Formalising Completeness Theorems

The constructive proof of classical completeness of Krivine à la Berard-Valentini

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- Work with Hugo Herbelin
- Themes of work so far:
  - Completeness
  - Abstract Stone duality (with P.Taylor and A.Bauer)
  - Coq Constructive Reals à la Stolzenberg (with B.Spitters)

Completeness – Motivation

- Reason model-theoretically instead of proof theoretically
- i.e. Reason *in Coq toplevel* about an object logical theory
- Derive rules which might only be admisible for an object theory
- Theoretical aspects: Still no constructive proof of completeness of IPC wrt Kripke models.

Completeness – Work done so far

- Krivine's constructive proof of classical completeness [Kri96]
  - 2nd order logic
  - classical proof with double-negation translation
- McCarthy's result [McCarty96] on imposibility of an intuitionistic proof
- Raffali's formalisation in PhoX
- The proof reworked by Berardi and Valentini [BV04]

The proof of Berardi-Valentini

- Direct style – no DN-translation
- Isolates the principle behind:

Theorem

A filter F over a countable boolean algebra can be extended to a complete filter equiconsistent with F.

- Our interest: Modularise: use the ideas for other completeness-es (replace Boolean with Heyting algebras?)

Formalising Berardi-Valentini

Completeness

A is valid iff A is provable

- A is valid: A is true in all models
- A is true in a model M: $A \in M$
- What is a model?

A *set of sentences* using $\bot, \to, \forall$ satisfying :

$\to$-faithfullness $(A \to B) \in \mathcal{M}$ iff $A \in \mathcal{M}$ implies $B \in \mathcal{M}$

$\forall$-faithfullness $(\forall x.A) \in \mathcal{M}$ iff for all closed terms t, $A[x := t] \in \mathcal{M}$

double negation $(\neg\neg A \to A) \in \mathcal{M}$

equality axioms $E \in \mathcal{M}$ for E an equality axiom

$\bot$-elimination $\bot \notin \mathcal{M}$

Let $\mathcal{L}$ be a language with at least one constant symbol.

Classical Model

A set of formulas $\mathcal{M}$ written in $\mathcal{L}$ satisfying the previous slide.

Minimal Model

A set of formulas $\mathcal{M}$ written in $\mathcal{L}$ satisfying the previous except $\bot$-elimination.

- Classically, exactly 1 model which is minimal and not classical
- Intuitionistically, at least 1 ...
- Classical Model $\cong$ Tarski Model

A, Γ closed formulas

Truth in a model $\mathcal{M}$

1. $\mathcal{M} \vDash A$: $A \in \mathcal{M}$
2. $\mathcal{M} \vDash \Gamma$: $\Gamma \subseteq \mathcal{M}$
3. $\mathcal{M} \vDash (\Gamma \vdash A)$: $\mathcal{M} \vDash \Gamma$ implies $\mathcal{M} \vDash A$

$\Gamma \vdash A$ is *minimally* valid ($\Gamma \Vdash_{min} A$)

For any minimal model $\mathcal{M}$, $\mathcal{M} \vDash (\Gamma \vdash A)$.

$\Gamma \vdash A$ is *classically* valid ($\Gamma \Vdash_{class} A$)

For any classical model $\mathcal{M}$, $\mathcal{M} \vDash (\Gamma \vdash A)$.

Minimal Completeness

$\Gamma \Vdash_{min} A$ implies $\Gamma \vdash A$

Classical Completeness

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Minimal Completeness

$\Gamma \Vdash_{min} A$ implies $\Gamma \vdash A$

Classical Completeness

$\Gamma \Vdash_{class} A$ implies $\Gamma \vdash A$ impossible! From [McCarty96]

"Model Existence Lemma"

If $\mathcal{T}$ is a theory, then there exists a *minimal* model $\mathcal{U}$ which extends T and is equiconsistent with T.
If T is consistent, then U is a *classical* model.
possible!

Proof.

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- $\mathcal{M}$ is $\rightarrow$-faithful, so $\Gamma \subseteq \mathcal{M}$ implies $A \in \mathcal{M}$

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The proof was constructive.

Model Existence Lemma

If $\mathcal{T}$ is a theory, then there exists a *minimal* model $\mathcal{U}$ which extends $\mathcal{T}$ and is equiconsistent with $\mathcal{T}$.
If $\mathcal{T}$ is consistent, then $\mathcal{U}$ is a *classical* model.

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Given a theory $\mathcal{T}$,

1. build a Henkin theory $\mathcal{H}$ which is an extension of $\mathcal{T}$ and equiconsistent with $\mathcal{T}$ – this gives us ∀-faithfullness

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3. dispense easily :) with double negation, equality axioms (and $\perp$-faithfullness) – this shows $\mathcal{U}$ is a (classical) model.

□

Step 2.

Given a theory $\mathcal{H}$, build a theory U which is an extension of $\mathcal{H}$, equiconsistent with $\mathcal{H}$ and satisfies (Meta-DN).

Meta-DN

A theory $\mathcal{T}$ *satisfies (Meta-DN)* iff for all closed A:

$$(\mathcal{T} \vdash \neg A \Rightarrow \mathcal{T} \vdash \bot) \Rightarrow \mathcal{T} \vdash A$$

Complete (classically equivalent to Meta-DN)

A theory $\mathcal{T}$ is *complete* iff for all closed A:

$$\mathcal{T} \vdash \neg A \text{ or } \mathcal{T} \vdash A$$

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Proof.

- define a fixpoint F_n over an enumeration of all sentences
- show that at each stage it satisfies the required
- the output is $Z := \lambda b.\exists n^{nat}, F_n b$

□

Speculation

- Replace `nat` with `formula`?
- Replace the Boolean algebra with a Heyting one?

Technical Remarks

- Finitelly many existentil quantifications and finitelly many meets expressed with `fold-left`
- Easier to write an abstract proof, than to reason about object logical systems directly
- Not to hard to make work for setoids – good Coq support
- To define $\texttt{formula} - \texttt{dec} : \forall x, y^{\texttt{formula}}.\ x = y \vee x \neq y$ not as easy as it should be
- Use of proof irrelevance

Step 2.

Given a theory $\mathcal{T}$, build a Henkin theory $\mathcal{H}$ which is an extension of $\mathcal{T}$ and equiconsistent with $\mathcal{T}$.

$\mathcal{H}$ is a Henkin theory when

$(\forall x.A) \in \mathcal{H}$ implies there exists a constant c s.t. $(A[x := c] \rightarrow \forall x.A) \in \mathcal{H}$.

Proof

Quite easy to define the extension.
But, we work in a super-language. Might become a problem later?

The Future

- Finish putting all pieces together
- Characterise it as a program
- Replace induction over nat with induction over formulas
- Look into the work of Danvy on program style completeness
- Give a constructive proof for comp. of IPC wrt Kripke models

References I

Jean-Louis Krivine.
Une preuve formelle et intuitionniste du théorème de complétude de la logique classique

Stefano Berardi and Silvio Valentini.
Krivine's intuitionistic proof of classical completeness (for countable languages)

D. C. McCarty.
Undecidability and Intuitionistic Incompleteness