or undeterminacy (no proof).
- This extension reveals the complex interactions between multiplicatives.
- The graph objects show symmetries of the focused proof system.
- Future work: fixed points; connect with ludics; introduce some concurrency in the game.