Radial Basis Function method for black-box optimization and application to architectural design

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Outline

1 Preliminaries

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2. Radial Basis Function method
   - 1. Selection of k points
   - 2. Computation of the interpolant
   - 3. Selection of next point (x_{k+1})
   - 4. Research questions

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4. Conclusions

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Find the (global) optimum for this problem

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- Ψ: feasible region.

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Goal

- Find the global optimum with the lowest possible number of evaluations of the objective function through the oracle.

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Some possibilities

- Kriging [Jones; JOGO, 2001]
- Efficient Global Optimization (EGO) [Jones et al.; JOGO, 1998]
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We consider the RBF method.

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- How to choose these k points (i.e., what are the x_i)?

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This means that $k \geq n + 1$.

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What is a radial function?

A function $g : \mathbb{R}^n \to \mathbb{R}$ is called radial if there exists a function $\phi : [0, +\infty) \to \mathbb{R}$ such that

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Intuitively, the function depends from the distance of a point $x \in \mathbb{R}^n$ from the origin.

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Hence, a point can be determined as a linear combination of $\phi(\|x - x_i\|)$, $\forall x_i \in X$.

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Given the set of points $X = \{x_1 \dots x_k\}$, $x_i \in \mathbb{R}^n$ and a RBF function ϕ, the RBF interpolant of the points in X is defined as:

$$s_k(x) := \sum_{i=1}^{k} \lambda_i \phi(\|x - x_i\|) + p(x),$$

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We may need $p(x)$ in order to ensure that the interpolant exists!

RBF interpolation - 2

The most common RBF functions are

$\phi(r)$		$d_{\min}$
r	(linear)	0
r^3	(cubic)	1
$r^2 \log r$	(thin splate spline)	1
$\sqrt{r^2+\gamma^2}$	(multiquadric)	0
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The parameter $\gamma > 0$ can be used to change the shape of these functions.

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Condition (2) is used to ensure the existence of the interpolant. It guarantees the conditionally positive definiteness of the RBF (see `[Jakobsson et al.; Optim and Eng., 2010]` for details).

RBF interpolation - 4

This is equivalent to solve the following linear system:

$$\begin{pmatrix} \Phi & P \\ P^T & 0_{Q\times Q} \end{pmatrix} \begin{pmatrix} \lambda \\ h \end{pmatrix} = \begin{pmatrix} F \\ 0_Q \end{pmatrix}$$

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$\Phi\lambda + Ph = F$ is condition (1); $P^T\lambda = 0$ is condition (2).

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We consider the case of cubic and thin splate RBF (i.e., $d_{\min} = 1$), which seem to give best results for our application.

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- Evaluate the function through the oracle in x_{k+1} obtaining f_{k+1} and compute the new RBF interpolant s_{k+1} which respect the conditions $\forall x_i \in X \, s_{k+1}(x_i) = f(x_i)$ and $s_{k+1}(x_{k+1}) = f_k^*$ (solving a linear system as done before)

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Why? Because the RBF interpolant has the property to be the less "bumpy" among all the functions which assume value $f(x_i)$ in x_i.

The bumpiness can be expressed as $(-1)^{d_{\min}+1}\lambda^T\Phi\lambda$.

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It can be shown [Gutmann; JOGO, 2001] that the point x_{k+1} for which s_y is less bumpy is found by solving the optimization problem:

$$x_{k+1} = \arg\max_{x \in \Psi} \frac{1}{(-1)^{d_{\min}+1} \mu_k(x) [s_k(x) - f_k^*]^2},$$

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where $\mu_k(x) = \dfrac{1}{\phi(0) - (u(x)\ \pi(x))^T \begin{pmatrix} \Phi & P \\ P^T & 0 \end{pmatrix}^{-1} (u(x)\ \pi(x))},$

with $u(x) = (\phi(\|x - x_1\|), \ldots, \phi(\|x - x_k\|))^T$ and $\pi(x)$ is $\begin{pmatrix} x^T & 1 \end{pmatrix}$ when $d_{\min} = 1$, 1 when $d_{\min} = 0$ and not used when $d_{\min} = -1$.

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- Step κ (Local search): If $s_k(y^*) < f_{\min} - 10^{-10}|f_{\min}|$ accept y^* as the new sample point x_{k+1}. Otherwise choose $f_k^* = f_{\min} - 10^{-2}|f_{\min}|$.

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How to compute the target value f_k^*? - 2

In general $f_k^* \in [-\infty, s_k(y^*)]$

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So there is no need to solve this problem.

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Last 2 questions: still under investigation. So far local search methods are employed. Is it worth to find the global optimum?

RBF choice: cross validation - 1

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We choose the RBF function which provides the smallest value of error.

RBF choice: cross validation - 2

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RBF choice: cross validation - 2

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Recall that the bumpiness can be expressed as $(-1)^{d_{\min}+1}\lambda^T\Phi\lambda$

It is possible to find the RBF by solving a mathematical programming problem instead of solving a system.

Mathematical programming formulation - 1

The problem of finding the parameters of the interpolant can be modeled as follows:

$$\begin{aligned} \min \quad & f(x) & & \\ \text{s.t.:} \quad & \Phi\lambda + Ph & = & F \\ & P^T\lambda & = & 0_{n+1}, \end{aligned}$$

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Because there is only 1 feasible point for this problem (the same obtained by solving the linear system), with the property to minimize the bumpiness.

Mathematical programming formulation - 2

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\forall i \in \{1,\dots,k\} \setminus \{j\} \quad & 0 \le \xi_i &\le& 0 \\
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ξ_j free, so the constraint $\Phi\lambda + Ph + \xi = F$ involving F_j is deactivated.

Reformulation - 1

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Moreover, since $\forall i \neq j \ \xi_i = 0$ and $\lambda_j = 0$, we have

$$\lambda^T \xi = \sum_{i=1,\, i \neq j}^{k} \lambda_i \xi_i + \lambda_j \xi_j = 0.$$

Reformulation - 2

The resulting model is the following;

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We can use the same model, that is a LP, and only change the bounds for some variables (i.e., ξ_j and λ_j) for cross validation. A solver could solve the k problems needed for the cross validation effciently by warm-starting the dual simplex method.

Where are we?

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2. Radial Basis Function method
   - 1. Selection of k points
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The project

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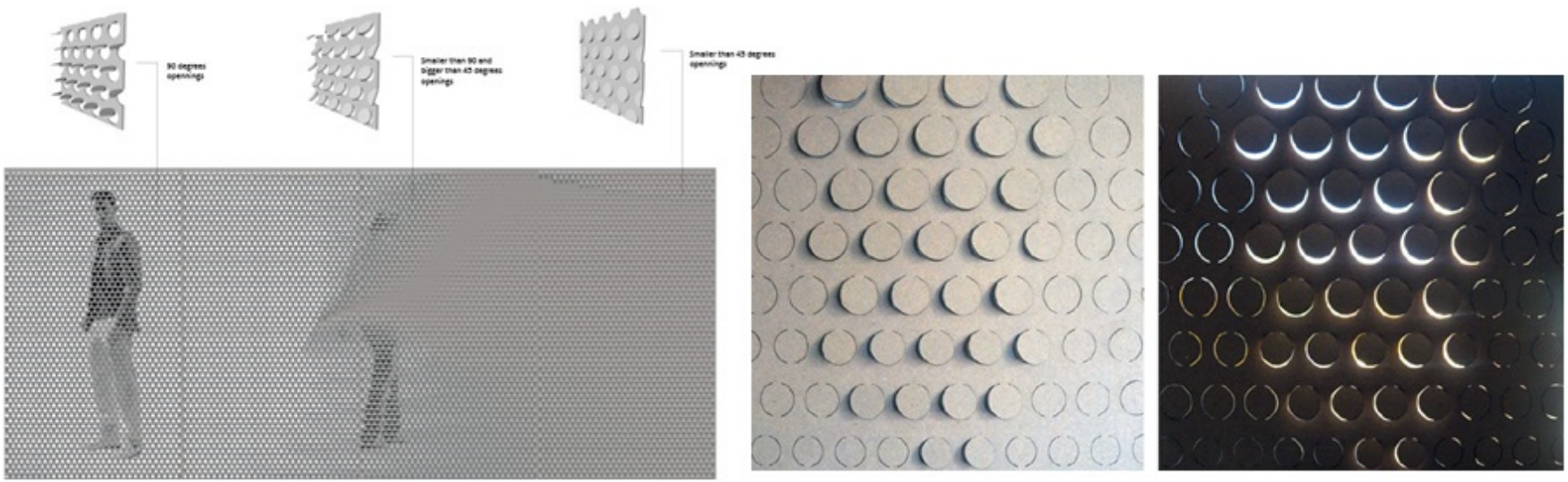
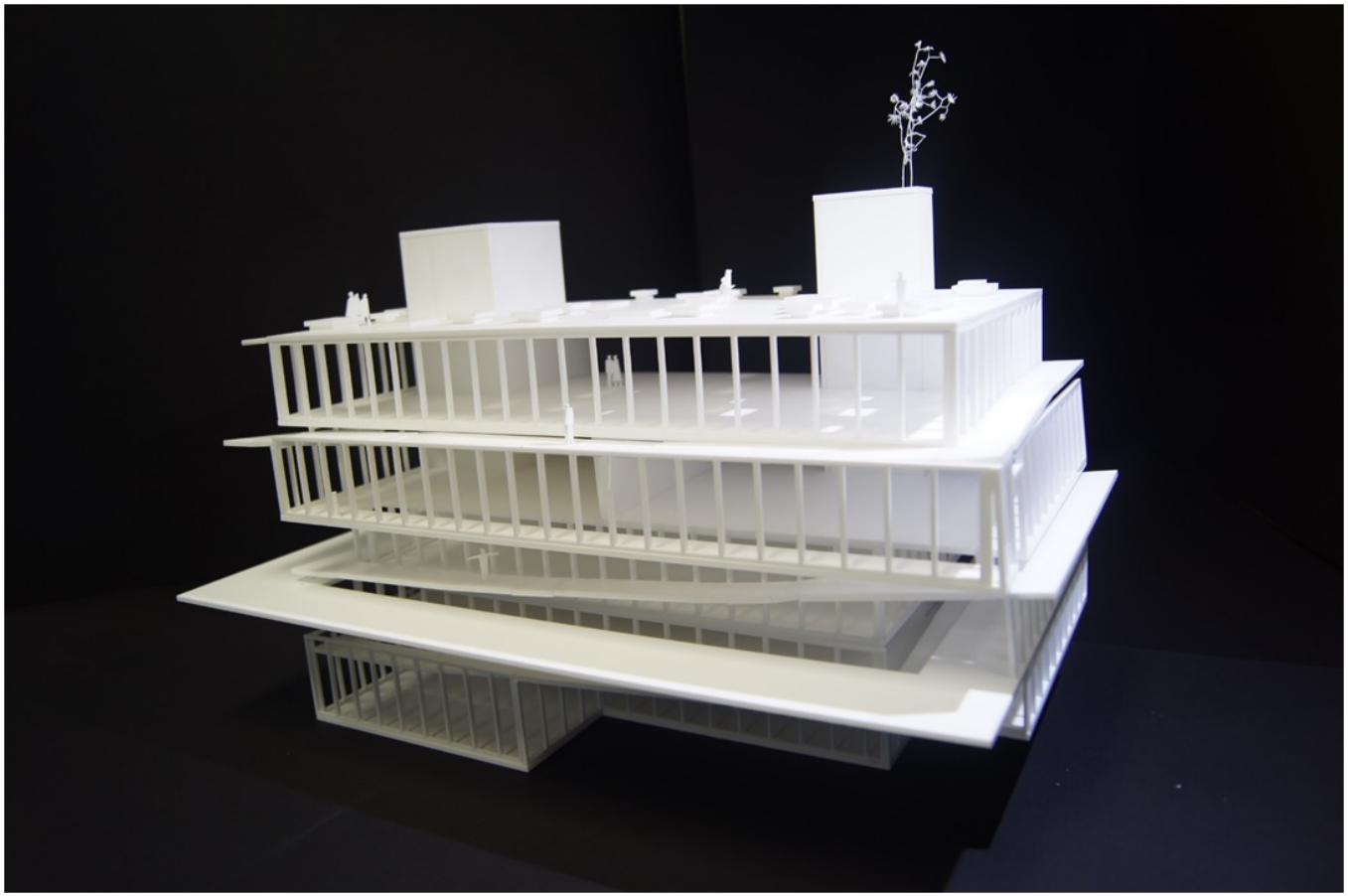
This work is based on the exploration of a performative facade aiming at offering natural light quality.

Facade



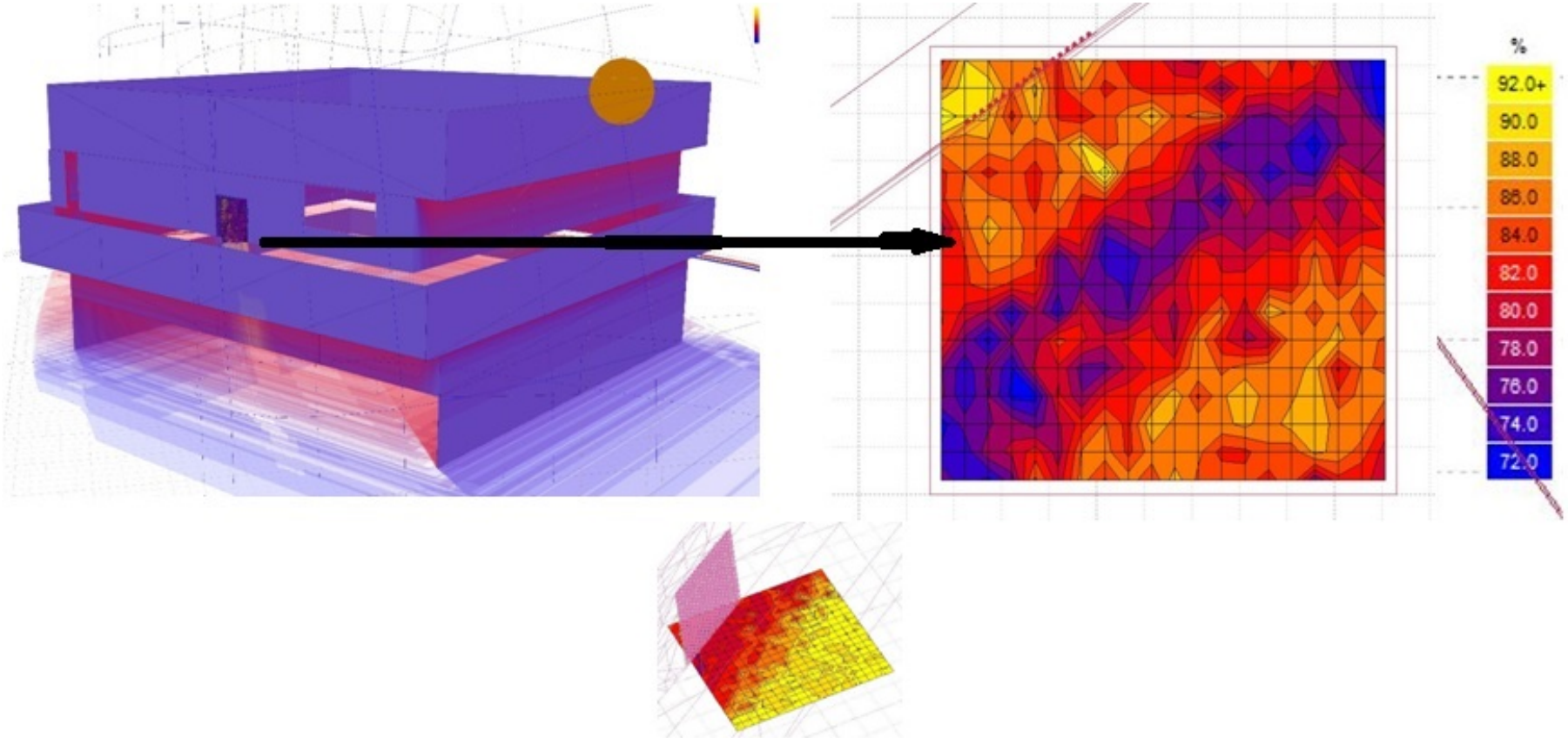
The envelope of the building is a powerful shadowing and filtering light device, depending on the angle of the circled louver.

The church



Simulations

Simulation of the lighthing on the floor of a room (using Ecotect (Autodesk)).



Objective functions

How can we decide about the quality of a solution given by the simulator?

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Different criteria

- Satisfaction to maximize
- Uniformity (neighbors) to minimize
- Uniformity (mean) to minimize
- Uniformity (variance) to minimize

Satisfaction - 1

For each area p of the building that we consider (e.g., a room), we define a satisfaction function $s_p(x) : \mathbb{R}^+ \to [0, 1]$.

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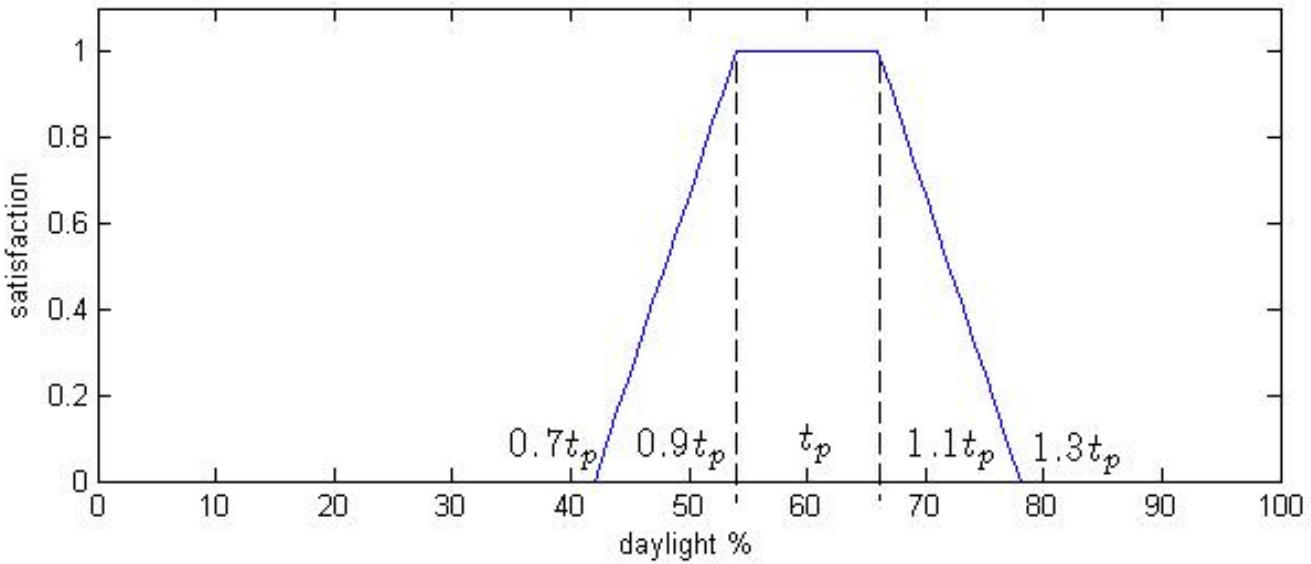
Given a target value t_p of illuminance for the area p, the satisfaction is equal to 1 (i.e., 100%) in the range $t_p \pm 10\%$, 0 if more than or less than 30% from t_p. For the other points, the satisfaction is defined by a linear function.

Satisfaction - 2

Example, $t_p = 60\%$



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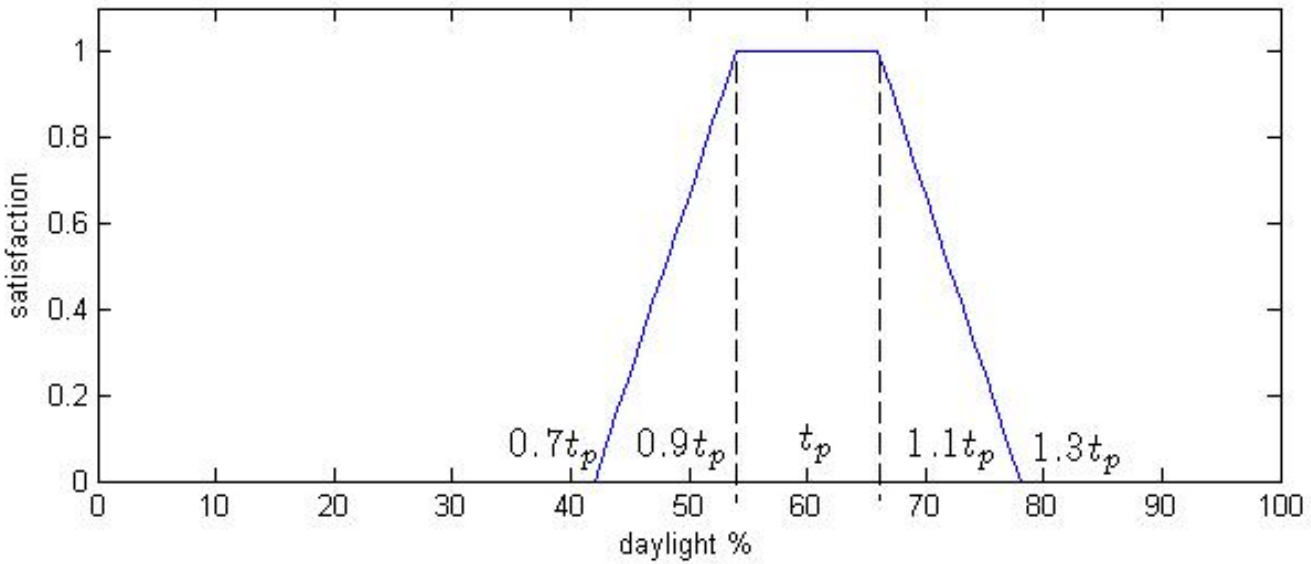

The average satisfaction over an area is the performance measure:

$$S(p) := \frac{1}{|p|} \sum_{i \in p} s_p(x_i).$$

Uniformity (neighbors)

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We can minimize the average difference between the illuminance value of each point in p and those of its neighbors:

$$U_n(p) = \frac{\sum_{i \in p,\, j \in N_p(i) \,:\, i<j} |x_i - x_j|}{\sum_{i \in p,\, j \in N_p(i) \,:\, i<j} 1},$$

where $N_p(i)$ represents the neighbors of the point i in p.

Uniformity (mean)

We can minimize the average difference between the illuminance value of each point in p and the average illuminance in p ($\bar{x}_p$):

$$U_a(p) = \frac{1}{|p|} \sum_{i \in p} |x_i - \bar{x}_p|.$$

Uniformity (variance)

We can alternatively minimize the variance:

$$U_v(p) = \frac{1}{|p|} \sum_{i \in p} (x_i - \bar{x}_p)^2.$$

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- Create a software to integrate to simulator.