Least and greatest fixed points in linear logic

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Introduction

We are interested in (first-order) reasoning over (co)inductive specifications: arithmetic, various computational and logical systems, etc. This is provided for example by LINC, an extension of intuitionistic logic.

We shall exhibit useful structure in derivations, even though the subformula property does not hold, by extending focusing to fixed points.

We will work on a simple extension of linear logic, and by the way:

- *get an elegant system, more dualities and less technical restrictions;*
- *prove cut-elimination and completeness of a focused proof-system;*
- *explore (partial) alternatives for dealing with infinity in linear logic.*

But this work is not only about linear logic, and we will conclude by outlining how it impacts intuitionistic logic.

(Co)induction in intuitionistic logic

The logic LINC does it as follows:

$$\frac{\Gamma, S\,\vec{x} \vdash C \quad BS\,\vec{y} \vdash S\,\vec{y}}{\Gamma, \mu B\vec{x} \vdash C} \qquad \frac{\Gamma \vdash B(\mu B)\vec{x}}{\Gamma \vdash \mu B\vec{x}}$$

$$\frac{\Gamma, B(\nu B)\vec{x} \vdash C}{\Gamma, \nu B\vec{x} \vdash C} \qquad \frac{\Gamma \vdash S\,\vec{x} \quad S\,\vec{y} \vdash BS\,\vec{y}}{\Gamma \vdash \nu B\vec{x}}$$

For example:

$$nat\ x \overset{def}{=} \mu(\lambda nat \lambda x.\ x = z \vee \exists y.\ x = s\ y \wedge nat\ y)x$$

We derive:

$$\frac{}{\Gamma \vdash nat\ z} \qquad \frac{\Gamma \vdash nat\ x}{\Gamma \vdash nat\ (s\ x)}$$

$$\frac{\vdash I\ z \quad I\ y \vdash I\ (s\ y) \quad \Gamma, I\ x \vdash G}{\Gamma, nat\ x \vdash G}$$

μMALL$^=$

The logic μMALL$^=$ is MALL…

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q} \qquad \frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \parr Q}$$

$$\frac{\vdash \Gamma, P_i}{\vdash \Gamma, P_0 \oplus P_1} \qquad \frac{\vdash \Gamma, P \quad \vdash \Gamma, Q}{\vdash \Gamma, P \,\&\, Q}$$

…plus first-order quantifiers and equality…

$$\frac{\vdash \Gamma, P\ t}{\vdash \Gamma, \exists x\,.\,P\ x} \qquad \frac{\vdash \Gamma, P\ y}{\vdash \Gamma, \forall x\,.\,P\ x}\ y \text{ fresh}$$

$$\frac{}{\vdash t = t} \qquad \frac{\{\vdash \Gamma\theta : \theta \in csu(s \doteq t)\}}{\vdash \Gamma, s \neq t}$$

$$(s = t)^{\perp} \overset{def}{=} s \neq t$$

μMALL$^=$

…plus fixed points (but no atom).

$$\frac{\vdash \Gamma, B(\mu B)\vec{x}}{\vdash \Gamma, \mu B\vec{x}}\ \mu \qquad \frac{\vdash \Gamma, S\,\vec{x} \quad \vdash BS\,\vec{x}, (S\,\vec{x})^\perp}{\vdash \Gamma, \nu B\vec{x}}\ \nu$$

$$\frac{}{\vdash \mu B\vec{x}, \nu\overline{B}\vec{x}}\ \mu\nu$$

$$(\mu B\vec{x})^\perp \overset{def}{=} \nu\overline{B}\vec{x}$$

$$\overline{B} \overset{def}{=} \lambda p.\lambda\vec{x}.(B(\lambda\vec{x}.(p\vec{x})^\perp)\vec{x})^\perp$$

Definition 1. Bodies are required to be *monotonic*: there should be no negative occurence of p in (the negation normal form of) $Bp\vec{x}$.

Invalid: $\mu(\lambda even \lambda x.\ x = z \oplus \exists y.\ x = s\ y \otimes (even\ y \multimap \mathbf{0}))x$

Valid: $\mu(\lambda even \lambda x.\ x = z \oplus \exists y.\ x = s\ (s\ y) \otimes even\ y)x$

Admissible rules

Proposition 1. *The general initial rule is admissible:*

$$\frac{}{\vdash P, P^\perp}\ init$$

Proposition 2. *The other unfolding rule is admissible:*

$$\frac{\vdash \Gamma, B(\nu B)\vec{x}}{\vdash \Gamma, \nu B\vec{x}}\ \nu R$$

The following are provable:

- $2 = 2 \multimapboth \mathbf{1}$ and $2 = 3 \multimapboth \mathbf{0}$
- $\mu(\lambda x.x) \multimapboth \mathbf{0}$
- $\mu B\vec{t} \multimap \nu B\vec{t}$
- $\nu B\vec{t} \multimap \mu B\vec{t}$ if all unfoldings of B and $\overline{B}$ terminate.

Cut-elimination

We provide a proof of cut-elimination, and thus consistency, without restricting the system at all.

The proof relies on the *normalization of full second-order linear logic* (we cheat regarding equality) and the following translation (which relies on monotonicity):

$$
\begin{aligned}
\lceil \mu B \vec{x} \rceil &= \forall S \;.\; !(\forall \vec{y} \;.\; \lceil B \rceil S \vec{y} \multimap S \vec{y}) \multimap S \vec{x} \\
\lceil \nu B \vec{x} \rceil &= \lceil \mu \overline{B} \vec{x} \rceil^{\perp}
\end{aligned}
$$

The backward translation from second-order cut-free proofs to first-order cut-free proofs is quite simple thanks to *focusing* of second-order linear logic.

Exponentials, non-monotonicity

Exponentials As shown above, μMALL$^=$ can be encoded using exponentials and second-order quantifiers. But at first-order, exponentials and fixed points are incomparable.

We could add exponentials in further work, but conjecture that the essential observations done in this work would stay the same.

Non-monotonicity and consistency There has been previous work on induction, considering logics with left and right unfoldings, but no (co)induction rule. In linear logic, Girard observed that both non-monotonicity and exponentials were needed to break consistency.

Here, it seems unlikely to obtain the left unfolding for non-monotonic definitions. Hence, we don't know any counter-example to consistency even with non-monotonic definitions and exponentials.

Asynchrony / Synchrony

We classify the following connectives as *asynchronous*:

$$⅋, \&, \forall, \neq \text{ and } \nu$$

The others are *synchronous*:

$$\otimes, \oplus, \exists, = \text{ and } \mu$$

Proposition 3. *The following structural rules are admissible provided that B is fully asynchronous:*

$$\frac{\vdash \Gamma, \nu B\vec{x}, \nu B\vec{x}}{\vdash \Gamma, \nu B\vec{x}}\ \nu C \qquad \frac{\vdash \Gamma}{\vdash \Gamma, \nu B\vec{x}}\ \nu W$$

If we had exponentials, the following would hold for any fully synchronous P and fully asynchronous Q:

$$P \multimapboth\, !P \quad \text{and} \quad Q \multimapboth\, ?Q$$

Focusing for MALL

Asynchronous phase (a is an asynchronous atom)

$$\frac{\vdash \Gamma \Uparrow A, B, \Delta}{\vdash \Gamma \Uparrow A \parr B, \Delta} \quad \frac{\vdash \Gamma \Uparrow A, \Delta \quad \vdash \Gamma \Uparrow B, \Delta}{\vdash \Gamma \Uparrow A \mathbin{\&} B, \Delta} \quad \frac{\vdash \Gamma \Uparrow A\, c, \Delta}{\vdash \Gamma \Uparrow \forall x \,.\, A\, x, \Delta} \quad \frac{\vdash \Gamma, a \Uparrow \Delta}{\vdash \Gamma \Uparrow a, \Delta}$$

Synchronous phase (a is a synchronous atom)

$$\frac{\vdash \Gamma \Downarrow A_i}{\vdash \Gamma \Downarrow A_0 \oplus A_1} \quad \frac{\vdash \Gamma \Downarrow A \quad \vdash \Gamma' \Downarrow B}{\vdash \Gamma, \Gamma' \Downarrow A \otimes B} \quad \frac{\vdash \Gamma \Downarrow A\, t}{\vdash \Gamma \Downarrow \exists x \,.\, A\, x} \quad \frac{}{\vdash a \Downarrow a}$$

Switching (P synchronous, N asynchronous)

$$\frac{\vdash \Gamma, P \Uparrow \Delta}{\vdash \Gamma \Uparrow P, \Delta} \quad \frac{\vdash \Gamma \Downarrow P}{\vdash \Gamma, P \Uparrow} \quad \frac{\vdash \Gamma \Uparrow N}{\vdash \Gamma \Downarrow N}$$

Theorem 1 (Andreoli)**.**

$$\vdash \Gamma \quad \Leftrightarrow \quad \vdash \Uparrow \Gamma$$

Focusing for μMALL$^=$

Asynchronous phase

$$\frac{\vdash \Gamma \Uparrow A, B, \Delta}{\vdash \Gamma \Uparrow A \parr B, \Delta} \qquad \frac{\vdash \Gamma \Uparrow A, \Delta \quad \vdash \Gamma \Uparrow B, \Delta}{\vdash \Gamma \Uparrow A \mathbin{\&} B, \Delta} \qquad \frac{\vdash \Gamma \Uparrow A\,c, \Delta}{\vdash \Gamma \Uparrow \forall x \,.\, A\,x, \Delta}$$

$$\frac{\{\vdash \Gamma\theta \Uparrow \Delta\theta : \theta \in csu(s \doteq t)\}}{\vdash \Gamma \Uparrow s \neq t, \Delta} \qquad \frac{\vdash \Gamma \Uparrow S\,\vec{x}, \Delta \quad \vdash \Uparrow BS\,\vec{x}, S\,\vec{x}^{\perp}}{\vdash \Gamma \Uparrow \nu B\vec{x}, \Delta} \qquad \frac{\vdash \Gamma, \nu B\vec{x} \Uparrow \Delta}{\vdash \Gamma \Uparrow \nu B\vec{x}, \Delta}$$

Synchronous phase

$$\frac{\vdash \Gamma \Downarrow A_i}{\vdash \Gamma \Downarrow A_0 \oplus A_1} \qquad \frac{\vdash \Gamma \Downarrow A \quad \vdash \Gamma' \Downarrow B}{\vdash \Gamma, \Gamma' \Downarrow A \otimes B} \qquad \frac{\vdash \Gamma \Downarrow A\,t}{\vdash \Gamma \Downarrow \exists x \,.\, A\,x}$$

$$\frac{}{\vdash \Downarrow t = t} \qquad \frac{\vdash \Gamma \Downarrow B(\mu B)\vec{x}}{\vdash \Gamma \Downarrow \mu B\vec{x}} \qquad \frac{}{\vdash \nu \overline{B}\vec{x} \Downarrow \mu B\vec{x}}$$

Switching (where P is synchronous, Q asynchronous)

$$\frac{\vdash \Gamma, P \Uparrow \Delta}{\vdash \Gamma \Uparrow P, \Delta} \qquad \frac{\vdash \Gamma \Downarrow P}{\vdash \Gamma, P \Uparrow} \qquad \frac{\vdash \Gamma \Uparrow Q}{\vdash \Gamma \Downarrow Q}$$

Focusing for μMALL$^=$

This is a LaTeX bug…

Completeness proof

We prove the completeness of the focused proof system, using Saurin's polarization graph technique.

The core of the proof is a bunch of size-preserving permutations. Both μ and ν actually commute with everything. An interesting step is the commuting of & and ν:

$$
\dfrac{
\dfrac{
\dfrac{\Pi}{\vdash \Gamma, P, S\vec{t}}
\qquad
\dfrac{\Pi_S}{\vdash BS\vec{x}, S\vec{x}^{\perp}}
}{\vdash \Gamma, P, \nu B\vec{t}}
\qquad
\dfrac{
\dfrac{\Pi'}{\vdash \Gamma, P', S'\vec{t}}
\qquad
\dfrac{\Pi_{S'}}{\vdash BS'\vec{x}, S'\vec{x}^{\perp}}
}{\vdash \Gamma, P', \nu B\vec{t}}
}{\vdash \Gamma, P \mathbin{\&} P', \nu B\vec{t}}
$$

$$\Downarrow$$

$$
\dfrac{
\dfrac{
\dfrac{\dfrac{\Pi}{\vdash \Gamma, P, S\vec{t}}}{\vdash \Gamma, P, S\vec{t} \oplus S'\vec{t}}
\qquad
\dfrac{\dfrac{\Pi'}{\vdash \Gamma, P', S'\vec{t}}}{\vdash \Gamma, P', S\vec{t} \oplus S'\vec{t}}
}{\vdash \Gamma, P \mathbin{\&} P', S\vec{t} \oplus S'\vec{t}}
\qquad
\dfrac{
\dfrac{\phi_1(\Pi_S)}{\vdash B(S \oplus S')\vec{x}, (S\vec{x})^{\perp}}
\qquad
\dfrac{\phi_2(\Pi_{S'})}{\vdash B(S \oplus S')\vec{x}, (S'\vec{x})^{\perp}}
}{\vdash B(S \oplus S')\vec{x}, ((S \oplus S')\vec{x})^{\perp}}\ \&
}{\vdash \Gamma, P \mathbin{\&} P', \nu B\vec{t}}
$$

Focusing: discussion

The focused system is satisfying, convenient to use to (manually) search for a proof, or to reason about non-provability (e.g. $\nvdash (nat\ x \parr nat\ x) \multimap nat\ x$).

Choices in the asynchronous phase It is a bit surprising to find several choices in the asynchronous phase. One way to look at that is to consider annotating the ν in order to guide the choices:

$$\frac{\vdash \Gamma \Uparrow S\,\vec{t}, \Delta \quad \vdash \Uparrow BS\,\vec{x}, S\,\vec{x}^\perp}{\vdash \Gamma \Uparrow \nu_S\,B\vec{t}, \Delta}\ \vec{x}\text{ new} \qquad \frac{\vdash \Gamma, \nu_\epsilon B\vec{t} \Uparrow \Delta}{\vdash \Gamma \Uparrow \nu_\epsilon B\vec{t}, \Delta} \qquad \frac{}{\vdash \nu_\epsilon \overline{B}\vec{t} \Downarrow \mu B\vec{t}}$$

Conjecture: a formula is provable iff it has an annotation which is provable in the annotated focusing system.

Focusing: discussion

Some flexibility in assigning polarities We conjecture a completeness result with μ treated among the asynchronous, ν among the synchronous:

$$\frac{\vdash \Gamma \Uparrow B(\mu B)\vec{t}, \Delta}{\vdash \Gamma \Uparrow \mu B\vec{t}, \Delta} \qquad \frac{\vdash \Gamma, \mu B\vec{t} \Uparrow \Delta}{\vdash \Gamma \Uparrow \mu B\vec{t}, \Delta}$$

$$\frac{\vdash \Gamma \Downarrow S\,\vec{t} \quad \vdash \Gamma \Uparrow BS\,\vec{x}, (S\,\vec{x})^{\perp}}{\vdash \Gamma \Downarrow \nu B\vec{t}} \qquad \frac{}{\vdash \mu\overline{B}\vec{t} \Downarrow \nu B\vec{t}}$$

We can probably allow some mixing of both choices, a bit like for atoms in Andreoli's result. What does it mean? What can it be used for ?

Application to intuitionistic logic

Focusing is not restricted to linear logic. It has been extended to intuitionistic and classical logics. There are two approaches for doing so: either start from scratch, or use an encoding.

$$\begin{tikzcd}
\vdash [F] \arrow[d] & \vdash F \arrow[l] \arrow[d, dotted] \\
\vdash\Uparrow [F] \arrow[r] & \vdash\Uparrow F
\end{tikzcd}$$

A known strategy for getting a better behaved focused system is to minimize the use of exponentials in the encoding, by pushing ? (resp. !) through asynchronous (resp. synchronous) connectives.

We do not have exponentials yet, but we can already handle a surprising fragment.

Back to LINC

Goal: design a fragment such that all formulas appearing negatively can be encoded in a fully asynchronous way.

$$\begin{array}{rcl} \mathcal{H} & ::= & \mathcal{H} \wedge \mathcal{H} \mid \mathcal{H} \vee \mathcal{H} \mid s = t \mid \mu\mathcal{H}\vec{t} \mid \exists x.\mathcal{H}x \\ \mathcal{G} & ::= & \mathcal{G} \wedge \mathcal{G} \mid \mathcal{G} \vee \mathcal{G} \mid s = t \mid \mu\mathcal{G}\vec{t} \mid \exists x.\mathcal{G}x \mid \forall x.\mathcal{G}x \mid \mathcal{H} \supset \mathcal{G} \mid \nu\mathcal{G}\vec{t} \end{array}$$

We can encode it without exponentials as follows:

$$\begin{array}{rcl} [A \wedge B] & \overset{def}{=} & [A] \otimes [B] \\ [A \vee B] & \overset{def}{=} & [A] \oplus [B] \\ [s = t] & \overset{def}{=} & s = t \\ [\mu B\vec{t}] & \overset{def}{=} & \mu[B]\vec{t} \\ [\exists x.Ax] & \overset{def}{=} & \exists x.[Ax] \end{array}$$

$$\begin{array}{rcl} [\forall x.Ax] & \overset{def}{=} & \forall x.[Ax] \\ [\nu B\vec{t}] & \overset{def}{=} & \nu[B]\vec{t} \\ [A \supset B] & \overset{def}{=} & [A] \multimap [B] \\ [\lambda p\lambda\vec{x}.Bpx] & \overset{def}{=} & \lambda p\lambda\vec{x}.[Bpx] \end{array}$$

Back to LINC

Proposition 4. *For any* $P \in \mathcal{G}$*,* P *is provable in* $\mu LJ^{=}$ *if and only if* $[P]$ *is provable in* $\mu MALL^{=}$*, under the restrictions that (co)invariants* $\lambda\vec{x}.S\,\vec{x}$ *in* $\mu MALL^{=}$ *(resp.* $\mu LJ^{=}$*) are such that* $S\,\vec{x}$ *is in* $[\mathcal{H}]$ *(resp.* $\mathcal{H}$*).*

From that proposition follows a strongly focused proof system for the fragment $\mathcal{G}$ (under the restriction that. . .)

The fragment $\mathcal{H}$ contains (slightly more than) Prolog — and it is handled in a single focusing phase! Thus, $\mathcal{G}$ can express that a set p described by Horn clauses satisfy some other Horn clause property:

$$\forall\vec{x}.p\ \vec{x} \supset q\ \vec{x}$$

When the fixed points are noetherian, the focusing discipline yields the re-discovery of Bedwyr, our logic programming engine for this fragment. We have used it to specify logics and calculi, perform model-checking, check bisimulation for π-calculus, etc.

Another open question

Adding exponentials is required to encode full $\mu\mathrm{LJ}^=$.

Is there a need to restrict the possible invariants used on the linear side, in order to be able to encode them as intuitionistic invariants?

- It seems difficult to translate any linear invariant into an intuitionistic one.

$$\frac{\vdash [B]^- S\,\vec{x}^\perp, S\,\vec{x} \quad \vdash [\Gamma]^-, S\,\vec{t}^\perp, [G]^+}{\vdash [\Gamma]^-, \nu[B]^-\vec{t}, [G]^+} \quad\longrightarrow\quad \frac{?}{\Gamma, \mu B\vec{t} \vdash G}$$

- On the other hand, as soon as this derivation is cut on the ν-expression, the linear invariant has to disappear – the subformula property holds if you get rid of the ν rule.

Answering this question could lead to a Logic of Unity with fixed points.

Conclusion

Contributions:

- An expressive extension of MALL with a complete focused system;
- insights on the treatment of atoms and infinity;
- encodes (a fragment of) LINC, without as much technical restrictions.

Further work:

- Extension with exponentials and atoms;
- study (use?) systems with non-monotonic definitions;
- understand (use?) the flexibility in the design of the focused system;
- study the relationship with full LINC;
- use partial focusing annotations in order to guide an automated prover.