

# Least and greatest fixed points in linear logic

David Baelde & Dale Miller

Parsifal team, INRIA / École Polytechnique

LPAR 2007

# Background / Linear logic

We are interested in linear logic as **the logic behind logic**:

- ▶ it is simpler: more symmetry, cleaner separation of concepts;
- ▶ it encodes other logics, such as intuitionistic and classical; and
- ▶ meta-theorems of LL (e.g., cut-elimination, focusing) can often be used to infer similar meta-theorems for encoded logics.

Linear logic as the **logic of resource accounting** (such as in linear logic programming) is **not** our interest here.

## Background / Focusing

It provides structure to the role of non-determinism in the search for cut-free proofs.

- ▶ **Asynchrony**: invertible rules can be introduced greedily in any order;
- ▶ **Synchrony**: non-invertible rules are hereditarily applied.

Focusing is the most transparent in linear logic (Andreoli, 1992).

The completeness of focusing in linear logic can be used to explain the completeness of other “focusing-like” proof systems, such as SLD-resolution, uniform proofs, systems mixing backward and forward chaining, etc.

## Background / Fixed points

(Co)Inductive definitions are **omnipresent**, but their proof-theory is **challenging** (e.g., no subformula property). They are also quite interesting from a proof-theoretical point-of-view, since they provide:

- ▶ an alternative to atoms: one can see a fixed point (e.g.,  $\text{nat } x$ ) as an atom **with structure**;
- ▶ an alternative point-of-view on infinity, more controlled than the opaque exponentials (!,?).

Both atoms and exponentials have a non-canonical and puzzling status in focusing.

The logic  $\mu\text{MALL}^\perp$  is **MALL**...

$$\frac{\vdash \Gamma, P \quad \vdash \Delta, Q}{\vdash \Gamma, \Delta, P \otimes Q} \quad \frac{\vdash \Gamma, P, Q}{\vdash \Gamma, P \wp Q}$$

$$\frac{\vdash \Gamma, P_i}{\vdash \Gamma, P_0 \oplus P_1} \quad \frac{\vdash \Gamma, P \quad \vdash \Gamma, Q}{\vdash \Gamma, P \& Q}$$

... plus first-order quantifiers and **equality**...

$$\frac{\vdash \Gamma, P \ t}{\vdash \Gamma, \exists x . P \ x} \quad \frac{\vdash \Gamma, P \ y}{\vdash \Gamma, \forall x . P \ x} \quad y \text{ fresh}$$

$$\frac{}{\vdash t = t} \quad \frac{\{\vdash \Gamma \theta : \theta \in \text{csu}(s \doteq t)\}}{\vdash \Gamma, s \neq t}$$

$$(s = t)^\perp \stackrel{\text{def}}{=} s \neq t$$

... plus **fixed points** (but no atom).

$$\frac{\frac{\vdash \Gamma, B(\mu B)\vec{x}}{\vdash \Gamma, \mu B\vec{x}} \quad \mu}{\vdash \mu B\vec{x}, \nu \overline{B}\vec{x}} \quad \frac{\frac{\vdash \Gamma, S\vec{x} \quad \vdash BS\vec{x}, (S\vec{x})^\perp}{\vdash \Gamma, \nu B\vec{x}} \quad \nu}{\vdash \mu B\vec{x}, \nu \overline{B}\vec{x}} \quad \mu\nu$$

$$(\mu B\vec{x})^\perp \stackrel{\text{def}}{=} \nu \overline{B}\vec{x}$$

$$\overline{B} \stackrel{\text{def}}{=} \lambda p. \lambda \vec{x}. (B(\lambda \vec{x}. (p\vec{x})^\perp)\vec{x})^\perp$$

All fixed points must be formed over **monotonic** bodies  $B$ : negation is not even part of the syntax!

# Examples

## Natural numbers

$$\text{nat } x \stackrel{\text{def}}{=} \mu(\lambda \text{nat} \lambda x. x = 0 \oplus \exists y. x = \text{s } y \otimes \text{nat } y)x$$

One can derive:

$$\text{nat } 0$$

$$\forall x. (\text{nat } x \multimap \text{nat } (\text{s } x))$$

and for any  $P$ :

$$P \ 0 \multimap (\forall x. P \ x \multimap P \ (\text{s } x)) \multimap \forall x. (\text{nat } x \multimap P \ x)$$

Theorems, assuming a standard definition of *plus*:

- ▶  $\vdash \forall x. \text{nat } x \multimap \text{plus } x \ 0 \ x$
- ▶  $\vdash \forall x. \text{nat } x \multimap \forall y \exists z. \text{plus } x \ y \ z$
- ▶  $\vdash \forall x. \text{nat } x \multimap \forall y. \text{nat } y \multimap \forall z. \text{plus } x \ y \ z \multimap \text{nat } z$

# Preliminary observations

## Proposition

*The general initial rule, as well as the unfolding of greatest fixed points, are admissible:*

$$\frac{}{\vdash P, P^\perp} \text{init} \qquad \frac{\vdash \Gamma, B(\nu B)\vec{X}}{\vdash \Gamma, \nu B\vec{X}} \nu R$$

## Theorem (cut-elimination)

*The cut rule is admissible:*

$$\frac{\vdash \Gamma, P \quad \vdash P^\perp, \Delta}{\vdash \Gamma, \Delta} \text{cut}$$

# Asynchrony / Synchrony

We classify the following connectives as **asynchronous**:

$$\wp, \&, \forall$$

The others are **synchronous**:

$$\otimes, \oplus, \exists$$

## Theorem

For any **fully asynchronous** formula  $Q$ , the following structural rules are admissible:

$$\frac{\vdash \Gamma, Q, Q}{\vdash \Gamma, Q} \quad \frac{\vdash \Gamma}{\vdash \Gamma, Q}$$

With exponentials, the following would hold for any fully synchronous  $P$  and fully asynchronous  $Q$ :

$$P \multimap !P \quad \text{and} \quad Q \multimap ?Q$$

# Asynchrony / Synchrony

We classify the following connectives as **asynchronous**:

$$\wp, \&, \forall, \neq \text{ and } \nu$$

The others are **synchronous**:

$$\otimes, \oplus, \exists, = \text{ and } \mu$$

## Theorem

For any **fully asynchronous** formula  $Q$ , the following structural rules are admissible:

$$\frac{\vdash \Gamma, Q, Q}{\vdash \Gamma, Q} \quad \frac{\vdash \Gamma}{\vdash \Gamma, Q}$$

With exponentials, the following would hold for any fully synchronous  $P$  and fully asynchronous  $Q$ :

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# Focusing for MALL

## Asynchronous phase

$$\frac{\vdash \Gamma \uparrow P, Q, \Delta}{\vdash \Gamma \uparrow P \wp Q, \Delta} \quad \frac{\vdash \Gamma \uparrow P, \Delta \quad \vdash \Gamma \uparrow Q, \Delta}{\vdash \Gamma \uparrow P \& Q, \Delta} \quad \frac{\vdash \Gamma \uparrow P c, \Delta}{\vdash \Gamma \uparrow \forall x . P x, \Delta}$$

## Synchronous phase

$$\frac{\vdash \Gamma \Downarrow P_i}{\vdash \Gamma \Downarrow P_0 \oplus P_1} \quad \frac{\vdash \Gamma \Downarrow P \quad \vdash \Gamma' \Downarrow Q}{\vdash \Gamma, \Gamma' \Downarrow P \otimes Q} \quad \frac{\vdash \Gamma \Downarrow P t}{\vdash \Gamma \Downarrow \exists x . P x}$$

**Switching** ( $P$  synchronous,  $Q$  asynchronous)

$$\frac{\vdash \Gamma, P \uparrow \Delta}{\vdash \Gamma \uparrow P, \Delta} \quad \frac{\vdash \Gamma \Downarrow P}{\vdash \Gamma, P \uparrow} \quad \frac{\vdash \Gamma \uparrow Q}{\vdash \Gamma \Downarrow Q}$$

Theorem (Completeness of focusing for MALL)

$$\vdash \Gamma \quad \Leftrightarrow \quad \vdash \uparrow \Gamma$$

# Focusing for MALL

**Asynchronous phase** ( $a$  is an asynchronous atom)

$$\frac{\vdash \Gamma \uparrow P, Q, \Delta}{\vdash \Gamma \uparrow P \wp Q, \Delta} \quad \frac{\vdash \Gamma \uparrow P, \Delta \quad \vdash \Gamma \uparrow Q, \Delta}{\vdash \Gamma \uparrow P \& Q, \Delta} \quad \frac{\vdash \Gamma \uparrow P c, \Delta}{\vdash \Gamma \uparrow \forall x . P x, \Delta} \quad \frac{\vdash \Gamma, a \uparrow \Delta}{\vdash \Gamma \uparrow a, \Delta}$$

**Synchronous phase** ( $a$  is a synchronous atom)

$$\frac{\vdash \Gamma \Downarrow P_i}{\vdash \Gamma \Downarrow P_0 \oplus P_1} \quad \frac{\vdash \Gamma \Downarrow P \quad \vdash \Gamma' \Downarrow Q}{\vdash \Gamma, \Gamma' \Downarrow P \otimes Q} \quad \frac{\vdash \Gamma \Downarrow P t}{\vdash \Gamma \Downarrow \exists x . P x} \quad \frac{}{\vdash a^\perp \Downarrow a}$$

**Switching** ( $P$  synchronous,  $Q$  asynchronous)

$$\frac{\vdash \Gamma, P \uparrow \Delta}{\vdash \Gamma \uparrow P, \Delta} \quad \frac{\vdash \Gamma \Downarrow P}{\vdash \Gamma, P \uparrow} \quad \frac{\vdash \Gamma \uparrow Q}{\vdash \Gamma \Downarrow Q}$$

**Theorem (Completeness of focusing for MALL)**

$$\vdash \Gamma \quad \Leftrightarrow \quad \vdash \uparrow \Gamma$$

# Focusing for $\mu$ MALL<sup>=</sup>

## Asynchronous phase

$$\frac{\vdash \Gamma \uparrow A, B, \Delta}{\vdash \Gamma \uparrow A \wp B, \Delta} \quad \frac{\vdash \Gamma \uparrow A, \Delta \quad \vdash \Gamma \uparrow B, \Delta}{\vdash \Gamma \uparrow A \& B, \Delta} \quad \frac{\vdash \Gamma \uparrow A \text{ c}, \Delta}{\vdash \Gamma \uparrow \forall x. A \ x, \Delta}$$

$$\frac{\{\vdash \Gamma \theta \uparrow \Delta \theta : \theta \in \text{csu}(s \doteq t)\}}{\vdash \Gamma \uparrow s \neq t, \Delta} \quad \frac{\vdash \Gamma \uparrow S \vec{x}, \Delta \quad \vdash \uparrow BS \vec{x}, S \vec{x}^\perp}{\vdash \Gamma \uparrow \nu B \vec{x}, \Delta} \quad \frac{\vdash \Gamma, \nu B \vec{x} \uparrow \Delta}{\vdash \Gamma \uparrow \nu B \vec{x}, \Delta}$$

## Synchronous phase

$$\frac{\vdash \Gamma \Downarrow A_i}{\vdash \Gamma \Downarrow A_0 \oplus A_1} \quad \frac{\vdash \Gamma \Downarrow A \quad \vdash \Gamma' \Downarrow B}{\vdash \Gamma, \Gamma' \Downarrow A \otimes B} \quad \frac{\vdash \Gamma \Downarrow A \ t}{\vdash \Gamma \Downarrow \exists x. A \ x}$$

$$\frac{}{\vdash \Downarrow t = t} \quad \frac{\vdash \Gamma \Downarrow B(\mu B) \vec{x}}{\vdash \Gamma \Downarrow \mu B \vec{x}} \quad \frac{}{\vdash \nu \bar{B} \vec{x} \Downarrow \mu B \vec{x}}$$

Switching (where  $P$  is synchronous,  $Q$  asynchronous)

$$\frac{\vdash \Gamma, P \uparrow \Delta}{\vdash \Gamma \uparrow P, \Delta} \quad \frac{\vdash \Gamma \Downarrow P}{\vdash \Gamma, P \uparrow} \quad \frac{\vdash \Gamma \uparrow Q}{\vdash \Gamma \Downarrow Q}$$

## Some flexibility in assigning polarities !?

We conjecture a completeness result with **flipped polarities**, i.e. with  $\mu$  treated among the asynchronous,  $\nu$  among the synchronous:

$$\frac{\vdash \Gamma \uparrow B(\mu B)\vec{t}, \Delta}{\vdash \Gamma \uparrow \mu B\vec{t}, \Delta} \quad \frac{\vdash \Gamma, \mu B\vec{t} \uparrow \Delta}{\vdash \Gamma \uparrow \mu B\vec{t}, \Delta}$$
$$\frac{\vdash \Gamma \Downarrow S\vec{t} \quad \vdash \Gamma \uparrow BS\vec{x}, (S\vec{x})^\perp}{\vdash \Gamma \Downarrow \nu B\vec{t}} \quad \frac{}{\vdash \mu \overline{B}\vec{t} \Downarrow \nu B\vec{t}}$$

We can probably allow some **mixing** of both choices, **as for atoms** in Andreoli's result. What does it mean? What can it be used for ?

# Application to intuitionistic logic

The focusing result can be translated to intuitionistic logic:

$$\begin{array}{ccc} \vdash [F] & \longleftarrow & \vdash F \\ \downarrow & & \downarrow \\ \vdash \uparrow [F] & \longrightarrow & \vdash \uparrow F \end{array}$$

Guide line for the encoding: **fewer exponentials** (!,?) means **stronger focusing behaviour**.

We do not have exponentials (yet), but we can already handle a surprising fragment, and the focusing behaviour is very structured on it.

# Design of the encoding

**Goal:** design a fragment such that all formulas appearing negatively can be encoded in a **fully asynchronous** way.

$$\mathcal{H} ::= \mathcal{H} \wedge \mathcal{H} \mid \mathcal{H} \vee \mathcal{H} \mid s = t \mid \mu \mathcal{H} \vec{t} \mid \exists x. \mathcal{H} x$$

$$\mathcal{G} ::= \mathcal{G} \wedge \mathcal{G} \mid \mathcal{G} \vee \mathcal{G} \mid s = t \mid \mu \mathcal{G} \vec{t} \mid \exists x. \mathcal{G} x \mid \forall x. \mathcal{G} x \mid \mathcal{H} \supset \mathcal{G} \mid \nu \mathcal{G} \vec{t}$$

We can encode it **without exponentials** as follows:

$$[A \wedge B] \stackrel{\text{def}}{=} [A] \otimes [B]$$

$$[A \vee B] \stackrel{\text{def}}{=} [A] \oplus [B]$$

$$[s = t] \stackrel{\text{def}}{=} s = t$$

$$[\mu B \vec{t}] \stackrel{\text{def}}{=} \mu [B] \vec{t}$$

$$[\exists x. Ax] \stackrel{\text{def}}{=} \exists x. [Ax]$$

$$[\forall x. Ax] \stackrel{\text{def}}{=} \forall x. [Ax]$$

$$[\nu B \vec{t}] \stackrel{\text{def}}{=} \nu [B] \vec{t}$$

$$[A \supset B] \stackrel{\text{def}}{=} [A] \multimap [B]$$

$$[\lambda p \lambda \vec{x}. B p x] \stackrel{\text{def}}{=} \lambda p \lambda \vec{x}. [B p x]$$

# Focusing for $\mathcal{G}$

## Proposition

*For any  $P \in \mathcal{G}$ ,  $P$  is provable in  $\mu\text{LJ}^\perp$  if and only if  $[P]$  is provable in  $\mu\text{MALL}^\perp$  (under the restrictions ...).*

This yields a nice focusing system for  $\mathcal{G}$ .

## Expressivity of $\mathcal{G}$

$\mathcal{G}$  can express that a set  $p$  described by *Horn* clauses satisfies some other Horn clause property:

$$\forall \vec{x}. p \vec{x} \supset q \vec{x}$$

When the fixed points are noetherian, the **focusing discipline** yields the re-discovery of Bedwyr, our **logic programming engine** for this fragment. We have used it to specify logics and calculi, perform model-checking, check bisimulation for  $\pi$ -calculus, etc.

# Conclusion

## Contributions

- ▶ An expressive extension of MALL with a complete focused system;
- ▶ insights on the treatment of atoms and infinity;
- ▶ encodes (a fragment of) LINC, without as much technical restrictions.

## Further work

- ▶ Extension with exponentials and atoms;
- ▶ expressivity of linear invariants in an intuitionistic proof;
- ▶ study (use?) systems with non-monotonic definitions;
- ▶ understand (use?) the flexibility in the design of the focused system;
- ▶ use our focusing system(s) to understand or design heuristics for (co)inductive theorem-proving .