Kripke Models For Classical Logic

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Natural Deduction and Sequent Calculus

introduction and elimination rules for connectives

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A\ true}u \quad \dfrac{}{B\ true}w}{A \wedge B\ true}\wedge I}{B \supset (A \wedge B)\ true}\supset I^w}{A \supset (B \supset (A \wedge B))\ true}\supset I^u$$

left and right introduction rules for connectives

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{A \vdash A}(I)}{\vdash \neg A, A}(\neg R)}{\vdash A \vee \neg A, A}(\vee R_2)}{\vdash A, A \vee \neg A}(PR)}{\vdash A \vee \neg A, A \vee \neg A}(\vee R_1)}{\vdash A \vee \neg A}(CR)$$

Properties of Sequent Calculi

- cut-elimination
- subformula property
- consistency (the empty sequenct can be only a conclusion of a cut)
- left-right rules / program-environment behaviour
- closer to abstract machines (explicit evaluation context)

Typed λ-calculus and $\bar{\lambda}$-calculus

- proof-as-programs correspondence: simply-typed λ-calculus and natural deduction
- correspondence for sequent calculus: $\bar{\lambda}$ (Herbelin, 1994)

This was for intuitionistic logic.

Semantics

- Proof theoretic semantics (ex. *meaning explanations* of Martin-Löf (1984, 1996))
- Explicit semantics

  Soundness: The formal system gets assigned meaning which is consistent within the meta-language

  Completeness: Any reasoning done with the meanings at the meta-level, can be projected into the object language

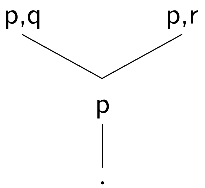
Semantics: What is it good for?

- a model can give non-provability results
- normalisation-by-evaluation
- working with models may simply be more convenient (in Coq)

Semantics: Kinds

- Boolean semantics (Classical Model Theory)
- Kripke semantics
- Beth semantics, Topological semantics, ...

Kripke Semantics (Possible Worlds Semantics)



- a possible world is determined by atomic formulas known to hold so far
- at any later world we enrich our knowledge

Kripke Semantics: Formal Definition

A *Kripke model* is given by:

- a partial order (K,$\leq$) of **worlds**
- a relation "$\Vdash$" between worlds and *atomic* formulas, called **forcing**

We extend the forcing relation to composite formulas inductivelly by:

w $\Vdash$	
$A \wedge B$	$w \Vdash A$ and $w \Vdash B$
$A \vee B$	$w \Vdash A$ or $w \Vdash B$
$A \rightarrow B$	for any $w' \geq w$, if $w' \Vdash A$ then $w' \Vdash B$
$\forall x P(x)$	for any $a \in D(w)$, $w \Vdash P(a)$
$\exists x P(x)$	there is $a \in D(w)$ such that $w \Vdash P(a)$
$\bot$	is never forced

Almost Tarski's truth definition, only implication standing out.

Examples of Kripke Counter-Models

Sentences not provable intuitionistically:

1. $p \vee \neg p$
2. $\neg\neg p \rightarrow p$
3. 2 implies 1
4. $(p \rightarrow q) \rightarrow (\neg p \vee q)$
5. $(p \rightarrow (q \vee r)) \rightarrow ((p \rightarrow q) \vee (p \rightarrow r))$
6. $\neg\forall x \neg P(x) \rightarrow \exists x P(x)$

Proof Systems For Classical Logic

- call/cc and Pierce's Law (Griffin, 1990)
- $\lambda\mu$-calculus (Parigot, 1992)
- $\bar{\lambda}\mu$-calculus (Herbelin, 1995)
- $\bar{\lambda}\mu\tilde{\mu} - calculus$ (Curien-Herbelin, 2000)

$\bar{\lambda}\mu\tilde{\mu}$: Syntax and Reduction Rules

Syntax 3 categories: **commands**, **terms** and **evaluation contexts**

$$c ::= \langle t \| e \rangle$$
$$t ::= x \mid \mu\alpha.c \mid \lambda x.t$$
$$e ::= \alpha \mid \tilde{\mu}x.c \mid t \cdot e$$

Reduction
(μ) $\langle \mu\alpha.c \| e \rangle \to c[e/\alpha]$
$(\tilde{\mu})$ $\langle t \| \tilde{\mu}x.c \rangle \to c[t/x]$
(β) $\langle \lambda x.t \| t' \cdot e \rangle \to \langle t' \| \tilde{\mu}x.\langle t \| e \rangle \rangle$

Critical pair $\langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle$: CBN, CBV strategies

The Sequent Calculus $\mathrm{LK}_{\mu\tilde{\mu}}$

$$\frac{\Gamma \vdash t : A | \Delta \qquad \Gamma | e : A \vdash \Delta}{\langle v \| e \rangle : (\Gamma \vdash \Delta)} \;(\mathrm{Cut})$$

$$\frac{}{(x : A), \Gamma \vdash x : A | \Delta} \;(\mathrm{Ax}_R) \qquad \frac{}{\Gamma | \alpha : A \vdash (\alpha : A), \Delta} \;(\mathrm{Ax}_L)$$

$$\frac{c : (\Gamma \vdash (\alpha : A), \Delta)}{\Gamma \vdash \mu\alpha.c : A | \Delta} \;(\mu) \qquad \frac{c : ((x : A), \Gamma \vdash \Delta)}{\Gamma | \tilde{\mu}x.c : A \vdash \Delta} \;(\tilde{\mu})$$

$$\frac{(x : A), \Gamma \vdash (t : B) | \Delta}{\Gamma \vdash \lambda x.t : A \to B | \Delta} \;(\to_R) \qquad \frac{\Gamma \vdash t : A | \Delta \qquad \Gamma | e : B \vdash \Delta}{\Gamma | t \cdot e : A \to B \vdash \Delta} \;(\to_L)$$

Kripke Semantics for $LK_{\mu\tilde{\mu}}$: Motivation

2 "new" ingredients for the model cake:

exploding nodes

- Intuitionistic completeness for Kripke semantics has seen only a **classical** proof
- Actually, the proof **cannot be** intuitionistic (Gödel, Kreisel, 1962)
- Yet, an **intuitionistic** proof by Veldman (1976), introducing exploding nodes (also Friedman, ????)
- The same relaxation happens for the intuitionistic proof for classical logic (Krivine, 1996)

refinement of $\Vdash$

- identify "strongly refutes" as primitive, define "forcing" and "refutation" by orthogonality

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Kripke Semantics for Classical Logic

A *Kripke model* is given by:

- a partial order (K,$\leq$) of **worlds**
- a relation "$\Vdash$" between worlds and *atomic* formulas, called ~~**forcing**~~ **strong refutation**
- a marker of **exploding** worlds $\Vdash_\perp$

Than we define by orthogonality:

forcing $w :\Vdash A$ iff for all $w' \geq w$, $w' : |A \Vdash$ implies $w' :\Vdash_\perp$

refutation $w : A \Vdash$ iff for all $w' \geq w$, $w' :\Vdash A$ implies $w' :\Vdash_\perp$

mutually with the extension of *strong* refutation to composite formulas:

$\mathbf{w} : \lvert\Box \Vdash$	
$A \vee B$	w refutes A and w refutes B
$A \rightarrow B$	w forces A and w refutes B
$\forall x P(x)$	there is $a \in D(w)$ such that w refutes $P(a)$
$\perp$	is ?

Primitive Forcing vs Defined Forcing

- Implication is OK

$$w :\Vdash A \to B \iff \text{for any } w' \geq w, \text{ if } w' :\Vdash A \text{ then } w' :\Vdash B$$

- Disjunction is NOT (reported by Gyesik)

$$w :\Vdash A \vee B \quad \not\Longleftarrow\!\!\Longrightarrow \quad w :\Vdash A \text{ or } w :\Vdash B$$

Soundness of $LK_{\mu\tilde{\mu}}$ for Kripke Semantics

$$c : (\Gamma \vdash \Delta) \implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w :\Vdash_{\bot}$$
$$\Gamma \vdash t : A|\Delta \implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w :\Vdash A$$
$$\Gamma|e : A \vdash \Delta \implies \text{for any } w, w :\Vdash \Gamma \text{ and } w : \Delta \Vdash \text{ implies } w : A \Vdash$$

Proof.

By mutual induction on the derivations. □

The Universal Kripke Model

Context Semantics

We prove Completeness for a **universal** Kripke model. From that, Completeness for any Kripke model follows.

The Universal Kripke model **U** is obtained by putting

possible worlds K is $\{(\Gamma,\Delta)|\Gamma : \text{tvar} \to \text{typ}, \Delta : \text{evar} \to \text{typ}\}$

partial order $(\Gamma,\Delta) \leq (\Gamma',\Delta')$ iff $\Gamma \subseteq \Gamma'$ and $\Delta \subseteq \Delta'$

exploding nodes $(\Gamma,\Delta) :\Vdash_\bot$ iff $\Gamma \vdash \Delta$

Completeness of Kripke Semantics for $\text{LK}_{\mu\tilde{\mu}}$

$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \text{there is a term } t \text{ such that } \Gamma \vdash t : A|\Delta$$
$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \text{there is an ev. context } e \text{ such that } \Gamma|e : A \vdash \Delta$$

$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \text{there is a term } t \text{ such that } \Gamma \vdash t : A|\Delta$$
$$\text{Neutral}_{\text{trm}}(x, A, \Gamma, \Delta) \Longrightarrow (\Gamma, \Delta) :\Vdash A$$
$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \text{there is an ev. context } e \text{ such that } \Gamma|e : A \vdash \Delta$$
$$\text{Neutral}_{\text{ect}}(\alpha, A, \Gamma, \Delta) \Longrightarrow (\Gamma, \Delta) : A \Vdash$$

Proof.

By induction on the type A, but it is necessary to talk about **neutral** terms and **neutral** evaluation contexts. □

Definition ($\text{Neutral}_{\text{trm}}(x, A, \Gamma, \Delta)$)

Future Work

- Check if we have *strong* completeness à la Okada
- If so, prove the correctness of the NbE algorithm w.r.t. an axiomatised reduction relation for $\bar{\lambda}\mu\tilde{\mu}$
- Work on the *call-by-value* variant of Kripke semantics
- Generalize from $\to, \land, \forall$ to generic connectives
- Check for connections with the problem of disjunction for intuitionistic logic
- Try to use the theorems for "applied formal meta-theory in Coq"

Coq Formalisation Issues

- `Combined Scheme` working only in `Prop`
- `Fixpoint with ... with Definition`